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EDGEWORTH EXPANSION FOR TWO-SAMPLE U-STATISTICS*

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Abstract

Formal Edgeworth expansion with remainder term $o(N^{-1})$ is established for two-sample U-statistics. And the conditions which ensure the validity of the expansion are also discussed.

1. Introduction

Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be independent random samples from distributions with c. d. f.'s(cumulative distribution functions) F(x) and G(y), respectively. Let $h(x_1, \cdots, x_r; y_1, \cdots, y_s)$ be symmetric in its x_i components and separately symmetric in its y_j components, satisfying $E[h(X_1, \cdots, X_r; Y_1, \cdots, Y_s)]=0$ with $r \leqq m$ and $s \leqq n$. h is called a kernel and (r, s) are called its degree. We shall define a two-sample U-statistic with a kernel of degree (r, s), h, by

$$U_{m,n}=\binom{m}{r}^{-1}\binom{n}{s}^{-1}\sum_{C_{m,r}}\sum_{C_{n,s}} h(X_{i_1}, \cdots, X_{i_r}; Y_{j_1}, \cdots, Y_{j_s})$$

where $\sum_{C_{m,r}}$ indicates that the summation is over $1 \leqq i_1 < \cdots < i_r \leqq n$.

In this paper, putting $N=m+n$, we shall discuss an asymptotic expansion under the assumption

$$\text{(A)} \qquad 0<\lambda=\lim_{N\to\infty} m/N<1.$$

This assumption means that $m=0(N)$ and $n=0(N)$.

Callaert, Janssen and Veraverbeke[1] have obtained the asymptotic expansion of one-sample U-statistic with a kernel of degree two. And Maesono[6] has obtained it with a kernel of arbitrary degree.

In Section 2 we state a representation for $U_{m,n}$ in terms of forward martingales and get the bounds of absolute moments of martingales. In Section 3, using the martingale representation of $U_{m,n}$, we obtain formal Edgeworth expansion of $U_{m,n}$ with remainder term $o(N^{-1})$. In Section 4, we discuss the conditions for the valid expansion.

2. Preliminaries

We shall represent $U_{m,n}$ in terms of forward martingales. Hoeffding[4] (cf.

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Serfling[7] p178) has obtained the similar representation for one-sample U-statistics. Under the assumption $\mathrm{E}|h(X_1, \cdots, X_r; Y_1, \cdots, Y_s)|<\infty$, let us define the following notations:

for $0\leqq a\leqq r$ and $0\leqq b\leqq s$

$$w_{a,b}(x_1, \cdots, x_a; y_1, \cdots, y_b)$$
$$=\mathrm{E}[h(X_1, \cdots, X_r; Y_1, \cdots Y_s)|X_1=x_1, \cdots, X_a=x_a, Y_1=y_1, \cdots, Y_b=y_b],$$

$$g_{0,0}=0, \quad g_{1,0}(x_1)=w_{1,0}(x_1), \quad g_{0,1}(y_1)=w_{0,1}(y_1),$$

$$g_{2,0}(x_1, x_2)=w_{2,0}(x_1, x_2)-\sum_{i=1}^{2}g_{1,0}(x_i), \quad g_{1,1}(x_1; y_1)=w_{1,1}(x_1; y_1)-g_{1,0}(x_1)-g_{0,1}(y_1),$$

$$g_{0,2}(y_1, y_2)=w_{0,2}(y_1, y_2)-\sum_{j=1}^{2}g_{0,1}(y_j),$$
$$\vdots$$
$$g_{r,s}(x_1, \cdots, x_r; y_1, \cdots, y_s)=w_{r,s}(x_1, \cdots, x_r; y_1, \cdots, y_s)$$
$$-\sum_{b=0}^{s-1}\sum_{C_{s,b}}g_{r,b}(x_1, \cdots, x_r; y_{j_1}, \cdots, y_{j_b})-\sum_{a=0}^{r-1}\sum_{C_{r,a}}g_{a,s}(x_{i_1}, \cdots, x_{i_a}; y_1, \cdots, y_s)$$
$$-\sum_{a=0}^{r-1}\sum_{b=0}^{s-1}\sum_{C_{r,a}}\sum_{C_{s,b}}g_{a,b}(x_{i_1}, \cdots, x_{i_a}; y_{j_1}, \cdots, y_{j_b}),$$

for $0\leqq a\leqq r$ and $0\leqq b\leqq s$

$$A_{a,b}=\sum_{C_{m,a}}\sum_{C_{n,b}}g_{a,b}(X_{i_1}, \cdots, X_{i_a}; Y_{j_1}, \cdots, Y_{j_b}).$$

Then $U_{m,n}$ can be rewritten as

$$U_{m,n}=\binom{m}{r}^{-1}\binom{n}{s}^{-1}\sum_{a=0}^{r}\sum_{b=0}^{s}\binom{m-a}{r-a}\binom{n-b}{s-b}A_{a,b}.$$

It is shown in the proof of Lemma 2.3 that $A_{a,b}$ is a forward martingale for each a and b ($a=0, 1, \cdots, r$; $b=0, 1, \cdots, s$).

By the definition of $g_{a,b}$, Lemma 2.1 follows.

LEMMA 2.1. *Assume that* $\mathrm{E}|h(X_1, \cdots, X_r; Y_1, \cdots, Y_s)|<\infty$. *If one of* $\{i_1, \cdots, i_a\}$ *is not contained in* $\{p_1, \cdots, p_c\}$, *or one of* $\{j_1, \cdots, j_b\}$ *is not contained in* $\{q_1, \cdots, q_d\}$, *then*

$$\mathrm{E}[g_{a,b}(X_{i_1}, \cdots, X_{i_a}; Y_{j_1}, \cdots, Y_{j_b})|X_{p_1}, \cdots, X_{p_c}; Y_{q_1}, \cdots, Y_{q_d}]=0. \tag{2.1}$$

Proof. By double induction on a and b we can prove (2.1) directly.

Using Lemma 2.1, we can prove the useful two lemmas.

LEMMA 2.2. *Assume the assumptions in Lemma* 2.1. *Then for any function* f *which satisfies* $\mathrm{E}|fg_{a,b}|<\infty$, *we have*

$$\mathrm{E}[f(X_{p_1}, \cdots, X_{p_c}; Y_{q_1}, \cdots, Y_{q_d})g_{a,b}(X_{i_1}, \cdots, X_{i_a}; Y_{j_1}, \cdots, Y_{j_d})]=0. \tag{2.2}$$

Proof. Taking the conditional expectation, we have the desired result from (2.1).

Before describing the next lemma, we prepare notations. For $1\leqq m_1<\cdots<m_a\leqq m$, $1\leqq n_1<\cdots<n_b\leqq n$, $0\leqq a\leqq r$ and $0\leqq b\leqq s$, let us define

$$B_{a,b}(m_1, \cdots, m_a; n_1, \cdots, n_b)$$
$$=\sum_{i_1=1}^{m_1}\cdots\sum_{i_a=i_{a-1}+1}^{m_a}\sum_{j_1=1}^{n_1}\cdots\sum_{j_b=j_{b-1}+1}^{n_b} g_{a,b}(X_{i_1}, \cdots, X_{i_a}; Y_{j_1}, \cdots, Y_{j_b}).$$

Then we have the upper bound of the pth absolute moment of $B_{a,b}$.

Lemma 2.3. *Given the existence of the pth* $(p\geqq 2)$ *absolute moment of kernel* h, *there exist a positive constant* C *such that*

$$\mathrm{E}|B_{a,b}(m_1, \cdots, m_a; n_1, \cdots, n_b)|^p\leqq C(\prod_{i=1}^{a} m_i\prod_{j=1}^{b} n_j)^{\frac{p}{2}}. \tag{2.3}$$

If the second moment of kernel h *is finite, the inequality* (2.3) *also holds with* $p=1$.

Proof. The latter part of the lemma immediately follows from the former. Therefore we consider the case $p\geqq 2$. By double induction on c and d, we shall prove the following inequality : for $0\leqq c\leqq a$, $0\leqq d\leqq b$, $1\leqq u_1<\cdots<u_c<i_{c+1}, \cdots, i_a$, and $1\leqq q_1<\cdots<q_d<j_{d+1}, \cdots, j_b$,

$$\begin{aligned}&\mathrm{E}|\sum_{i_1=1}^{u_1}\cdots\sum_{i_c=i_{c-1}+1}^{u_c}\sum_{j_1=1}^{q_1}\cdots\sum_{j_d=j_{d-1}+1}^{q_d}\\ &\qquad g_{a,b}(X_{i_1}, \cdots, X_{i_c}, \cdots, X_{i_a}; Y_{j_1}, \cdots, Y_{j_d}, \cdots, Y_{j_b})|^p\\ &\qquad\leqq (C_p)^{c+d}\mathrm{E}|g_{a,b}(X_1, \cdots, X_a; Y_1, \cdots, Y_b)|^p \quad (\prod_{i=1}^{c} u_i\prod_{j=1}^{d} q_j)^{\frac{p}{2}}\end{aligned} \tag{2.4}$$

where $C_p=\{8(p-1)\max(1, 2^{p-3})\}^p$.

When $c=1$ and $d=0$, let $Z_k=\sum_{i_1=1}^{k} g_{a,b}(X_{i_1}, \cdots, X_{i_a}; Y_{j_1}, \cdots, Y_{j_b})$ for $k=1, 2, \cdots, u_1$. Then we have $Z_k-Z_{k-1}=g_{a,b}(X_k, X_{i_2}, \cdots, X_{i_a}; Y_{j_1}, \cdots, Y_{j_b})$ and $k<i_2, \cdots, i_a$, where $Z_0=0$. Since $Z_1, \cdots, Z_{k-1}$ are functions of $X_1, \cdots, X_{k-1}, X_{i_2}, \cdots, X_{i_a}$ and $Y_{j_1}, \cdots, Y_{j_b}$, we find from lemma 2.1 that

$$\begin{aligned}\mathrm{E}[Z_k-Z_{k-1}|Z_1, \cdots, Z_{k-1}]=\mathrm{E}[\mathrm{E}\{&g_{a,b}(X_k, X_{i_2}, \cdots, X_{i_a}; Y_{j_1}, \cdots, Y_{j_b})\\ &|X_1, \cdots, X_{k-1}, X_{i_2}, \cdots, X_{i_a}, Y_{j_1}, \cdots, Y_{j_b}\}|Z_1, \cdots, Z_{k-1}]=0.\end{aligned}$$

Therefore $\{Z_k\}_{0\leqq k\leqq u_1}$ is a forward martingale. Applying an upper bound for moments of martingale obtained by Dharmadhikari, Fabian and Jogdeo[2], we have the inequality (2.4), when $c=1$ and $d=0$.

Using equation (2.1), the rest of the proof is similarly obtained.

3. Formal Edgeworth expansion

Let us define the following notations :

$$\sigma_{m,n}^2=\mathrm{Var}(U_{m,n}),$$

for $0\leqq a\leqq r$ and $0\leqq b\leqq s$

$$k_{a,b}(m, n)=\binom{m}{r}^{-1}\binom{n}{s}^{-1}\sigma_{m,n}^{-1}\binom{m-a}{r-a}\binom{n-b}{s-b},$$

$$\xi_{a,b}^2=\mathrm{E}[g_{a,b}(X_1, \cdots, X_a; Y_1, \cdots, Y_b)]^2,$$

$$\tau_{m,n}^2=\frac{r^2}{m}\xi_{1,0}^2+\frac{s^2}{n}\xi_{0,1}^2,$$

$$\eta(t)=\mathrm{E}[\exp\{itg_{1,0}(X_1)\}],\quad \nu(t)=\mathrm{E}[\exp\{itg_{0,1}(Y_1)\}].$$

Note that from the equation (2.2) in Lemma 2.2,

$$\sigma_{m,n}^2=\binom{m}{r}^{-2}\binom{n}{s}^{-2}\sum_{a=0}^{r}\sum_{b=0}^{s}\binom{m-a}{r-a}^2\binom{n-b}{s-b}^2\binom{m}{a}\binom{n}{b}\xi_{a,b}^2.$$

In this section we assume the following condition

(C1) $$\mathrm{E}|h(X_1,\ \cdots,\ X_r;\ Y_1,\ \cdots,\ Y_s)|^5<\infty.$$

Before we obtain the expansion, we prepare the useful lemma.

LEMMA 3.1. *If* (C1) *is satisfied, then there exist positive constants* ε *and* δ *such that for* $0\leqq t\leqq\varepsilon N^{\frac{1}{2}}$ *and fixed integers* u *and* v,

(3.1) $$\left|I_{u,v}-\eta^{m-u}\left(\frac{rt}{m\sigma_{m,n}}\right)\nu^{n-v}\left(\frac{st}{n\sigma_{m,n}}\right)\right|\leqq o(N^{-1})P_1(t)e^{-\delta t^2},$$

where $P_1(t)$ *is a polynomial in* t *and*

$$\begin{aligned}I_{u,v}=\mathrm{e}^{-\frac{t^2}{2}}\Big[1&-\frac{(it)^2}{2\tau_{m,n}^2}\left\{\frac{r^2s^2}{mn}\xi_{1,1}^2+\frac{[r(r-1)]^2}{2m^2}\xi_{2,0}^2+\frac{[s(s-1)]^2}{2n^2}\xi_{0,2}^2\right.\\&\left.+\frac{ur^2}{m^2}\xi_{1,0}^2+\frac{vs^2}{n^2}\xi_{0,1}^2\right\}+\frac{(it)^3}{6\tau_{m,n}^3}\left\{\frac{r^3}{m^2}\mathrm{E}[g_{1,0}^3(X_1)]+\frac{s^3}{n^2}\mathrm{E}[g_{0,1}^3(Y_1)]\right\}\\&+\frac{(it)^4}{24\tau_{m,n}^4}\left\{\frac{r^4}{m^3}(\mathrm{E}[g_{1,0}^4(X_1)]-3\xi_{1,0}^4)+\frac{s^4}{n^3}(\mathrm{E}[g_{0,1}^4(Y_1)]-3\xi_{0,1}^4)\right\}\\&+\frac{(it)^6}{72\tau_{m,n}^6}\left\{\frac{r^3}{m^2}\mathrm{E}[g_{1,0}^3(X_1)]+\frac{s^3}{n^2}\mathrm{E}[g_{0,1}^3(Y_1)]\right\}^2\Big].\end{aligned}$$

Proof. By the same way of Lemma 2 as Callaert et al.[1], we have the desired result easily.

In order to obtain a formal Edgeworth expansion, we shall find the function $\tilde{\chi}_{m,n}(t)$ which satisfies

(3.2) $$\int_0^{N^{\frac{1}{4}}/\log N}t^{-1}|\chi_{m,n}(t)-\tilde{\chi}_{m,n}(t)|\,dt=o(N^{-1})$$

where $\chi_{m,n}(t)=\mathrm{E}[\exp(it\sigma_{m,n}^{-1}U_{m,n})]$. (See Callaert et al. [1] and Maesono[6].)

Note that

$$\sigma_{m,n}^{-1}U_{m,n}=\sum_{a=0}^{r}\sum_{b=0}^{s}k_{a,b}(m,\ n)A_{a,b}.$$

Let

$$_1\chi_N(t)=\mathrm{E}[\exp\{it\sum_{1\leqq a+b\leqq 3}k_{a,b}(m,\ n)A_{a,b}\}].$$

Then from Lemma 2.3 and the fact $k_{a,b}(m,\ n)=O(N^{\frac{1}{2}-(a+b)})$, we have

$$\int_0^{N^{\frac{1}{4}}/\log N} t^{-1}|\chi_{m,n}(t)-{}_1\chi_N(t)|\,dt=o(N^{-1}).$$

Furthermore let us define

$$\begin{aligned}\chi_N^*(t)=&\mathrm{E}[\exp\{it(k_{1,0}A_{1,0}+k_{0,1}A_{0,1})\}]+itk_{1,1}\mathrm{E}[A_{1,1}\exp\{it(k_{1,0}A_{1,0}+k_{0,1}A_{0,1})\}]\\&+\mathrm{E}[\exp\{it(k_{1,0}A_{1,0}+k_{0,1}A_{0,1})\}\{itk_{2,0}A_{2,0}+itk_{0,2}A_{0,2}+(it)^2k_{1,1}A_{1,1}k_{2,0}A_{2,0}\\&+(it)^2k_{1,1}A_{1,1}k_{0,2}A_{0,2}+(it)^2k_{2,0}A_{2,0}k_{0,2}A_{0,2}\\&+\frac{(it)^2}{2}k_{1,1}^2A_{1,1}^2+\frac{(it)^2}{2}k_{2,0}^2A_{2,0}^2+\frac{(it)^2}{2}k_{0,2}^2A_{0,2}^2\\&+itk_{2,1}A_{2,1}+itk_{1,2}A_{1,2}+itk_{3,0}A_{3,0}+itk_{0,3}A_{0,3}\}],\end{aligned}\tag{3.3}$$

where $k_{a,b}$ is an abbreviation of $k_{a,b}(m, n)$. Then by the similar way of Callaert et al. [1] and Maesono[6], we have

$$\int_0^{N^{\frac{1}{4}}/\log N} t^{-1}|{}_1\chi_N(t)-\chi_N^*(t)|\,dt=o(N^{-1}).$$

Since $k_{1,0}(m, n)=r/(m\sigma_{m,n})$ and $k_{0,1}(m, n)=s/(n\sigma_{m,n})$, from Lemma 3.1 the approximation of the first term of (3.3) is $I_{0,0}$. And the second term of (3.3) is approximated as follows. From Lemma 2.2, we have

$$\begin{aligned}&itk_{1,1}\mathrm{E}[A_{1,1}\exp\{it(k_{1,0}A_{1,0}+k_{0,1}A_{0,1})\}]\\&=itk_{1,1}mn\eta^{m-1}\Big(\frac{rt}{m\sigma_{m,n}}\Big)\nu^{n-1}\Big(\frac{st}{n\sigma_{m,n}}\Big)\\&\quad\times\mathrm{E}[g_{1,1}(X_1\,;\,Y_1)\exp\{it(k_{1,0}g_{1,0}(X_1)+k_{0,1}g_{0,1}(Y_1))\}]\end{aligned}$$

which will be denoted by $itk_{1,1}mnI_{1,1}^*E_1^*$. From Lemma 3.1, $I_{1,1}^*$ is approximated by $I_{1,1}$. Taking the first few terms of the Taylor series for approximating E_1^* and using Lemma 2.2, we have an approximation E_1 such that

$$\begin{aligned}E_1=&\frac{(it)^2rs}{mn\sigma_{m,n}^2}\mathrm{E}[g_{1,0}(X_1)g_{0,1}(Y_1)g_{1,1}(X_1\,;\,Y_1)]\\&+\frac{(it)^3}{6\sigma_{m,n}^3}\Big\{\frac{3r^2s}{m^2n}\mathrm{E}[g_{1,0}^2(X_1)g_{0,1}(Y_1)g_{1,1}(X_1\,;\,Y_1)]+\frac{3rs^2}{mn^2}\mathrm{E}[g_{1,0}(X_1)g_{0,1}^2(Y_1)g_{1,1}(X_1\,;\,Y_1)]\Big\}.\end{aligned}$$

Since $\sigma_{m,n}^2=\tau_{m,n}^2(1+O(N^{-1}))$ and $k_{1,1}(m,n)=rs/(mn\tau_{m,n})(1+O(N^{-1}))$, we can obtain an approximation $\rho(t)$ of the second term as follows:

$$\begin{aligned}\rho(t)=&e^{-\frac{1}{2}t^2}\Big(\frac{r^2s^2(it)^3}{mn\tau_{m,n}^3}\mathrm{E}[g_{1,0}(X_1)g_{0,1}(Y_1)g_{1,1}(X_1\,;\,Y_1)]\\&+\frac{(it)^4}{6\tau_{m,n}^4}\Big\{\frac{3r^3s^2}{m^2n}\mathrm{E}[g_{1,0}^2(X_1)g_{0,1}(Y_1)g_{1,1}(X_1\,;\,Y_1)]+\frac{3r^2s^3}{mn^2}\mathrm{E}[g_{1,0}(X_1)g_{0,1}^2(Y_1)g_{1,1}(X_1\,;\,Y_1)]\Big\}\\&+\frac{r^2s^2(it)^6}{6mn\tau_{m,n}^6}\mathrm{E}[g_{1,0}(X_1)g_{0,1}(Y_1)g_{1,1}(X_1\,;\,Y_1)]\Big\{\frac{r^3}{m^2}\mathrm{E}[g_{1,0}^3(X_1)]+\frac{s^3}{n^2}\mathrm{E}[g_{0,1}^3(Y_1)]\Big\}\Big).\end{aligned}$$

From the condition (C1) and $\sigma_{m,n}^2=O(N^{-1})$, we have

$$|E_1^*-E_1|\leqq P_2(t)O(N^{-2}),$$

where $P_2(t)$ is a Polynomial in t. Then from (3.1) and above inequality, we have

$$\int_0^{N^{\frac{1}{4}}/\log N} t^{-1}|\,itk_{1,1}mnI^*_{1,1}E^*_1-\rho(t)|\,dt=o(N^{-1}).$$

Similarly we can obtain the approximations of the rest terms of (3.3). Hence we have an approximation of $\chi^*_N(t)$ as follows :

$$\tilde{\chi}_{m,n}(t)=e^{-\frac{1}{2}t^2}\left\{1+\frac{K_3}{6}(it)^3+\frac{K_4}{24}(it)^4+\frac{K_3^2}{72}(it)^6\right\}$$

where

$$K_3=\frac{1}{\tau^3_{m,n}}\left\{\frac{r^3}{m^2}E[g^3_{1,0}(X_1)]+\frac{s^3}{n^2}E[g^3_{0,1}(Y_1)]+\frac{6r^2s^2}{mn}E[g_{1,0}(X_1)g_{0,1}(Y_1)g_{1,1}(X_1\,;\,Y_1)]\right.$$
$$+\frac{3r^3(r-1)}{m^2}E[g_{1,0}(X_1)g_{1,0}(X_2)g_{2,0}(X_1,\ X_2)]$$
$$\left.+\frac{3s^3(s-1)}{n^2}E[g_{0,1}(Y_1)g_{0,1}(Y_2)g_{0,2}(Y_1,\ Y_2)]\right\}$$

and

$$K_4=\frac{1}{\tau^4_{m,n}}\left\{\frac{r^4}{m^3}\left(E[g^4_{1,0}(X_1)]-3\xi^4_{1,0}\right)+\frac{s^4}{n^3}\left(E[g^4_{0,1}(Y_1)]-3\xi^4_{0,1}\right)\right.$$
$$+\frac{12r^3s^2}{m^2n}E[g^2_{1,0}(X_1)g_{0,1}(Y_1)g_{1,1}(X_1\,;\,Y_1)]+\frac{12r^2s^3}{mn^2}E[g_{1,0}(X_1)g^2_{0,1}(Y_1)g_{1,1}(X_1\,;\,Y_1)]$$
$$+\frac{12r^4(r-1)}{m^3}E[g^2_{1,0}(X_1)g_{1,0}(X_2)g_{2,0}(X_1,\,X_2)]+\frac{12s^4(s-1)}{n^3}E[g^2_{0,1}(Y_1)g_{0,1}(Y_2)g_{0,2}(Y_1,\,Y_2)]$$
$$+\frac{12r^3(r-1)s^2}{m^2n}E[g_{1,0}(X_1)g_{1,0}(X_2)g_{0,1}(Y_1)g_{2,1}(X_1,\,X_2\,;\,Y_1)]+\frac{12r^2s^3(s-1)}{mn^2}E[g_{1,0}(X_1)g_{0,1}(Y_1)$$
$$\times g_{0,1}(Y_2)g_{1,2}(X_1\,;\,Y_1,\,Y_2)]+\frac{4r^4(r-1)(r-2)}{m^3}E[g_{1,0}(X_1)g_{1,0}(X_2)g_{1,0}(X_3)g_{3,0}(X_1,\,X_2,\,X_3)]$$
$$+\frac{4s^4(s-1)(s-2)}{n^3}E[g_{0,1}(Y_1)g_{0,1}(Y_2)g_{0,1}(Y_3)g_{0,3}(Y_1,\,Y_2,\,Y_3)]+\frac{24r^3(r-1)s^2}{m^2n}E[g_{1,0}(X_2)g_{0,1}(Y_1)$$
$$\times g_{1,1}(X_1\,;\,Y_1)g_{2,0}(X_1,\,X_2)]+\frac{24r^2s^3(s-1)}{mn^2}E[g_{1,0}(X_1)g_{0,1}(Y_2)g_{1,1}(X_1\,;\,Y_1)g_{0,2}(Y_1,\ Y_2)]$$
$$+\frac{12r^2s^4}{mn^2}E[g_{0,1}(Y_1)g_{0,1}(Y_2)g_{1,1}(X_1\,;\,Y_1)g_{1,1}(X_1\,;\,Y_2)]+\frac{12r^4s^2}{m^2n}E[g_{1,0}(X_1)g_{1,0}(X_2)$$
$$\times g_{1,1}(X_1\,;\,Y_1)g_{1,1}(X_2\,;\,Y_1)]+\frac{12r^4(r-1)^2}{m^3}E[g_{1,0}(X_2)g_{1,0}(X_3)g_{2,0}(X_1,\,X_2)g_{2,0}(X_1,\,X_3)]$$
$$\left.+\frac{12s^4(s-1)^2}{n^3}E[g_{0,1}(Y_2)g_{0,1}(Y_3)g_{0,2}(Y_1,\,Y_2)g_{0,2}(Y_1,\,Y_3)]\right\}.$$

This $\tilde{\chi}_{m,n}(t)$ satisfies the equation (3.2). Inverting $\tilde{\chi}_{m,n}(t)$, we have a formal Edgeworth expansion $Q_{m,n}(x)$ such that

$$Q_{m,n}(x)=\Phi(x)-\phi(x)\left[\frac{K_3}{6}(x^2-1)+\frac{K_4}{24}(x^3-3x)+\frac{K_3^2}{72}(x^5-10x^3+15x)\right]$$

where $\Phi(x)$ and $\phi(x)$ denote the distribution function and density of the standard normal distribution. Note that $K_3=O(N^{-\frac{1}{2}})$ and $K_4=O(N^{-1})$. Then $Q_{m,n}(x)$ is the expansion with remainder term $o(N^{-1})$.

4. Conditions for the valid expansion

In order to prove the validity of the formal Edgeworth expansion $Q_{m,n}(x)$, we shall apply Esseen's smoothing lemma[3]. From smoothing lemma, we have

$$\sup_x |P(\sigma_{m,n}^{-1}U_{m,n}\leqq x)-Q_{m,n}(x)|\leqq\frac{1}{\pi}\int_{-N\log N}^{N\log N}|t|^{-1}|\chi_{m,n}(t)-\tilde{\chi}_{m,n}(t)|\,dt+o(N^{-1})$$

where $Q_{m,n}(x)$, $\chi_{m,n}(t)$ and $\tilde{\chi}_{m,n}(t)$ are defined in the previous section. Since the proof for the negative part of t is similar to that for positive one, we shall find the conditions which ensure

$$\int_0^{N\log N}t^{-1}|\chi_{m,n}(t)-\tilde{\chi}_{m,n}(t)|\,dt=o(N^{-1}).$$

Let us define

$${}_2\chi_N(t)=\mathrm{E}[\exp\{it\sum_{1\leqq a+b\leqq 4}k_{a,b}(m,\ n)A_{a,b}\}]$$

and

$${}_3\chi_N(t)=\mathrm{E}[\exp\{it\sum_{1\leqq a+b\leqq 5}k_{a,b}(m,\ n)A_{a,b}\}].$$

Then from Lemma 2.3 and $k_{a,b}(m,\ n)=O(N^{\frac{1}{2}-(a+b)})$, we have

$$\int_{N^{\frac{1}{4}}/\log N}^{N^{\frac{3}{4}}/\log N}t^{-1}|\chi_{m,n}(t)-{}_2\chi_N(t)|\,dt=o(N^{-1})$$

and

$$\int_{N^{\frac{3}{4}}/\log N}^{N\log N}t^{-1}|\chi_{m,n}(t)-{}_3\chi_N(t)|\,dt=o(N^{-1}).$$

Then putting

$$(\mathrm{I})=\int_0^{N^{\frac{1}{4}}/\log N}t^{-1}|\chi_{m,n}(t)-\tilde{\chi}_{m,n}(t)|\,dt,$$

$$(\mathrm{II})=\int_{N^{\frac{1}{4}}/\log N}^{N^{\frac{3}{4}}/\log N}t^{-1}|{}_2\chi_N(t)|\,dt,\qquad(\mathrm{III})=\int_{N^{\frac{3}{4}}/\log N}^{N\log N}t^{-1}|{}_3\chi_N(t)|\,dt$$

and

$$(\mathrm{IV})=\int_{\log N}^{\infty}t^{-1}|\tilde{\chi}_{m,n}(t)|\,dt,$$

we have

$$\int_0^{N\log N}t^{-1}|\chi_{m,n}(t)-\tilde{\chi}_{m,n}(t)|\,dt\leqq(\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})+o(N^{-1}).$$

In Section 2, we have proved that (I) is $o(N^{-1})$ under the condition (C1). It immediately follows that $(\mathrm{IV})=o(N^{-1})$ under the same condition (C1).

To obtain an order of (II), we shall consider following decomposition. For $0\leqq u\leqq m-a+1$ and $0\leqq v\leqq n-b+1$, let us define

$$D_{a,b}(u,\ v)=B_{a,b}(u,\ m-a+2,\ \cdots,\ m\ ;\ v,\ n-b+2,\ \cdots,\ n),$$

$$H_{a,b}(u,\ v)=\sum_{i_1=1}^{u}\sum_{i_2=i_1+1}^{m-a+2}\cdots\sum_{i_a=i_{a-1}+1}^{m}\sum_{j_1=v+1}^{n-b+1}\sum_{j_2=j_1+1}^{n-b+2}\cdots\sum_{j_b=j_{b-1}+1}^{n} g_{a,b}(X_{i_1},\ \cdots,\ X_{i_a}\ ;\ Y_{j_1},\ \cdots,\ Y_{j_b}),$$

$$L_{a,b}(u,\ v)=\sum_{i_1=u+1}^{m-a+1}\sum_{i_2=i_1+1}^{m-a+2}\cdots\sum_{i_a=i_{a-1}+1}^{m}\sum_{j_1=1}^{v}\sum_{j_2=j_1+1}^{n-b+2}\cdots\sum_{j_b=j_{b-1}+1}^{n} g_{a,b}(X_{i_1},\ \cdots,\ X_{i_a}\ ;\ Y_{j_1},\ \cdots,\ Y_{j_b})$$

and

$$R_{a,b}(u,\ v)=\sum_{i_1=u+1}^{m-a+1}\sum_{i_2=i_1+1}^{m-a+2}\cdots\sum_{i_a=i_{a-1}+1}^{m}\sum_{j_1=v+1}^{n-b+1}\sum_{j_2=j_1+1}^{n-b+2}\cdots\sum_{j_b=j_{b-1}+1}^{n} g_{a,b}(X_{i_1},\ \cdots,\ X_{i_a}\ ;\ Y_{j_1},\ \cdots,\ Y_{j_b}).$$

Then $A_{a,b}=D_{a,b}(u,\ v)+H_{a,b}(u,\ v)+L_{a,b}(u,\ v)+R_{a,b}(u,\ v)$. Here $H_{a,b}(u,\ v)$ and $\{Y_1,\ \cdots,\ Y_v\}$ are independent. Similarly $L_{a,b}(u,\ v)$ and $\{X_1,\ \cdots,\ X_u\}$ are independent. Furthermore $\mathrm{R}_{a,b}(u,v)$ and $\{X_1,\ \cdots,\ X_u,\ Y_1,\ \cdots,\ Y_v\}$ are also independent. From Lemma 2.3, we have

$$\mathrm{E}|D_{a,b}(u,\ v)|^p\leqq 0((uv)^{\frac{p}{2}}N^{\frac{p}{2}(a+b-2)}).$$

And we get

$$\begin{aligned}\mathrm{E}|H_{a,b}(u,\ v)|^p&\leqq\mathrm{E}|H_{a,b}(u,\ v)+D_{a,b}(u,\ v)-D_{a,b}(u,\ v)|^p\\&\leqq 2^{p-1}\{E|H_{a,b}(u,\ v)+D_{a,b}(u,\ v)|^p+\mathrm{E}|D_{a,b}(u,\ v)|^p\}\\&=0(u^{\frac{p}{2}}N^{\frac{p}{2}(a+b-1)}).\end{aligned}$$

Similarly we have

$$\mathrm{E}|L_{a,b}(u,\ v)|^p\leqq 0(v^{\frac{p}{2}}N^{\frac{p}{2}(a+b-1)}).$$

Hence we can obtain an appropriate upper bound for ${}_2\chi_N(t)$ by the similar way of Lemma 4 as Callaert et al. [1] and Lemma 3 as Maesono[6]. The bound and the proof of it are rather complicated, and will be omitted here.

In addition to (C1), we assume that

(C2) $$\lim_{|t|\to\infty}|\eta(t)|<1 \text{ and } \lim_{|t|\to\infty}|\nu(t)|<1.$$

Then by the same arguments which have been described in Callaert et al. [1] pp308–309, we can prove that (II) is $o(N^{-1})$ under the conditions (C1) and (C2).

Let us define

$$\zeta(x_1,\ x_2)=w_{2,0}(x_1,\ x_2)-\{(r-2)/(r-1)\}[w_{1,0}(x_1)+w_{1,0}(x_2)]$$
$$\mu(y_1,\ y_2)=w_{0,2}(y_1,\ y_2)-\{(s-2)/(s-1)\}[w_{0,1}(y_1)+w_{0,1}(y_2)].$$

Then if we assume the following complicated condition, it will be possible to show that (III) is $o(N^{-1})$.

(C3) There exist positive constants $c_1<1$ and $c_2<1$ such that for $u=[m^{\alpha}]$ and $v=[n^{\beta}]$, where $0<\alpha<1/8$ and $0<\beta<1/8$,

$$\mathrm{P}\Big(|\mathrm{E}[\exp\{it(k_{2,0}\sum_{j=u+1}^{m}\zeta(X_1,\ X_j)$$

$$+k_{1,1}\sum_{j=v+1}^{n} g_{1,1}(X_1\,;\,Y_j))\}|\,X_{u+1},\ \cdots,\ X_m,\ Y_{v+1},\ \cdots,\ Y_n]\,|\leqq c_1,$$

$$or,\ |\mathrm{E}[\exp\{\,it(k_{0,2}\sum_{j=v+1}^{n}\mu(Y_1,\ Y_j)$$

$$+k_{1,1}\sum_{j=u+1}^{m} g_{1,1}(X_j\,;\,Y_1))\}|\,X_{u+1},\ \cdots,\ X_m,\ Y_{v+1},\ \cdots,\ Y_n]\,|\leqq c_2\Big)$$

$$\geqq 1-o\Big(\frac{1}{N\log N}\Big)$$

uniformly for all $t\in[N^{\frac{3}{4}}/\log N,\ N\log N]$.

This is an extension of the conditions which are given in Callaert· et al. [1] and Maesono[6]. It may be hard to check the validity of (C3) in most of the examples encountered in statistics. Then it is desirable to obtain simple condition which ensures $(\mathrm{III})=o(N^{-1})$.

From the discussion above, we have

Theorem. *If the conditions* (C1), (C2) *and* (C3) *are satisfied, then*

$$\sup_x |\,P(\sigma_{m,n}^{-1}U_{m,n}\leqq x)-Q_{m,n}(x)\,|=o(N^{-1}).$$

Remark. Instead of condition (C1), the asymptotic expansion may be valid under the existence of a fourth moment of kernel h. Lin[5] has proved it in the case of one-sample U-statistics with kernel of degree two.

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