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SUPERGENERALIZED FINSLER SPACES

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Abstract

The present paper is a complete version of the lecture [2] presented by the authors to "Romanian-Japanese Colloquium on Finsler Geometry" held in Romania during 15–25 August, 1984. We discuss non-symmetric generalized Finsler metrics, and especially consider the problem of existence and arbitrariness of Finsler connections compatible with such a metric.

Introduction

As a generalized metric a non-symmetric tensor field $g_{ij}(\neq g_{ji})$ has been treated by some authors. For example, in order to obtain a unified field theory, A. Einstein [6] started from a (complex) tensor field g_{ij} with Hermitian symmetry $g_{ij}=\overline{g}_{ji}$. Also L. P. Eisenhart [7] discussed a non-symmetric tensor field $g_{ij}(x)$ as a generalized Riemann metric, and tried to solve a problem to find the set of all linear connections compatible with such a metric. R. Miron-Gh. Atanasiu [16] considered the problem of existence and arbitrariness of such connections, and solved Eisenhart's problem in a natural case.

The purpose of the present paper is to discuss a non-symmetric Finsler tensor fieid $g_{ij}(x,y)$ and to obtain the Finslerian results corresponding to [16]. As a *generalization* of a *generalized* Finsler space (R. Miron [15], M. Hashiguchi [11]), we shall call a space associated with such a generalized metric $g_{ij}(x,y)$ a *g. g. space* or a *supergeneralized Finsler space*, which is hoped to be interesting for physicists. The problem of existence and arbitrariness of Finsler connections compatible with such a metric is reduced to the study of a system of tensorial equations (Theorem 4.1), and in a natural case a condition that such a Finsler connection exists is given (Theorem 4.2, Theorem 4.3), and Eisenhart's problem is solved (Theorem 4.3, Theorem 5.1).

Our motive to the subject is a generalized Finsler metric $g_{ij}(x,y)=e^{2\sigma(x,y)}\gamma_{ij}(x)$ conformal to a Riemann one $\gamma_{ij}(x)$ (S. Ikeda [12], S. Numata [19], S. Watanabe [21], S. Watanabe-S. Ikeda-F. Ikeda [22], etc.). If we take Eisenhart's generalized Riemann metric as γ_{ij}, we have an example of a g. g. space. And the research is in line with Gh. Atanasiu [1], Gh. Atanasiu-B. B. Sinha-S. K. Singh [4], R. Miron [15], M.Hashiguchi [10, 11], and R. Miron-M. Hashiguchi [17, 18]. For a non-symmetric tensor field g_{ij} we have a symmet-

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ric tensor field $\underline{g}_{ij}$ and an alternate one $\underset{\vee}{g}_{ij}$ from the splitting $g_{ij}=\underline{g}_{ij}+\underset{\vee}{g}_{ij}$. Thus the study of g_{ij} is reduced to the study of the pair $(\underline{g}_{ij},\ \underset{\vee}{g}_{ij})$. The study of a complex tensor field g_{ij} with Hermitian symmetry is also reduced to the study of the pair $(s_{ij},\ a_{ij})$ of a symmetric tensor field s_{ij} and an alternate one a_{ij}, where $g_{ij}=s_{ij}+\sqrt{-1}\,a_{ij}$. Interestingly, the method developed in the present paper is also applicable to the pair of two symmetric tensor fields, which will be investigated in our appearing paper [3].

The terminology and notations are referred to M. Matsumoto's monograph [14]. And we also use some of notations in Eisenhart [7] and Miron-Atanasiu [16], under some modifications. In matrix notations $A=(a_{ij})$, $B=(b^{ij})$, $C=(c^i_j)$ we always assume i and j denote the respective positions of the row and the column of the component. Thus $b^{ij}a_{jk}=c^i_k$ and $a_{ij}b^{jk}=c^k_i$ are expressed as $BA=C$ and $AB={}^tC$ respectively, where ${}^tC$ denotes the transpose of C.

1. The notion of g. g. metric

Let M be an n-dimensional differentiable manifold of class C^∞, and $x=(x^i)$ and $y=(y^i)$ denote a point of M and a supporting element respectively. A non-symmetric Finsler tensor field $g_{ij}(x,y)$ of type (0,2) on M is uniquely expressed by the sum of the symmetric part $\underline{g}_{ij}(x,y)$ and the alternate part $\underset{\vee}{g}_{ij}(x,y)$:

(1.1) $$g_{ij}=\underline{g}_{ij}+\underset{\vee}{g}_{ij}\quad(\underline{g}_{ij}=\underline{g}_{ji},\ \underset{\vee}{g}_{ij}=-\underset{\vee}{g}_{ji}).$$

The notations $\underline{g}_{ij}$, $\underset{\vee}{g}_{ij}$, which were originally written as $g_{\underline{ij}}$, $g_{\underset{\vee}{ij}}$ by Eisenhart [7], will be used in the following without comment.

We define a *g. g. space* (or a *supergeneralized Finsler space*) $(M,\ g_{ij})$ as a space M associated with a non-symmetric g_{ij} defined by

Definition 1.1. A Finsler tensor field g_{ij} of type (0,2) on M is called a *g.g.metric* (or a *supergeneralized Finsler metric*) *of index k*, if it satisfies

1) $\det(\underline{g}_{ij})\neq 0$,

2) $\operatorname{rank}(\underset{\vee}{g}_{ij})=n-k=2p$,

where k, p are integers and $0\leqq k<n$.

The Finsler tensor field $\underline{g}_{ij}$ determines a generalized Finsler metric on M. The matrix $(\underline{g}_{ij})$ has the inverse $(\underline{g}^{jk})$, but the matrix $(\underset{\vee}{g}_{ij})$ is not regular except for a remarkable case $k=0$. So in general we shall define some matrix $(\underset{\vee}{g}^{jk})$ such that $(\underset{\vee}{g}^{jk})=(\underset{\vee}{g}_{ij})^{-1}$ for the case $k=0$. If $k>0$ and $(\underline{g}_{ij})$ is positive-definite, then on each local chart there are exactly k independent Finsler vector fields $\xi^i_1,\ \cdots,\ \xi^i_k$ with the properties

(1.2) $$\underset{\vee}{g}_{ij}\xi^j_a=0,\quad \underline{g}_{ij}\xi^i_a\xi^j_b=\delta_{ab}\quad(a,b=1,\ \cdots,\ k).$$

Introducing local Finsler covector fields

(1.3) $$\eta^a_i=\underline{g}_{ij}\xi^j_a\quad(a=1,\ \cdots,\ k),$$

we define local Finsler tensor fields l^i_j and m^i_j of type (1,1) by

(1.4) $$l^i_j=\xi^i_a\eta^a_j,\quad m^i_j=\delta^i_j-\xi^i_a\eta^a_j\qquad(\text{summed for } a).$$

Remark 1.1. In case of $k=0$ we put $l^i_j=0$, $m^i_j=\delta^i_j$. In case of $k>0$, if $(\underline{g}_{ij})$ is not positive-definite, we assume that there exist exactly k independent Finsler vector fields $\xi^i_1, \cdots, \xi^i_k$ with the properties (1.2).

On the other hand, if $(\underline{g}_{ij})$ is positive-definite, then (g_{ij}) is also so, but in case that $(\underline{g}_{ij})$ is not positive-definite, (g_{ij}) is not necessarily regular.

Then we have from (1.2), (1.3), (1.4)

$$g_{ij} l^j_k = 0, \quad g_{ij} m^j_k = g_{ik}, \tag{1.5}$$

$$\eta^a_j \xi^j_b = \delta^a_b, \quad \eta^a_j l^j_k = \eta^a_k, \quad \eta^a_j m^j_k = 0, \tag{1.6}$$

$$\underline{g}_{ij} l^j_k = \sum_a \eta^a_i \eta^a_k, \quad \underline{g}_{ij} m^j_k = \underline{g}_{ik} - \sum_a \eta^a_i \eta^a_k, \tag{1.7}$$

$$l^i_j \underline{g}^{jk} = \sum_a \xi^i_a \xi^k_a, \quad m^i_j \underline{g}^{jk} = \underline{g}^{ik} - \sum_a \xi^i_a \xi^k_a. \tag{1.8}$$

The formulas (1.7), (1.8) show

Proposition 1.1. $\underline{g}_{ij} l^j_k$, $\underline{g}_{ij} m^j_k$, $l^i_j \underline{g}^{jk}$ *and* $m^i_j \underline{g}^{jk}$ *are symmetric with respect to the indices* i, k *respectively.*

Now, for the module $\boldsymbol{X}$ of Finsler vector fields on $\boldsymbol{M}$, we consider two submodules $\boldsymbol{K}$ and $\boldsymbol{H}$:

$$\boldsymbol{K} = \{ \xi^j \in \boldsymbol{X} \mid g_{ij} \xi^j = 0 \}, \tag{1.9}$$

$$\boldsymbol{H} = \{ \zeta^j \in \boldsymbol{X} \mid \underline{g}_{ij} \xi^i \zeta^j = 0 \text{ for all } \xi^i \in \boldsymbol{K} \}. \tag{1.10}$$

$\boldsymbol{K}$ is globally defined as the kernel of the mapping $g_{ij} : \xi^j \to g_{ij}\xi^j$. $\boldsymbol{H}$ is orthogonal to $\boldsymbol{K}$, and is also globally defined. The Finsler distributions $\boldsymbol{K}$ and $\boldsymbol{H}$ are called the *kernel distribution of* g_{ij} and the *orthogonal distribution to* $\boldsymbol{K}$ respectively. Since $\boldsymbol{K}$ is generated by $\{ \xi^i_a \}$, we have the following Propositions.

Proposition 1.2. *The following three conditions are mutually equivalent*:

(1) $\zeta^j \in \boldsymbol{H}$, (2) $\eta^a_j \zeta^j = 0$, (3) $l^i_j \zeta^j = 0$.

Proposition 1.3. *The following system of equations has the trivial solution* $X^j = 0$ *only*:

$$g_{ij} X^j = 0, \quad \eta^a_j X^j = 0 \quad (a = 1, \cdots, k). \tag{1.11}$$

Now we have from (1.5), (1.6) and Proposition 1.2

$$l^i_j X^j \in \boldsymbol{K}, \quad m^i_j X^j \in \boldsymbol{H} \text{ for any } X^j \in \boldsymbol{X}, \tag{1.12}$$

and we have further

$$l^i_j + m^i_j = \delta^i_j, \tag{1.13}$$

$$l^i_j l^j_k = l^i_k, \quad m^i_j m^j_k = m^i_k, \quad l^i_j m^j_k = m^i_j l^j_k = 0, \tag{1.14}$$

$$\underline{g}_{ij} l^i_k m^j_l = 0. \tag{1.15}$$

It is shown from (1.12), (1.13), Proposition 1.3 and (1.10) that $\boldsymbol{K}$ and $\boldsymbol{H}$ are supplementarily orthogonal:

$$\boldsymbol{X} = \boldsymbol{K} + \boldsymbol{H}, \quad \boldsymbol{K} \cap \boldsymbol{H} = \{ 0 \}, \tag{1.16}$$

$$\underline{g}_{ij} \xi^i \zeta^j = 0 \text{ for any } \xi^i \in \boldsymbol{K}, \ \zeta^j \in \boldsymbol{H}. \tag{1.17}$$

Then (1.12), (1.13), (1.14) show that l^i_j, m^i_j are locally the projectors from $\boldsymbol{X}$ on $\boldsymbol{K}$ and $\boldsymbol{H}$ respectively. From the uniqueness of the projections, however, they are independent on the choice of ξ^i_a. Thus l^i_j, m^i_j are globally defined, and are the orthogonal projectors from $\boldsymbol{X}$ on $\boldsymbol{K}$ and $\boldsymbol{H}$.

In the following we shall sometimes use matrix expressions. Putting $\underline{g}=(\underline{g}_{ij})$, $g=(g_{ij})$, $\xi=(\xi^i_a)$, $\eta=(\eta^a_j)$, $l=(l^i_j)$, $m=(m^i_j)$, and $\delta=(\delta_{ab})$, (δ^i_j), etc., the formulas (1.2) ~ (1.8) are expressed as

(1.2′) $$g\xi=0,\ {}^t\xi\underline{g}\xi=\delta,$$

(1.3′) $${}^t\eta=\underline{g}\xi,$$

(1.4′) $$l=\xi\eta,\ m=\delta-l,$$

(1.5′) $$gl=0,\ gm=g,$$

(1.6′) $$\eta\xi=\delta,\ \eta l=\eta,\ \eta m=0,$$

(1.7′) $$\underline{g}l={}^t\eta\eta,\ \underline{g}m=\underline{g}-{}^t\eta\eta,$$

(1.8′) $$l\underline{g}^{-1}=\xi{}^t\xi,\ m\underline{g}^{-1}=\underline{g}-\xi{}^t\xi,$$

and (1.13) ~ (1.15) are expressed as

(1.13′) $$l+m=\delta,$$

(1.14′) $$l^2=l,\ m^2=m,\ lm=ml=0,$$

(1.15′) $${}^tl\underline{g}m=0.$$

And Proposition 1.1 is expressed by

(1.18) $$\underline{g}l={}^tl\underline{g},\ \underline{g}m={}^tm\underline{g},\ l\underline{g}^{-1}=\underline{g}^{-1}{}^tl,\ m\underline{g}^{-1}=\underline{g}^{-1}{}^tm.$$

In order to get some regular matrix from g, we extend g to the alternate $(n+k, n+k)$-matrix

$$\tilde{g}=\begin{pmatrix} g & -{}^t\eta \\ \eta & 0 \end{pmatrix}.$$

The following system of equations, with respect to unknown column vectors X, Y, has the trivial solution $X=Y=0$ only:

$$\begin{pmatrix} g & -{}^t\eta \\ \eta & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}=0, \text{ i.e., } \begin{cases} gX-{}^t\eta Y=0, \\ \eta X=0. \end{cases}$$

In fact, multiplying $gX-{}^t\eta Y=0$ by ${}^t\xi$, we get $Y=0$ from (1.2′), (1.6′), and then we get $X=0$ from Proposition 1.3. Hence $\tilde{g}$ is regular. Since the inverse of an alternate matrix is also alternate, the inverse $\tilde{g}^{-1}$ has a form $\begin{pmatrix} \breve{g} & \lambda \\ -{}^t\lambda & \nu \end{pmatrix}$, where the matrices $\breve{g}$ and ν are alternate. From $\tilde{g}\tilde{g}^{-1}=\delta$ we have

(1.19) $$g\breve{g}=\delta-{}^t(\lambda\eta),\ \eta\breve{g}=0,$$

(1.20) $$g\lambda-{}^t\eta\nu=0,\ \eta\lambda=\delta.$$

Since (1.20) is equivalent to $g(\lambda-\xi)-{}^t\eta\nu=0$, $\eta(\lambda-\xi)=0$, in the similar way as stated above we have $\nu=0$ and $\lambda=\xi$, and (1.19) becomes $g\breve{g}=\delta-{}^t(\xi\eta)$, $\eta\breve{g}=0$, which is equivalent to $g\breve{g}={}^tm$, $l\breve{g}=0$. Thus we have

$$\tilde{g}^{-1}=\begin{pmatrix} \breve{g} & \xi \\ -{}^t\xi & 0 \end{pmatrix},$$

where the alternate matrix $\breve{g}=(\breve{g}^{jk})$ is given by

Proposition 1.4. $\breve{g}=(\breve{g}^{jk})$ *is uniquely determined by*

$$\underset{\smile}{g}\breve{g}={}^t m,\quad l\breve{g}=0. \tag{1.21}$$

Since $\breve{g}^{jk}$ is uniquely determined by the global equations, it is independent on the choice of ξ^i_a, and globally defined. Especially, if $k=0$, then $l=0$, $m=\delta$ and $\breve{g}=\underset{\smile}{g}^{-1}$.

Remark 1.2. In the papers [1, 16] $\breve{g}$ is given by $\underset{\smile}{g}^t\breve{g}={}^t m$, $l\breve{g}=0$ and differs in the sign.

Remark 1.3. In Definition 1.1 it is assume that g_{ij} is really non-symmetric ($\underset{\smile}{g}_{ij}\neq 0$, i.e., $k<n$), but the above discussions hold still good for the symmetric case ($\underset{\smile}{g}_{ij}=0$, i.e., $k=n$). Then $\eta=\xi^{-1}$, $l=\delta$, $m=0$ and $\breve{g}=0$.

On the other hand, the alternate case ($\underline{g}_{ij}=0$) was investigated as an almost symplectic Finsler structure a_{ij} (Miron-Hashiguchi [18], $k=0$) and an almost horsymplectic Finsler structure ($\underset{\smile}{g}_{ij}$, η^a_i, ξ^i_a) (Atanasiu [1], $k>0$).

2. Obata's operators with respect to a g. g. metric

For a g. g. metric g_{ij} we have Obata's operators $\underset{1}{\Lambda}$, $\underset{2}{\Lambda}$ ([15], [20]) with respect to the corresponding generalized Finsler metric $\underline{g}_{ij}$:

$$\underset{1}{\Lambda}{}^{ir}_{sj}=\frac{1}{2}(\delta^i_s\delta^r_j-\underline{g}_{sj}\underline{g}^{ir}),\quad \underset{2}{\Lambda}{}^{ir}_{sj}=\frac{1}{2}(\delta^i_s\delta^r_j+\underline{g}_{sj}\underline{g}^{ir}). \tag{2.1}$$

These have the symmetry $\underset{\alpha}{\Lambda}{}^{ir}_{sj}=\underset{\alpha}{\Lambda}{}^{ri}_{js}$ ($\alpha=1,2$), and act on a tensor field K of type (1,2) as

$$(\underset{\alpha}{\Lambda}K)^i_{jk}=\underset{\alpha}{\Lambda}{}^{ir}_{sj}K^s_{rk}\quad (\alpha=1,2). \tag{2.2}$$

Since $(\underset{\alpha}{\Lambda}\underset{\beta}{\Lambda}K)^i_{jk}=\underset{\alpha}{\Lambda}{}^{ir}_{sj}\underset{\beta}{\Lambda}{}^{sm}_{nr}K^n_{mk}$, the product $\underset{\alpha}{\Lambda}\underset{\beta}{\Lambda}$ is defined by

$$(\underset{\alpha}{\Lambda}\underset{\beta}{\Lambda})^{im}_{nj}=\underset{\alpha}{\Lambda}{}^{ir}_{sj}\underset{\beta}{\Lambda}{}^{sm}_{nr}\quad (\alpha,\beta=1,2). \tag{2.3}$$

Proposition 2.1. $\underset{1}{\Lambda}$, $\underset{2}{\Lambda}$ *are the supplementary projectors on the module* $\boldsymbol{T}^1_2$ *of the tensor fields of type* (1,2):

$$\underset{1}{\Lambda}+\underset{2}{\Lambda}=I, \tag{2.4}$$

$$\underset{\alpha}{\Lambda}{}^2=\underset{\alpha}{\Lambda},\quad \underset{1}{\Lambda}\underset{2}{\Lambda}=\underset{2}{\Lambda}\underset{1}{\Lambda}=0\quad (\alpha=1,2), \tag{2.5}$$

where I is the identity given by $\delta^i_s\delta^r_j$: $IK=K$.

Further, with respect to the alternate part $\underset{\smile}{g}_{ij}$, we introduce the operators $\underset{1}{\phi}$, $\underset{2}{\phi}$:

$$\underset{1}{\phi}{}^{ir}_{sj}=\frac{1}{2}(\delta^i_s\delta^r_j+l^i_s l^r_j-\underset{\smile}{g}_{sj}\breve{g}^{ir}),\quad \underset{2}{\phi}{}^{ir}_{sj}=\frac{1}{2}(\delta^i_s\delta^r_j-l^i_s l^r_j+\underset{\smile}{g}_{sj}\breve{g}^{ir}). \tag{2.6}$$

These have also the symmetry $\underset{\alpha}{\phi}{}^{ir}_{sj}=\underset{\alpha}{\phi}{}^{ri}_{js}$ ($\alpha=1,2$), and act on $\boldsymbol{T}^1_2$ in the same way as (2.2). If we define the product of the operators in the same way as (2.3), we have

$$\underset{1}{\phi}+\underset{2}{\phi}=I, \tag{2.7}$$

(2.8) $$\underset{\alpha}{\phi}^2=\underset{\alpha}{\phi}-\frac{1}{2}\theta,\ \underset{1}{\phi}\underset{2}{\phi}=\underset{2}{\phi}\underset{1}{\phi}=\frac{1}{2}\theta\ (\alpha=1,2),$$

where θ is the operator defined by

(2.9) $$\theta_{sj}^{ir}=\frac{1}{2}(l_s^i m_j^r+m_s^i l_j^r).$$

Since θ satisfies

(2.10) $$\underset{\alpha}{\phi}\theta=\theta\underset{\alpha}{\phi}=\theta^2=\frac{1}{2}\theta\ (\alpha=1,2),$$

if we modify $\underset{\alpha}{\phi}$ by

(2.11) $$\underset{1}{O}=\underset{1}{\phi}-\theta,\ \underset{2}{O}=\underset{2}{\phi}+\theta,$$

we have

Proposition 2.2. *$\underset{1}{O}$, $\underset{2}{O}$ are the supplementary projectors on the module T_2^1 :*

(2.12) $$\underset{1}{O}+\underset{2}{O}=I,$$

(2.13) $$\underset{\alpha}{O}^2=\underset{\alpha}{O},\ \underset{1}{O}\underset{2}{O}=\underset{2}{O}\underset{1}{O}=0\ (\alpha=1,2).$$

$\underset{1}{O}$, $\underset{2}{O}$ are expressed as

(2.14) $$\underset{1}{O}{}_{sj}^{ir}=\frac{1}{2}(\delta_s^i\delta_j^r-\delta_s^i l_j^r-l_s^i\delta_j^r+3l_s^i l_j^r-g_{sj}g^{ir}),$$
$$\underset{2}{O}{}_{sj}^{ir}=\frac{1}{2}(\delta_s^i\delta_j^r+\delta_s^i l_j^r+l_s^i\delta_j^r-3l_s^i l_j^r+g_{sj}g^{ir}),$$

which are the operators $\underset{\alpha}{\Phi}{}_{sj}^{ir}$ of the structure $(g_{ij},\ \eta_i^a,\ \xi_a^i)$ of Atanasiu [1] (cf. Remark 1.3).

For later use it is noted

Proposition 2.3. *$\underset{1}{O}$ satisfies*

(2.15) $$l_i^r g_{sj}\underset{1}{O}{}_{nr}^{sm}=0,$$ (2.16) $$l_s^i m_j^r\underset{1}{O}{}_{nr}^{sm}=0.$$

From (2.10) we have

(2.17) $$\underset{1}{O}\theta=\theta\underset{1}{O}=0,\ \underset{2}{O}\theta=\theta\underset{2}{O}=\theta.$$

θ is commutable with $\underset{\alpha}{\phi}$ and $\underset{\alpha}{O}$. Paying attention to Proposition 1.1, it is directly verified that θ is also commutable with $\underset{\alpha}{\Lambda}$:

(2.18) $$\underset{\alpha}{\Lambda}\theta=\theta\underset{\alpha}{\Lambda}\ (\alpha=1,2).$$

Not $\underset{\alpha}{\phi}$ nor $\underset{\alpha}{O}$ is necessarily commutable with $\underset{\alpha}{\Lambda}$. For example,

(2.19) $$4(\underset{1}{\Lambda}\underset{1}{\phi}-\underset{1}{\phi}\underset{1}{\Lambda})_{nj}^{im}=g_{sj}\underset{\smile}{g}^{ir}g_{nr}\underset{\smile}{g}^{sm}-g_{sj}\underset{-}{g}^{ir}g_{nr}\underset{-}{g}^{sm}.$$

From (2.4), (2.7), (2.11), (2.12) and (2.18) we can show

Proposition 2.4. *The following eight commutativities hold, if any one of them holds :*

(2.20) $$\underset{\alpha}{\Lambda}\underset{\beta}{\phi}=\underset{\beta}{\phi}\underset{\alpha}{\Lambda},\ \underset{\alpha}{\Lambda}\underset{\beta}{O}=\underset{\beta}{O}\underset{\alpha}{\Lambda}\ (\alpha,\beta=1,2).$$

We shall here give some results about the tensorial equations. Let $X\in T_2^1$ be unknown, and U, V and $W\in \boldsymbol{T}^1_2$ be given.

Proposition 2.5. *$\underset{2}{\Lambda}X=0$ has solutions, and its general solutions are given by $X=\underset{1}{\Lambda}Y$, where $Y\in \boldsymbol{T}^1_2$ is arbitrary.*

Proposition 2.6. *The following two equations are mutually equivalent:*

(1) $\quad \underset{2}{\phi}X=W,\qquad$ (2) $\quad \underset{2}{O}X=W+2\theta W.$

Proof. Multiplying each of (1) and (2) by θ we have $\theta X=2\theta W$. Thus the proof of Proposition 2.6 follows from (2.11).

Theorem 2.1. *Let us suppose $\underset{1}{\Lambda}\underset{1}{O}=\underset{1}{O}\underset{1}{\Lambda}$. The system of equations*

$$\underset{2}{\Lambda}X=U,\quad \underset{2}{O}X=V \tag{2.21}$$

has solutions if and only if

$$\underset{1}{\Lambda}U=0,\quad \underset{1}{O}V=0,\quad \underset{2}{O}U=\underset{2}{\Lambda}V, \tag{2.22}$$

and then its general solutions are given by

$$X=U+\underset{1}{\Lambda}V+\underset{1}{\Lambda}\underset{1}{O}Y, \tag{2.23}$$

or equivalently

$$X=V+\underset{1}{O}U+\underset{1}{O}\underset{1}{\Lambda}Y, \tag{2.24}$$

where $Y\in \boldsymbol{T}^1_2$ is arbitrary.

Proof. The necessity (2.22) is obvious from (2.5), (2.13) and (2.20). Now, let us assume (2.22). Since $\underset{2}{\Lambda}X=U$ is equivalent to $\underset{2}{\Lambda}(X-U)=0$, so $\underset{2}{\Lambda}X=U$ has the solutions $X=U+\underset{1}{\Lambda}Y$ by Proposition 2.5. X is also a solution of $\underset{2}{O}X=V$ if and only if $\underset{2}{O}U+\underset{2}{O}\underset{1}{\Lambda}Y=V$, as follows from (2.12) which is equivalent to $U+\underset{1}{\Lambda}Y=V+\underset{1}{O}U+\underset{1}{O}\underset{1}{\Lambda}Y$. Thus X should have a form (2.24). In consequence of the last of (2.22) this is also expressed as (2.23). And it is evident from the forms of (2.23), (2.24) that such a tensor field X satisfies (2.21) really.

3. Natural g. g. metrics

In the proof of Theorem 2.1 the assumption $\underset{1}{\Lambda}\underset{1}{O}=\underset{1}{O}\underset{1}{\Lambda}$ is essential, and it seems natural to continue our discussions under the assumption.

Definition 3.1. A g. g. metric is called *natural* if the commutativities (2.20) hold.

Theorem 3.1. *A g. g. metric g_{ij} is natural if and only if there exists a non-vanishing Finsler function μ such that*

$$\breve{g}_{ij}=\mu g_{ij}, \tag{3.1}$$

where $\breve{g}_{ij}=\underline{g}_{ir}\underline{g}_{js}\breve{g}^{rs}$.

Proof. Because of (2.19) it follows that $\underset{1}{\Lambda}\underset{1}{\phi}=\underset{1}{\phi}\underset{1}{\Lambda}$ is equivalent to $\underline{g}_{sj}\breve{g}^{hr}\underline{g}_{nr}\breve{g}^{sm}=g_{sj}\breve{g}^{hr}\underline{g}_{nr}\underline{g}^{sm}$, whose contraction by $g_{im}\underline{g}_{th}$ implies

$$\breve{g}_{ij}g_{tn}=\breve{g}_{tn}g_{ij}. \tag{3.2}$$

Since $g_{tn}\breve{g}^{nt}=m^t_t=n-k\neq 0$, if we contract (3.2) by $\breve{g}^{nt}$, we have (3.1) with $\mu=$

$(\check{g}_{tn}g^{nt})/(n-k)$. Then $\mu \neq 0$. In fact, $\mu=0$ implies $\check{g}_{ij}=0$, i.e., $\check{g}^{rs}=0$, which means $m^i_j=0$ and contradicts $k<n$.

Conversely, if (3.1) holds good for some $\mu(\neq 0)$, then we have (3.2), which is equivalent to $\underset{1}{\Lambda}\underset{1}{\phi}=\underset{1}{\phi}\underset{1}{\Lambda}$.

Remark 3.1. The generalized Finsler metric $\underline{g}_{ij}$ serves for lowering of indices. Then (3.1) means that the tensor field $\check{g}_{ij}$ associate to $\check{g}^{rs}$ is proportional to $\underline{g}_{ij}$. (3.1) is also expressed as

$$\underline{g}\check{g}\underline{g}=\mu\underline{g}. \tag{3.1'}$$

Suggested by D.E.Blair-G.D.Ludden-K.Yano [5] and S.I.Goldberg-K.Yano [8, 9] we shall give typical examples for natural g. g. metrics.

Definition 3.2. In a generalized Finsler space (M, g_{ij}), let a Finsler tensor field $F=(F^i_j)$ (resp. $P=(P^i_j)$) of type (1,1), k Finsler vector fields ξ^i_a $(a=1, \ldots, k)$ and k Finsler covector fields η^a_i $(a=1, \ldots, k)$ be given.

The set $(F^i_j, \xi^i_a, \eta^a_i, \underline{g}_{ij})$ is called an *$(F, \xi, \eta, \underline{g})$-structure of index k* on M, if it satisfies

$$F^2=-\delta+\xi\eta,\quad \eta F=0,\quad F\xi=0,\quad \eta\xi=\delta,\quad {}^tF\underline{g}F=\underline{g}-{}^t\eta\eta. \tag{3.3}$$

The set $(P^i_j, \xi^i_a, \eta^a_i, \underline{g}_{ij})$ is called a *$(P, \xi, \eta, \underline{g})$-structure of index k* on M, if it satisfies

$$P^2=\delta-\xi\eta,\quad \eta P=0,\quad P\xi=0,\quad \eta\xi=\delta,\quad {}^tP\underline{g}P=-\underline{g}+{}^t\eta\eta. \tag{3.4}$$

Remark 3.2. In case of $k=0$, (3.3) and (3.4) are understood to be $F^2=-\delta$, ${}^tF\underline{g}F=\underline{g}$ and $P^2=\delta$, ${}^tP\underline{g}P=-\underline{g}$ respectively. In the former case we have a Finsler almost Hermitian structure. In the latter case we have a Finsler almost product structure. Hence there exists an eigen-vector u such that $Pu=\varepsilon u$ $(\varepsilon=\pm 1)$. Then ${}^tP\underline{g}P=-\underline{g}$ yields ${}^tu\underline{g}u=0$. Thus $\underline{g}_{ij}$ is not positive-definite, which was noted by Y.Ichijyō.

In general case F and P satisfy $F^3=-F$ and $P^3=P$ respectively, and give examples of a Finsler almost f-structure and a Finsler almost product structure.

Theorem 3.2. *Let an $(F, \xi, \eta, \underline{g})$-structure (resp. a $(P, \xi, \eta, \underline{g})$-structure) of index k be given on M. If we define $\underset{\vee}{g}=(\underset{\vee}{g}_{ij})$ by*

$$\underset{\vee}{g}=\frac{1}{c}\underline{g}F \quad (\textit{resp.}\ \underset{\vee}{g}=\frac{1}{c}\underline{g}P) \tag{3.5}$$

for some non-vanishing Finsler function c, then $\underset{\vee}{g}_{ij}$ is alternate, and $g_{ij}=\underline{g}_{ij}+\underset{\vee}{g}_{ij}$ is a natural g. g. metric of index k on M. In this case $\mu=-c^2$ (resp. $\mu=c^2$).

Proof. First, multiplying ${}^tF\underline{g}F=\underline{g}-{}^t\eta\eta$ by ξ, we have $\underline{g}\xi={}^t\eta$, and so ${}^tF\underline{g}\xi={}^t(\eta F)=0$. Thus, multiplying ${}^tF\underline{g}F=\underline{g}-{}^t\eta\eta$ by F and making use of $F^2=-\delta+\xi\eta$, we have ${}^tF\underline{g}=-\underline{g}F$, i.e., ${}^t\underset{\vee}{g}=-\underset{\vee}{g}$, which shows $\underset{\vee}{g}$ is alternate.

Now, let $X=(X^i)$ be any solution of $\underset{\vee}{g}X=0$, i.e., $FX=0$. From $F^2=-\delta+\xi\eta$ we have $X=\xi(\eta X)$. Hence X^i is a linear combination of ξ^i_a $(a=1, \ldots, k)$. Because of $\eta\xi=\delta$, ξ^i_a are linearly independent. Thus rank $\underset{\vee}{g}=n-k$, and so $g_{ij}=\underline{g}_{ij}+\underset{\vee}{g}_{ij}$ is a g.

g. metric of index k. Then $F^2=-m$, ${}^tFgF=\underline{g}m$, where $l=\xi\eta$ and $m=\delta-l$.

If we put $\rho=-cF\underline{g}^{-1}$, then it is shown from the second of (1.18) that ρ satisfies $g\rho={}^tm$, $l\rho=0$. By Proposition 1.4 this ρ is nothing but the matrix $\breve{g}$. Thus we have $\breve{g}=-cF\underline{g}^{-1}$, from which the naturality $\underline{g}\breve{g}\underline{g}=-c^2g$ follows.

In the other case the proof is given in the similar way. Then we have $\breve{g}=cP\underline{g}^{-1}$, from which the naturality $\underline{g}\breve{g}\underline{g}=c^2g$ follows.

It is noted that the converse of Theorem 3.2 is true. Paying attention to the second of (1.18) again, we can easily show

Theorem 3.3. *Let g_{ij} be a natural g. g. metric of index k on M with some non-vanishing Finsler function $\mu=-c^2$ (resp. $\mu=c^2$). If we put*

$$F=c\underline{g}^{-1}\breve{g} \quad (\textit{resp. } P=c\underline{g}^{-1}\breve{g}), \tag{3.6}$$

or equivalently

$$F=-\frac{1}{c}\breve{g}\underline{g} \quad (\textit{resp. } P=\frac{1}{c}\breve{g}\underline{g}), \tag{3.7}$$

then the set $(F^i_j,\ \xi^i_a,\ \eta^a_i,\ g_{ij})$ (resp. the set $(P^i_j,\ \xi^i_a,\ \eta^a_i,\ g_{ij})$) is an (F,ξ,η,g)-structure (resp. a (P,ξ,η,g)-structure) of index k on M.

4. Finsler connections compatible with a g. g. metric

An important problem concerning with a g. g. metric g_{ij} on M is to determine the existence and arbitrariness of Finsler connections with respect to which g_{ij} is covariantly constant. Here a Finsler connection $F\Gamma$ is defined as a triad of a V-connection Γ_V on the linear frame bundle $L(M)$, a non-linear connection N on the tangent bundle $T(M)$ and a vertical connection Γ^v on the Finsler bundle $F(M)$. A Finsler connection having N as the non-linear connection is denoted by $F\Gamma(N)=(F^{\ i}_{j\ k},\ C^{\ i}_{j\ k})$, where $F^{\ i}_{j\ k}$ and $C^{\ i}_{j\ k}$ are the respective coefficients of Γ_V and Γ^v. And the respective h- and v-covariant differentiations of a Finsler tensor field are denoted by a short line and a long line, e.g., $g_{ij|k}$, $g_{ij}|_k$ (with respect to $F\Gamma$), $g_{ij\overset{o}{|}k}$, $g_{ij}\overset{o}{|}_k$ (with respect to $F\overset{o}{\Gamma}$), etc.

For later use it is noted

Proposition 4.1. *With respect to any Finsler connection $F\overset{o}{\Gamma}$ a g. g. metric g_{ij} satisfies*

$$l^i_s l^r_j l^s_{r\overset{o}{|}k}=0,\quad l^i_s l^r_j l^s_r\overset{o}{|}_k=0, \tag{4.1}$$

$$m^i_r l^r_{j\overset{o}{|}k}=l^r_j l^i_{r\overset{o}{|}k},\quad m^i_r l^r_j\overset{o}{|}_k=l^r_j l^i_r\overset{o}{|}_k, \tag{4.2}$$

$$m^s_j l^i_{s\overset{o}{|}k}=l^i_s l^s_{j\overset{o}{|}k},\quad m^s_j l^i_s\overset{o}{|}_k=l^i_s l^s_j\overset{o}{|}_k, \tag{4.3}$$

$$l^r_i l^s_j \underline{g}_{rs\overset{o}{|}k}=0,\quad l^r_i l^s_j \underline{g}_{rs}\overset{o}{|}_k=0, \tag{4.4}$$

$$\underline{g}_{sj}\breve{g}^{ir} l^s_{r\overset{o}{|}k}=0,\quad \underline{g}_{sj}\breve{g}^{ir} l^s_r\overset{o}{|}_k=0, \tag{4.5}$$

$$l^s_j\breve{g}^{ir}\underline{g}_{rs\overset{o}{|}k}=-l^s_j l^i_{s\overset{o}{|}k},\quad l^s_j\breve{g}^{ir}\underline{g}_{rs}\overset{o}{|}_k=-l^s_j l^i_s\overset{o}{|}_k. \tag{4.6}$$

Proof. For a fixed subscript k we put $l_{\overset{o}{|}k}=(l^i_{j\overset{o}{|}k})$, etc. Then from $l^2=l$ we have

$l^{\circ}_{|k}l+ll^{\circ}_{|k}=l^{\circ}_{|k}$, $l\overset{\circ}{|}_{k}l+ll\overset{\circ}{|}_{k}=l\overset{\circ}{|}_{k}$, which imply

$$ll^{\circ}_{|k}l=0,\quad ll\overset{\circ}{|}_{k}l=0, \tag{4.1'}$$

$$ml^{\circ}_{|k}=l^{\circ}_{|k}l,\quad ml\overset{\circ}{|}_{k}=l\overset{\circ}{|}_{k}l, \tag{4.2'}$$

$$l^{\circ}_{|k}m=ll^{\circ}_{|k},\quad l\overset{\circ}{|}_{k}m=ll\overset{\circ}{|}_{k}. \tag{4.3'}$$

In the same way we have from $\underline{g}l=0$

$$^{t}l\underline{g}^{\circ}_{|k}l=0,\quad {}^{t}l\underline{g}\overset{\circ}{|}_{k}l=0, \tag{4.4'}$$

$$\underline{g}l^{\circ}_{|k}\underline{g}=0,\quad \underline{g}l\overset{\circ}{|}_{k}\underline{g}=0, \tag{4.5'}$$

$$g\underline{g}^{\circ}_{|k}l=-l^{\circ}_{|k}l,\quad g\underline{g}\overset{\circ}{|}_{k}l=-l\overset{\circ}{|}_{k}l. \tag{4.6'}$$

These are the matrix expressions of $(4.1)\sim(4.6)$.

Definition 4.1. Let g_{ij} be a g. g. metric. A Finsler connection $F\Gamma$ is called *compatible* with g_{ij} if g_{ij} is covariantly constant:

$$g_{ij|k}=0,\quad g_{ij}|_{k}=0, \tag{4.7}$$

or equivalently

$$g_{ij|k}=0,\quad \underline{g}_{ij|k}=0, \tag{4.8}$$

$$g_{ij}|_{k}=0,\quad \underline{g}_{ij}|_{k}=0. \tag{4.8'}$$

Proposition 4.2. *With respect to a Finsler connection $F\Gamma$ compatible with a g. g. metric g_{ij}, the tensor fields l^{i}_{j}, m^{i}_{j}, g^{ij} and $\underline{g}^{ij}$ are covariantly constant*:

$$l^{i}_{j|k}=0,\quad m^{i}_{j|k}=0, \tag{4.9}$$

$$l^{i}_{j}|_{k}=0,\quad m^{i}_{j}|_{k}=0, \tag{4.9'}$$

$$g^{ij}{}_{|k}=0,\quad \underline{g}^{ij}{}_{|k}=0, \tag{4.10}$$

$$g^{ij}|_{k}=0,\quad \underline{g}^{ij}|_{k}=0. \tag{4.10'}$$

Proof. Since $g_{ij}\xi^{j}_{a}=0$, $\underline{g}_{ij}\xi^{i}_{a}\xi^{j}_{b}=\delta_{ab}$, we have $g_{ij}\xi^{j}_{a|k}=0$, $\underline{g}_{ij}\xi^{i}_{a|k}\xi^{j}_{b}+\underline{g}_{ij}\xi^{i}_{a}\xi^{j}_{b|k}=0$. Hence $\xi^{j}_{a|k}$ is expressed as $\xi^{j}_{a|k}=h^{c}_{ak}\xi^{j}_{c}$, and it holds $h^{b}_{ak}+h^{a}_{bk}=0$. Then $\eta^{a}_{j}=\underline{g}_{jr}\xi^{r}_{a}$ implies $\eta^{a}_{j|k}=\underline{g}_{jr}\xi^{r}_{a|k}=\underline{g}_{jr}h^{b}_{ak}\xi^{r}_{b}=-h^{a}_{bk}\eta^{b}_{j}$. Thus we have $l^{i}_{j|k}=\xi^{i}_{a|k}\eta^{a}_{j}+\xi^{i}_{a}\eta^{a}_{j|k}=0$, and so $m^{i}_{j|k}=-l^{i}_{j|k}=0$.

$g^{ij}{}_{|k}=0$ is evident. $\underline{g}^{ij}{}_{|k}=0$ is also evident, since $g_{rs}\underline{g}^{sj}=m^{j}_{r}$ and $l^{i}_{s}\underline{g}^{sj}=0$ imply $m^{i}_{s}\underline{g}^{sj}{}_{|k}=0$ and $l^{i}_{s}\underline{g}^{sj}{}_{|k}=0$ respectively.

(4.9') and (4.10') are similarly proved.

In order to determine the existence and arbitrariness of Finsler connections compatible with a g. g. metric g_{ij}, we start from an arbitrary Finsler connection $\overset{\circ}{F\Gamma}$, and generalize the so-called *Kawaguchi method* [13] to the supergeneralized case as follows. In the following we fix an arbitrary non-linear connection N.

Theorem 4.1. *Let $\overset{\circ}{F\Gamma}(N)=(\overset{\circ}{F}{}^{i}_{jk}, \overset{\circ}{C}{}^{i}_{jk})$ be a fixed Finsler connection. For a g. g. metric g_{ij} we define the Finsler tensor fields U, V, $\tilde{U}$, $\tilde{V}$ by*

$$U^{i}_{jk}=-\frac{1}{2}\underline{g}^{ir}g_{rj}{}^{\circ}_{|k}, \tag{4.11}$$

$$\tilde{U}^{i}_{jk}=-\frac{1}{2}\underline{g}^{ir}g_{rj}\overset{\circ}{|}_{k}, \tag{4.11'}$$

$$V^{i}_{jk}=-\frac{1}{2}(\underline{g}^{ir}\underline{g}_{rj}{}^{\circ}_{|k}+3l^{i}_{s}l^{s\circ}_{j|k}-l^{i\circ}_{j|k}), \tag{4.12}$$

$$\tilde{V}^{i}_{jk}=-\frac{1}{2}(\underline{g}^{ir}\underline{g}_{rj}\overset{\circ}{|}_{k}+3l^{i}_{s}l^{s}_{j}\overset{\circ}{|}_{k}-l^{i}_{j}\overset{\circ}{|}_{k}). \tag{4.12'}$$

Then a Finsler connection $F\Gamma(N)=(F^{i}_{jk}, C^{i}_{jk})$ is compatible with g_{ij}, if and only if the dif-

ference tensor fields B and D given by

(4.13) $F^i_{jk}=\overset{\circ}{F}{}^i_{jk}-B^i_{jk}$, (4.13′) $C^i_{jk}=\overset{\circ}{C}{}^i_{jk}-D^i_{jk}$

are solutions of the following system of tensorial equations :

(4.14) $\underset{2}{\Lambda}B=U$, (4.14′) $\underset{2}{\Lambda}D=\tilde{U}$,

(4.15) $\underset{2}{O}B=V$, (4.15′) $\underset{2}{O}D=\tilde{V}$,

(4.16) $l^r_i\underline{g}_{sj}B^s_{rk}=-l^r_i\underline{g}_{rj\overset{\circ}{|}k}$, (4.16′) $l^r_i\underline{g}_{sj}D^s_{rk}=-l^r_i\underline{g}_{rj}\overset{\circ}{|}_{k}$,

(4.17) $l^i_sm^r_jB^s_{rk}=-l^i_sl^s_{j\overset{\circ}{|}k}$, (4.17′) $l^i_sm^r_jD^s_{rk}=-l^i_sl^s_j\overset{\circ}{|}_{k}$.

Proof. The conditions (4.8) are equivalent to

$$g_{rj\overset{\circ}{|}k}+\underline{g}_{rs}B^s_{jk}+\underline{g}_{sj}B^s_{rk}=0, \tag{4.18}$$

$$g_{rj\overset{\circ}{|}k}+g_{rs}B^s_{jk}+g_{sj}B^s_{rk}=0. \tag{4.19}$$

Contracting (4.18) by $\underline{g}^{ir}$, we have (4.14). And contracting (4.19) by g^{ir} and l^r_i, we have

$$m^i_sB^s_{jk}+g_{sj}g^{ir}B^s_{rk}=-g^{ir}g_{rj\overset{\circ}{|}k} \tag{4.20}$$

and (4.16) respectively. Conversely, we have (4.18) from (4.14), and also we have (4.19) from (4.20), (4.16).

Now, if $F\Gamma(N)$ is compatible with g_{ij}, then from $l^i_sl^s_{j|k}=0$ we have (4.17). The addition of (4.17) to (4.20) implies

$$\underset{2}{\phi}B=W, \tag{4.21}$$

where W is given by

$$W^i_{jk}=-\frac{1}{2}(g^{ir}g_{rj\overset{\circ}{|}k}+l^i_sl^s_{j\overset{\circ}{|}k}). \tag{4.22}$$

Conversely, we have (4.20) from (4.21), (4.17). By Proposition 2.6, (4.21) is equivalent to (4.15), where

$$V=W+2\theta W. \tag{4.23}$$

Paying attention to (4.1), (4.6), V^i_{jk} is easily reduced to (4.12).

The arguments on D are quite similar.

A g. g. metric g_{ij} does not necessarily admit a Finsler connection compatible with itself. In fact, we have

Theorem 4.2. *Let g_{ij} be a natural g. g. metric : $g_{ij}=\mu g_{ij}$. If there exists a Finsler connection $F\Gamma$ compatible with g_{ij}, the function μ is a non-zero constant.*

Proof. By Proposition 4.2 we have $\mu_{|k}g_{ij}=0$, $\mu|_kg_{ij}=0$, which are reduced to $\mu_{|k}=0$, $\mu|_k=0$ because of $g_{ij}g^{ji}=n-k\neq0$. Hence the non-vanishing function μ is constant.

From the above theorem we are led to consider the two structures $(F,\ \xi,\ \eta,\ \underline{g})$ and $(P,\ \xi,\ \eta,\ \underline{g})$ in which $c=\text{const.}\neq0$.

Definition 4.2. A natural g. g. metric g_{ij} is called *elliptic* if $\mu=-c^2$ and *hyperbolic* if $\mu=c^2$, where c is a positive constant.

We shall show that the converse of Theorem 4.2 is also true.

Proposition 4.3. *Let* g_{ij} *be a natural g. g. metric in which* $c=const.\neq 0$. *Then* U *and* V *(resp.* $\tilde{U}$ *and* $\tilde{V}$*) given by* (4.11) *and* (4.12) *(resp.* (4.11′) *and* (4.12′)*) satisfy*

(4.24) $\underset{1}{\Lambda} U=0,$ (4.24′) $\underset{1}{\Lambda} \tilde{U}=0,$

(4.25) $\underset{1}{O} V=0,$ (4.25′) $\underset{1}{O} \tilde{V}=0,$

(4.26) $\underset{2}{O} U=\underset{2}{\Lambda} V,$ (4.26′) $\underset{2}{O} \tilde{U}=\underset{2}{\Lambda} \tilde{V}.$

Proof. (4.24) is directly shown from (2.1), (4.11). Since from (2.17), (4.23) we have $\underset{1}{O} V=\underset{1}{O} W$, so (4.25) is also directly shown from (2.14), (4.22) if we pay attention to (4.1).

If for X and W of Proposition 2.6 we substitute U and $\underset{2}{\Lambda} W$ respectively, it is shown that $\underset{2}{\phi} U=\underset{2}{\Lambda} W$ is equivalent to $\underset{2}{O} U=\underset{2}{\Lambda} W+2\theta \underset{2}{\Lambda} W=\underset{2}{\Lambda} V$. So we shall show (4.26) by proving $\underset{2}{\phi} U=\underset{2}{\Lambda} W$.

We have directly

$$
\begin{aligned}
-4(\underset{2}{\phi} U)^{i}_{jk}&=(\delta^{i}_{s}\delta^{r}_{j}-l^{i}_{s}l^{r}_{j}+\underline{g}_{sj}\underline{g}^{ir})(\underline{g}^{st}\underline{g}_{tr\overset{o}{|}k})\\
&=-2U^{i}_{jk}-l^{i}_{s}l^{r}_{j}\underline{g}^{st}\underline{g}_{tr\overset{o}{|}k}+\underline{g}^{ts}\underline{g}_{sj}\underline{g}^{ir}\underline{g}_{rt\overset{o}{|}k},\\
-4(\underset{2}{\Lambda} W)^{i}_{jk}&=(\delta^{i}_{s}\delta^{r}_{j}+\underline{g}_{sj}\underline{g}^{ir})(\underline{g}^{st}\underline{g}_{tr\overset{o}{|}k}+l^{s}_{t}l^{t}{}_{r\overset{o}{|}k})\\
&=-2W^{i}_{jk}+\underline{g}^{ts}\underline{g}_{sj}\underline{g}^{ir}\underline{g}_{rt\overset{o}{|}k}+\underline{g}_{sj}\underline{g}^{ir}l^{s}_{t}l^{t}{}_{r\overset{o}{|}k}.
\end{aligned}
$$

First we shall treat the elliptic case. From (3.6) we have $c\underline{g}^{ts}\underline{g}_{sj}=F^{t}_{j}$. Multiplying (3.7) by $\underline{g}$ we have $c\underline{g}F=-{}^{t}m\underline{g}$, i.e., $\underline{g}={}^{t}l\underline{g}-c\underline{g}F$, from which we have $\underline{g}_{rt}=l^{m}_{r}\underline{g}_{mt}-c\underline{g}_{rm}F^{m}_{t}$. Thus we have

$$
\begin{aligned}
\underline{g}^{ts}\underline{g}_{sj}\underline{g}^{ir}\underline{g}_{rt\overset{o}{|}k}&=\underline{g}^{ts}\underline{g}_{sj}\underline{g}^{ir}(l^{m}_{r}\underline{g}_{mt}-c\underline{g}_{rm}F^{m}_{t})_{\overset{o}{|}k}\\
&=\underline{g}^{ts}\underline{g}_{sj}\underline{g}^{ir}(l^{m}_{r}\underline{g}_{mt})_{\overset{o}{|}k}-F^{t}_{j}\underline{g}^{ir}(\underline{g}_{rm}F^{m}_{t})_{\overset{o}{|}k}\\
&=\underline{g}_{sj}\underline{g}^{ir}l^{s}{}_{r\overset{o}{|}k}+m^{s}_{j}\underline{g}^{ir}\underline{g}_{rs\overset{o}{|}k}-m^{i}_{s}F^{t}_{j}F^{s}{}_{t\overset{o}{|}k},
\end{aligned}
$$

where the first term vanishes owing to (4.5), the middle term becomes $\underline{g}^{ir}\underline{g}_{rj\overset{o}{|}k}+l^{s}_{j}l^{i}{}_{s\overset{o}{|}k}$ owing to (4.6), and the last term becomes $-F^{t}_{j}F^{i}{}_{t\overset{o}{|}k}+l^{i}_{s}l^{s}{}_{j\overset{o}{|}k}$ owing to $l^{i}_{s}F^{s}_{t}=0$ and (4.3). Consequently, from (4.22) we have

$$
-4(\underset{2}{\phi} U)^{i}_{jk}=-2(U^{i}_{jk}+W^{i}_{jk})-F^{t}_{j}F^{i}{}_{t\overset{o}{|}k}+l^{s}_{j}l^{i}{}_{s\overset{o}{|}k}-l^{i}_{s}l^{r}_{j}\underline{g}^{st}\underline{g}_{tr\overset{o}{|}k}.
$$

On the other hand, from (3.7) we have $\underline{g}^{ts}\underline{g}_{sj}=-cF^{t}_{j}$. And from (3.6) we have $\underline{g}_{rt}=\dfrac{1}{c}\underline{g}_{rs}F^{s}_{t}$. Thus we have

$$
\underline{g}^{ts}\underline{g}_{sj}\underline{g}^{ir}\underline{g}_{rt\overset{o}{|}k}=-F^{t}_{j}\underline{g}^{ir}(\underline{g}_{rs}F^{s}_{t})_{\overset{o}{|}k}=m^{s}_{j}\underline{g}^{ir}\underline{g}_{rs\overset{o}{|}k}-F^{t}_{j}F^{i}{}_{t\overset{o}{|}k},
$$

that is,

$$
\underline{g}^{ts}\underline{g}_{sj}\underline{g}^{ir}\underline{g}_{rt\overset{o}{|}k}=\underline{g}^{ir}\underline{g}_{rj\overset{o}{|}k}-\underline{g}^{ir}l^{s}_{j}\underline{g}_{rs\overset{o}{|}k}-F^{t}_{j}F^{i}{}_{t\overset{o}{|}k}. \tag{4.27}
$$

Consequently,

$$
-4(\underset{2}{\Lambda} W)^{i}_{jk}=-2(U^{i}_{jk}+W^{i}_{jk})-F^{t}_{j}F^{i}{}_{t\overset{o}{|}k}-\underline{g}^{ir}l^{s}_{t}\underline{g}_{rs\overset{o}{|}k}+\underline{g}_{sj}\underline{g}^{ir}l^{s}_{t}l^{t}{}_{r\overset{o}{|}k}.
$$

Then, paying attention to Proposition 1.1, we have

$$-4(\underset{2}{\phi}U-\underset{2}{\Lambda}W)^i_{jk}=\underline{g}^{ir}l^s_j(l^t_s\underline{g}_{rt}-l^t_r\underline{g}_{st})_{\overset{o}{|}k}=0.$$

In the similar way $\underset{2}{\phi}U=\underset{2}{\Lambda}W$ is proved in the hyperbolic case. (4.24′) ~ (4.26′) are now evident.

Theorem 4.3. *For a natural g. g. metric g_{ij} in which $\mu=const.\neq 0$, there exists a Finsler connection $\overset{o}{F\Gamma}(N)$ compatible with g_{ij}.*

Let $\overset{o}{F\Gamma}(N)$ be a fixed Finsler connection, U, V, $\tilde{U}$, $\tilde{V}$ be the Finsler tensor fields defined by (4.11), (4.12), (4.11′), (4.12′), *and* $Y, Z\in \boldsymbol{T}^1_2$ *be arbitrary. The set of all Finsler connections $F\Gamma(N)$ compatible with g_{ij} is given by* (4.13), (4.13′), *where the difference tensor fields B and D are given by*

$$B=U+\underset{1}{\Lambda}V+\underset{1}{\Lambda}\underset{1}{O}Y,\quad D=\tilde{U}+\underset{1}{\Lambda}\tilde{V}+\underset{1}{\Lambda}\underset{1}{O}Z, \tag{4.28}$$

or equivalently

$$B=V+\underset{1}{O}U+\underset{1}{O}\underset{1}{\Lambda}Y,\quad D=\tilde{V}+\underset{1}{O}\tilde{U}+\underset{1}{O}\underset{1}{\Lambda}Z. \tag{4.29}$$

Proof. By Theorem 2.1 and Proposition 4.3, the system of equations (4.14), (4.15), (4.14′), (4.15′) with respect to B and D has a solution, and the general solutions are given by (4.28) or (4.29). Thus, the theorem follows from Theorem 4.1, if we show that B and D satisfy the conditions (4.16), (4.17), (4.16′), (4.17′), which are directly proved by using the expression (4.29) as follows.

(4.16) follows from (2.15) and

$$\begin{aligned}-2l^r_i g_{sj}V^s_{rk}&=l^r_i g_{sj}(g^{st}g_{tr\overset{o}{|}k}+3l^s_t l^{to}_{r|k}-l^{so}_{r|k})\\&=l^r_i m^t_j g_{rt\overset{o}{|}k}-l^r_i g_{sj}l^{so}_{r|k}\\&=2l^r_i g_{rj\overset{o}{|}k}\ \text{(owing to (4.4))}.\end{aligned}$$

(4.17) follows from (2.16) and

$$\begin{aligned}-2l^i_s m^r_j V^s_{rk}&=l^i_s m^r_j(g^{st}g_{tr\overset{o}{|}k}+3l^s_t l^{to}_{r|k}-l^{so}_{r|k})\\&=2l^i_s m^r_j l^{so}_{r|k}\\&=2l^i_t l^{to}_{j|k}\ \text{(owing to (4.1))}.\end{aligned}$$

(4.16′), (4.17′) are similarly verified.

5. Solutions of Eisenhart's problem in the natural g. g. spaces

Eisenhart's problem, to find the set of all Finsler connections compatible with a g. g. metric, is already solved for the natural case in Theorem 4.2 and Theorem 4.3. We shall give here other expressiors for the solutions.

If we put $Y=Z=0$ in (4.28), we have an example $F\Gamma^*(N)=(F^{*i}_{jk}, C^{*i}_{jk})$ of a Finsler connection compatible with an elliptic or hyperbolic g. g. metric g_{ij} :

$$\begin{aligned}F^{*i}_{jk}&=\overset{o}{F}{}^i_{jk}+\frac{1}{2}\{\underline{g}^{ir}\underline{g}_{rj\overset{o}{|}k}+\underset{1}{\Lambda}{}^{ir}_{sj}(g^{st}g_{tr\overset{o}{|}k}+3l^s_t l^{to}_{r|k}-l^{so}_{r|k})\},\\C^{*i}_{jk}&=\overset{o}{C}{}^i_{jk}+\frac{1}{2}\{\underline{g}^{ir}\underline{g}_{rj}\overset{o}{|}_k+\underset{1}{\Lambda}{}^{ir}_{sj}(g^{st}g_{tr}\overset{o}{|}_k+3l^s_t l^t_r\overset{o}{|}_k-l^s_r\overset{o}{|}_k)\}.\end{aligned}\tag{5.1}$$

Making use of (4.27), in the elliptic case, (5.1) is also expressed as

$$
(5.2)\quad
\begin{aligned}
F^{*}{}^{i}_{jk} &= \overset{o}{F}{}^{i}_{jk} + \frac{1}{4}\{\underline{g}^{ir}(\underline{g}_{rj\overset{o}{|}k} + l^{s}_{j}\underline{g}_{rs\overset{o}{|}k}) + \underset{\vee}{g}^{ir}\underset{\vee}{g}_{rj\overset{o}{|}k} \\
&\qquad + F^{t}_{j}F^{i}{}_{t\overset{o}{|}k} + 2\underset{1}{\Lambda}{}^{ir}_{sj}(3\,l^{s}_{t}\,l^{t}{}_{r\overset{o}{|}k} - l^{s}{}_{r\overset{o}{|}k})\}, \\
C^{*}{}^{i}_{jk} &= \overset{o}{C}{}^{i}_{jk} + \frac{1}{4}\{\underline{g}^{ir}(\underline{g}_{rj}\overset{o}{|}_{k} + l^{s}_{j}\underline{g}_{rs}\overset{o}{|}_{k}) + \underset{\vee}{g}^{ir}\underset{\vee}{g}_{rj}\overset{o}{|}_{k} \\
&\qquad + F^{t}_{j}F^{i}_{t}\overset{o}{|}_{k} + 2\underset{1}{\Lambda}{}^{ir}_{sj}(3\,l^{s}_{t}\,l^{t}_{r}\overset{o}{|}_{k} - l^{s}_{r}\overset{o}{|}_{k})\}.
\end{aligned}
$$

In the similar way, in the hyperbolic case, (5.1) is also expressed as

$$
(5.2')\quad
\begin{aligned}
F^{*}{}^{i}_{jk} &= \overset{o}{F}{}^{i}_{jk} + \frac{1}{4}\{\underline{g}^{ir}(\underline{g}_{rj\overset{o}{|}k} + l^{s}_{j}\underline{g}_{rs\overset{o}{|}k}) + \underset{\vee}{g}^{ir}\underset{\vee}{g}_{rj\overset{o}{|}k} \\
&\qquad - P^{t}_{j}P^{i}{}_{t\overset{o}{|}k} + 2\underset{1}{\Lambda}{}^{ir}_{sj}(3\,l^{s}_{t}\,l^{t}{}_{r\overset{o}{|}k} - l^{s}{}_{r\overset{o}{|}k})\}, \\
C^{*}{}^{i}_{jk} &= \overset{o}{C}{}^{i}_{jk} + \frac{1}{4}\{\underline{g}^{ir}(\underline{g}_{rj}\overset{o}{|}_{k} + l^{s}_{j}\underline{g}_{rs}\overset{o}{|}_{k}) + \underset{\vee}{g}^{ir}\underset{\vee}{g}_{rj}\overset{o}{|}_{k} \\
&\qquad - P^{t}_{j}P^{i}_{t}\overset{o}{|}_{k} + 2\underset{1}{\Lambda}{}^{ir}_{sj}(3\,l^{s}_{t}\,l^{t}_{r}\overset{o}{|}_{k} - l^{s}_{r}\overset{o}{|}_{k})\}.
\end{aligned}
$$

Then Theorem 4.3 is restated as follows.

Theorem 5.1. *The set of all Finsler connections* $F\Gamma(N)=(F^{i}_{jk},\ C^{i}_{jk})$ *compatible with an elliptic or hyperbolic g. g. metric* g_{ij} *is given by*

$$
(5.3)\qquad F^{i}_{jk} = F^{*}{}^{i}_{jk} + \underset{1}{\Lambda}{}^{ir}_{sj}\underset{1}{O}{}^{sm}_{nr}Y^{n}_{mk},\quad C^{i}_{jk} = C^{*}{}^{i}_{jk} + \underset{1}{\Lambda}{}^{ir}_{sj}\underset{1}{O}{}^{sm}_{nr}Z^{n}_{mk},
$$

where $F\Gamma^{*}(N)=(F^{*}{}^{i}_{jk},\ C^{*}{}^{i}_{jk})$ *is the Finsler connection given by* (5.1) (*equivalent to* (5.2) *or* (5.2')), *and* Y^{i}_{jk}, Z^{i}_{jk} *are arbitrary Finsler tensor fields.*

There are some particular important cases. One of them is given by

Definition 5.1. A g. g. metric g_{ij} is called *regular*, if the symmetric part $\underline{g}_{ij}$ is regular, that is, $\underline{g}_{ij}$ satisfies the regularity conditions of Miron [15]:

(1) $(\partial \underline{g}_{ij}/\partial y^{k})y^{i}y^{j}=0$, (2) $\det(A^{i}_{k})\neq 0$,

where $A^{i}_{k}=\delta^{i}_{k}+\underline{g}^{ir}(\partial \underline{g}_{rj}/\partial y^{k})y^{j}$.

In the regular case, the matrix (A^{i}_{k}) has the inverse (B^{i}_{k}), and the equations of the geodesics with respect to $\underline{g}_{ij}$ are expressed in the from

$$
(5.4)\qquad d^{2}x^{i}/ds^{2} + B^{i}_{r}\gamma^{r}_{jk}(dx^{j}/ds)(dx^{k}/ds)=0,
$$

where

$$
(5.5)\qquad \gamma^{i}_{jk} = \frac{1}{2}\underline{g}^{ir}(\partial \underline{g}_{jr}/\partial x^{k} + \partial \underline{g}_{kr}/\partial x^{j} - \partial \underline{g}_{jk}/\partial x^{r}).
$$

Thus we can take $\overset{c}{N}$ given by $\overset{c}{N}{}^{i}_{k}=\frac{1}{2}\partial(B^{i}_{r}\gamma^{r}_{st}y^{s}y^{t})/\partial y^{k}$ as a canonical non-linear connection. It is noted that $\overset{c}{N}$ is determined by $\underline{g}_{ij}$ only. Then a canonical Finsler connec. tion $\overset{c}{F\Gamma}(\overset{c}{N})=(\overset{c}{F}{}^{i}_{jk},\ \overset{c}{C}{}^{i}_{jk})$ is defined by

(5.6)
$$\overset{c}{F}{}^i_{jk}=\frac{1}{2}g^{ir}(\delta g_{jr}/\delta x^k+\delta g_{kr}/\delta x^j-\delta g_{jk}/\delta x^r),$$
$$\overset{c}{C}{}^i_{jk}=\frac{1}{2}g^{ir}(\partial g_{jr}/\partial y^k+\partial g_{kr}/\partial y^j-\partial g_{jk}/\partial y^r),$$

where $\delta/\delta x^k=\partial/\partial x^k-\overset{c}{N}{}^i_k\,(\partial/\partial y^i)$. $\overset{c}{F\Gamma}(\overset{c}{N})$ was introduced by Miron [15], and was called the *Miron-Cartan connection* by Hashiguchi [11]. It is also noted that this connection is characterized as $F\Gamma(\overset{c}{N})$ satisfying the Cartan-like conditions:

(5.7)
$$g_{ij\overset{c}{|}k}=0,\quad g_{ij}\overset{c}{|}_k=0,\quad \overset{c}{F}{}^i_{jk}=\overset{c}{F}{}^i_{kj},\quad \overset{c}{C}{}^i_{jk}=\overset{c}{C}{}^i_{kj}.$$

If we take $\overset{c}{F\Gamma}(\overset{c}{N})$ as $\overset{o}{F\Gamma}(N)$ in (5.1) (equivalent to (5.2) or (5.2′)), we have a remarkable example $\overset{\Omega}{F\Gamma}(\overset{c}{N})$, which we shall call *canonical.*

Theorem 5.2. *Let g_{ij} be a regular elliptic or hyperbolic g. g. metric, and $\overset{c}{F\Gamma}(\overset{c}{N})$ be the Miron-Cartan connection with respect to g_{ij}. The following Finsler connection $\overset{\Omega}{F\Gamma}(\overset{c}{N})=(\overset{\Omega}{F}{}^i_{jk}, \overset{\Omega}{C}{}^i_{jk})$ is compatible with g_{ij}:*

(5.8)
$$\overset{\Omega}{F}{}^i_{jk}=\overset{c}{F}{}^i_{jk}+\frac{1}{2}\underset{1}{\Lambda}{}^{ir}_{sj}(g^{st}g_{tr\overset{c}{|}k}+3\,l^s_t\,l^t_{r\overset{c}{|}k}-l^s_{r\overset{c}{|}k}),$$
$$\overset{\Omega}{C}{}^i_{jk}=\overset{c}{C}{}^i_{jk}+\frac{1}{2}\underset{1}{\Lambda}{}^{ir}_{sj}(g^{st}g_{tr}\overset{c}{|}_k+3\,l^s_t\,l^t_r\overset{c}{|}_k-l^s_r\overset{c}{|}_k),$$

which is expressed in the elliptic case as

(5.9)
$$\overset{\Omega}{F}{}^i_{jk}=\overset{c}{F}{}^i_{jk}+\frac{1}{4}\{g^{ir}g_{rj\overset{c}{|}k}+F^t_jF^i_{t\overset{c}{|}k}+2\underset{1}{\Lambda}{}^{ir}_{sj}(3\,l^s_t\,l^t_{r\overset{c}{|}k}-l^s_{r\overset{c}{|}k})\},$$
$$\overset{\Omega}{C}{}^i_{jk}=\overset{c}{C}{}^i_{jk}+\frac{1}{4}\{g^{ir}g_{rj}\overset{c}{|}_k+F^t_jF^i_t\overset{c}{|}_k+2\underset{1}{\Lambda}{}^{ir}_{sj}(3\,l^s_t\,l^t_r\overset{c}{|}_k-l^s_r\overset{c}{|}_k)\},$$

and in the hyperbolic case as

(5.9′)
$$\overset{\Omega}{F}{}^i_{jk}=\overset{c}{F}{}^i_{jk}+\frac{1}{4}\{g^{ir}g_{rj\overset{c}{|}k}-P^t_jP^i_{t\overset{c}{|}k}+2\underset{1}{\Lambda}{}^{ir}_{sj}(3\,l^s_t\,l^t_{r\overset{c}{|}k}-l^s_{r\overset{c}{|}k})\},$$
$$\overset{\Omega}{C}{}^i_{jk}=\overset{c}{C}{}^i_{jk}+\frac{1}{4}\{g^{ir}g_{rj}\overset{c}{|}_k-P^t_jP^i_t\overset{c}{|}_k+2\underset{1}{\Lambda}{}^{ir}_{sj}(3\,l^s_t\,l^t_r\overset{c}{|}_k-l^s_r\overset{c}{|}_k)\}.$$

Theorem 5.3. *For a regular elliptic or hyperbolic g. g. metric g_{ij}, the set of all Finsler connections $F\Gamma(\overset{c}{N})$ compatible with g_{ij} is given by*

(5.10)
$$F^i_{jk}=\overset{\Omega}{F}{}^i_{jk}+\underset{1}{\Lambda}{}^{ir}_{sj}\underset{1}{O}{}^{sm}_{nr}Y^n_{mk},\quad C^i_{jk}=\overset{\Omega}{C}{}^i_{jk}+\underset{1}{\Lambda}{}^{ir}_{sj}\underset{1}{O}{}^{sm}_{nr}Z^n_{mk},$$

where $\overset{\Omega}{F\Gamma}(\overset{c}{N})=(\overset{\Omega}{F}{}^i_{jk}, \overset{\Omega}{C}{}^i_{jk})$ is the canonical Finsler connection, and Y^i_{jk}, Z^i_{jk} are arbitrary Finsler tensor fields.

Remark 5.1. As is noted in Remark 1.1, for a remarkable case $k=0$ ($\det(g_{ij})\neq0$), we put $l^i_j=0$. Thus (4.1) ~ (4.6), (4.16) and (4.17) become trivial, and the formulas given in this section are simplified.

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