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LAGRANGE CONNECTIONS COMPATIBLE WITH A PAIR OF GENERALIZED LAGRANGE METRICS

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Abstract

This article is a revised note of the lecture presented by the authors to "The XXth National Symposium on Finsler Geometry" held at Kagoshima, with Romanian and Korean participants, during July 29 ~ August 1, 1985. We discuss generalized Lagrange metrics, and especially consider the problem of existence and arbitrariness of Finsler connections compatible with a pair of such metrics.

A generalized Lagrange metric is a generalized Finsler metric (cf. Miron [10] , Hashiguchi [6]) which is not necessarily assumed to be positively homogeneous, and the geometry based on such metrics is a generalization of the geometry based on Lagrangians which was named Lagrange Geometry by Kern [8] . Generalized Lagrange metrics and their applications have been treated in Miron [11] in detail, which was also presented at this Symposium.

This research is in line with Einstein [3] , Eisenhart [4] , Ghinea [5] , Miron-Atanasiu [12] , Ikeda [7] and Atanasiu-Hashiguchi-Miron [1, 2] , etc. Since the treatment proceeds in the same way as in our previous paper [2] , proofs are omitted. As to the terminology and notaions we use also those in [2] , which are essentially based on Matsumoto [9].

1. Generalized Lagrange metrics and Lagrange connections

Let M be an n-dimensional differentiable manifold, and $x=(x^i)$ and $y=(y^i)$ denote a point of M and a supporting element respectively. We put $\partial_i=\partial/\partial x^i$, $\dot\partial_i=\partial/\partial y^i$.

A Finsler tensor field $g_{ij}(x,y)$ of type (0,2) in M is called a *generalized Lagrange metric* if it is symmetric and non-degenerate:

$$(1.1)\qquad g_{ij}=g_{ji}, \qquad\qquad (1.2)\qquad \det(g_{ij})\neq 0.$$

Especially, a generalized Lagrange metric $g_{ij}(x,y)$ is called a *Lagrange metric* if there exists a Finsler function $L(x,y)$ in M such that $g_{ij}=(\dot\partial_i\dot\partial_j L)/2$.

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A generalized Lagrange metric $g_{ij}(x, y)$ is called a *generalized Finsler metric*, if it is positively homogeneous of degree 0:

$$g_{ij}(x, \lambda y)=g_{ij}(x, y) \text{ for } \lambda>0. \tag{1.3}$$

Especially, a Lagrange metric $g_{ij}=(\dot{\partial}_i\dot{\partial}_j L)/2$ is called a *Finsler metric*, if L is given by $L=F^2$, where $F(x, y)$ is positively homogeneous of degree 1.

For uniformity of the terms we shall call a Finsler connection in the sense of Matsumoto [9] a *Lagrange connection*, if it is not necessarily assumed to be positively homogeneous, and when we use the term "a Finsler connection", we assume it is positively homogeneous, that is, the coefficients $N^i{}_k$, $F_j{}^i{}_k$, $C_j{}^i{}_k$ satisfy the conditions $N^i{}_k(x, \lambda y)=\lambda N^i{}_k(x, y)$, $F_j{}^i{}_k(x, \lambda y)=F_j{}^i{}_k(x, y)$, $C_j{}^i{}_k(x, \lambda y)=\lambda^{-1}C_j{}^i{}_k(x, y)$ for $\lambda>0$.

We shall express a Lagrange connection $L\Gamma$ in terms of its coefficients as $L\Gamma=(N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$. A Lagrange connection having a fixed non-linear connection N is also denoted by $L\Gamma(N)=(F_j{}^i{}_k, C_j{}^i{}_k)$. And the respective h- and v-covariant differentiations are denoted by short and long bars, e.g., $g_{ij|k}$, $g_{ij}|_k$ (with respect to $L\Gamma$), $g_{ij\overset{o}{|}k}$, $g_{ij}\overset{o}{|}_k$ (with respect to $L\overset{o}{\Gamma}$), etc.

Given a generalized Lagrange metric g_{ij}, a Lagrange connection $L\Gamma$ is called *metrical*, if it satisfies

$$g_{ij|k}=0, \qquad g_{ij}|_k=0. \tag{1.4}$$

For a generalized Lagrange metric g_{ij}, we have so-called Obata's operators:

$$\underset{1}{\Lambda}{}^{ir}_{sj}=(\delta^i_s\delta^r_j-g_{sj}g^{ir})/2, \quad \underset{2}{\Lambda}{}^{ir}_{sj}=(\delta^i_s\delta^r_j+g_{sj}g^{ir})/2, \tag{1.5}$$

where $(g^{ij})=(g_{ij})^{-1}$. Then we have

Theorem 1.1. *Let $L\overset{o}{\Gamma}(N)=(\overset{o}{F}_j{}^i{}_k, \overset{o}{C}_j{}^i{}_k)$ be a fixed Lagrange connection. For a generalized Lagrange metric g_{ij}, we define Finsler tensor fields $U_j{}^i{}_k$, $\tilde{U}_j{}^i{}_k$ by*

$$U_j{}^i{}_k=-g^{ir}g_{rj\overset{o}{|}k}/2, \quad \tilde{U}_j{}^i{}_k=-g^{ir}g_{rj}\overset{o}{|}_k/2. \tag{1.6}$$

Then a Lagrange connection $L\Gamma(N)=(F_j{}^i{}_k, C_j{}^i{}_k)$ is metrical, if and only if the difference tensor fields $B_j{}^i{}_k$, $D_j{}^i{}_k$ given by

$$F_j{}^i{}_k=\overset{o}{F}_j{}^i{}_k-B_j{}^i{}_k, \qquad C_j{}^i{}_k=\overset{o}{C}_j{}^i{}_k-D_j{}^i{}_k \tag{1.7}$$

are solutions of the equations

$$\underset{2}{\Lambda}{}^{ir}_{sj}B_r{}^s{}_k=U_j{}^i{}_k, \qquad \underset{2}{\Lambda}{}^{ir}_{sj}D_r{}^s{}_k=\tilde{U}_j{}^i{}_k. \tag{1.8}$$

The above equations have solutions, and their general forms are given by

Theorem 1.2. *Let $L\overset{o}{\Gamma}(N)=(\overset{o}{F}_j{}^i{}_k, \overset{o}{C}_j{}^i{}_k)$ be a fixed Lagrange connection. For a generalized Lagrange metric g_{ij}, there exists a metrical Lagrange connection $L\Gamma(N)=(F_j{}^i{}_k, C_j{}^i{}_k)$*

and the set of all such connections is given by

$$(1.9)\qquad \begin{aligned} F_j{}^i{}_k &= \overset{o}{F}_j{}^i{}_k + g^{ir} g_{rj\overset{o}{|}k}/2 + \underset{1}{\Lambda}{}^{ir}_{sj} X_r{}^s{}_k, \\ C_j{}^i{}_k &= \overset{o}{C}_j{}^i{}_k + g^{ir} g_{rj}\overset{o}{|}_k/2 + \underset{1}{\Lambda}{}^{ir}_{sj} Y_r{}^s{}_k, \end{aligned}$$

where $X_j{}^i{}_k$, $Y_j{}^i{}_k$ *are arbitrary Finsler tensor fields.*

2. Lagrange connections compatible with a pair of generalized Lagrange metrics

Let g_{ij} and a_{ij} be two given generalized Lagrange metrics. A Lagrange connection is called *compatible* with the pair (g_{ij}, a_{ij}), if it is metrical with respect to both g_{ij} and a_{ij}:

$$(2.1)\qquad g_{ij|k}=0, \quad g_{ij}|_k=0, \quad a_{ij|k}=0, \quad a_{ij}|_k=0.$$

The results of Ghinea [5] about Finsler connections compatible with a pair of metrical or almost symplectical structures still hold for the pair of generalized Lagrange metrics. We define Obata's operators by (1.5) and

$$(2.2)\qquad \underset{1}{O}{}^{ir}_{sj}=(\delta^i_s\delta^r_j - a_{sj}a^{ir})/2, \quad \underset{2}{O}{}^{ir}_{sj}=(\delta^i_s\delta^r_j + a_{sj}a^{ir})/2,$$

where $(a^{ij})=(a_{ij})^{-1}$. Then we have

Theorem 2.1. *Let* $L\Gamma(N)=(\overset{o}{F}_j{}^i{}_k, \overset{o}{C}_j{}^i{}_k)$ *be a fixed Lagrange connection. For a pair of generalized Lagrange metrics* g_{ij}, a_{ij} *we define Finsler tensor fields* $U_j{}^i{}_k$, $\tilde{U}_j{}^i{}_k$, $V_j{}^i{}_k$, $\tilde{V}_j{}^i{}_k$ *by* (1.6) *and*

$$(2.3)\qquad V_j{}^i{}_k = -a^{ir}a_{rj\overset{o}{|}k}/2, \qquad \tilde{V}_j{}^i{}_k = -a^{ir}a_{rj}\overset{o}{|}_k/2.$$

Then a Lagrange connection $L\Gamma(N)=(F_j{}^i{}_k, C_j{}^i{}_k)$ *is compatible with the pair* (g_{ij}, a_{ij}), *if and only if the difference tensor fields* $B_j{}^i{}_k$, $D_j{}^i{}_k$ *given by* (1.7) *are solutions of the equations* (1.8) *and*

$$(2.4)\qquad \underset{2}{O}{}^{ir}_{sj}B_r{}^s{}_k = V_j{}^i{}_k, \qquad \underset{2}{O}{}^{ir}_{sj}D_r{}^s{}_k = \tilde{V}_j{}^i{}_k.$$

It is terribly complicated to solve the above equations, as a proverb says "If you run after two hares, you will catch neither". We shall show the case the equations have solutions. A pair of two generalized Lagrange metrics g_{ij}, a_{ij} is called *natural*, if there exists a non-vanishing Finsler function $\mu(x, y)$ such that

$$(2.5)\qquad g_{ir}g_{js}a^{rs}=\mu a_{ij},$$

or equivalently, if the commutativities

$$(2.6)\qquad \underset{\alpha}{\Lambda}{}^{ir}_{sj}\underset{\beta}{O}{}^{sm}_{nr} = \underset{\beta}{O}{}^{ir}_{sj}\underset{\alpha}{\Lambda}{}^{sm}_{nr} \quad (\alpha, \ \beta=1, \ 2)$$

hold. Then we have

Proposition 2.1. *All the commutativities* (2.6) *hold if any one of them holds.*

Proposition 2.2. *Let* $(g_{ij},\ a_{ij})$ *be a natural pair of generalized Lagrange metrics. If there exists a Lagrange connection compatible with the pair, the function* μ *in* (2.5) *is constant.*

Proposition 2.3. *Let* g_{ij} *be a generalized Lagrange metric. There exists a generalized Lagrange metric* a_{ij} *such that the pair* $(g_{ij},\ a_{ij})$ *is natural by a constant* $\mu=\varepsilon c^2$ $(\varepsilon=\pm 1,\ c>0)$, *if and only if there exists a Finsler tensor field* $F^i{}_j$ *of type* (1,1) *satisfying*

$$\varepsilon F^i{}_r F^r{}_j=\delta^i_j, \qquad \varepsilon g_{rs}F^r{}_i F^s{}_j=g_{ij}. \tag{2.7}$$

The correspondence between $F^i{}_j$ and a_{ij} in Proposition 2.3 is given by

$$F^i{}_j=cg^{ir}a_{rj}, \qquad a_{ij}=g_{ir}F^r{}_j/c. \tag{2.8}$$

Using Proposition 2.3 we can show that for a natural pair with a constant $\mu\neq 0$ the equations (1.8), (2.4) have solutions, and their general forms are given by

Theorem 2.2. *Let* $L\overset{o}{\Gamma}(N)=(\overset{o}{F}{}_j{}^i{}_k,\ \overset{o}{C}{}_j{}^i{}_k)$ *be a fixed Lagrange connection. For a natural pair with a constant* $\mu\neq 0$ *of generalized Lagrange metrics* $g_{ij},\ a_{ij}$, *there exists a Lagrange connection* $L\Gamma(N)=(F_j{}^i{}_k,\ C_j{}^i{}_k)$ *compatible with the pair and the set of all such connections is given by*

$$\begin{aligned} F_j{}^i{}_k&=\overset{o}{F}{}_j{}^i{}_k+(g^{ir}g_{rj\overset{o}{|}k}+\underset{1}{\Lambda}{}^{ir}_{sj}a^{st}a_{tr\overset{o}{|}k})/2+\underset{1}{\Lambda}{}^{ir}_{sj}\underset{1}{O}{}^{sm}_{nr}X_m{}^n{}_k,\\ C_j{}^i{}_k&=\overset{o}{C}{}_j{}^i{}_k+(g^{ir}g_{rj}\overset{o}{|}_k+\underset{1}{\Lambda}{}^{ir}_{sj}a^{st}a_{tr}\overset{o}{|}_k)/2+\underset{1}{\Lambda}{}^{ir}_{sj}\underset{1}{O}{}^{sm}_{nr}Y_m{}^n{}_k, \end{aligned} \tag{2.9}$$

where $X_j{}^i{}_k,\ Y_j{}^i{}_k$ *are arbitrary Finsler tensor fields.*

3. The case of a generalized Lagrange metric with an additional structure

The previous results for a pair of generalized Lagrange metrics $g_{ij},\ a_{ij}$ are generalized to the case a_{ij} is degenerate. A differentiable manifold M endowed with a generalized Lagrange metric g_{ij} is called a *generalized Lagrange space*. Let a generalized Lagrange space $(M,\ g_{ij})$ admit a symmetric (or alternate) and degenerate Finsler tensor field a_{ij} :

$$a_{ij}=\tau a_{ji}, \tag{3.1}$$

$$\operatorname{rank}(a_{ij})=n-k, \tag{3.2}$$

where $\tau=\pm 1$ and k is an integer and $0<k<n$. Then $(M,\ g_{ij})$ is called to have an *additional structure of index k*. The case of a generalized Lagrange metric a_{ij} is contained in the following discussions as the exceptional case $k=0$.

The results of our previous paper [2] about a generalized Finsler space $(M,\ g_{ij})$

with an alternate additional strucure a_{ij} still hold for the case with a symmetric one a_{ij}.

The matrix (g_{ij}) has the inverse (g^{jk}), but the matrix (a_{ij}) is not regular. So we shall construct some matrix (a^{jk}), which plays the role similar to the inverse matrix. If (g_{ij}) is positive-definite, then on each local chart there are exaktly k independent Finsler vector fields ξ^i_a $(a=1, \cdots, k)$ with the properties

$$a_{ij}\xi^j_a=0, \quad g_{ij}\xi^i_a\xi^j_b=\delta_{ab} \ (a, \ b=1, \ \cdots, k). \tag{3.3}$$

If (g_{ij}) is not positive-definite, we assume that there exist such vector fields ξ^i_a. Then we define local Finsler covector fields η^a_i $(a=1, \cdots, k)$ by

$$\eta^a_i=g_{ij}\xi^j_a. \tag{3.4}$$

If we define local Finsler tensor fields $l^i{}_j$ and $m^i{}_j$ by

$$l^i{}_j=\sum_a \xi^i_a\eta^a_j, \qquad m^i{}_j=\delta^i_j-l^i{}_j, \tag{3.5}$$

then $l^i{}_j$ and $m^i{}_j$ are independent on the choice of ξ^i_a and globally defined as the respective projectors on the kernel $\boldsymbol{K}$ of the mapping $a_{ij}: \xi^j \rightarrow a_{ij}\xi^j$ and the orthoganal $\boldsymbol{H}$ to $\boldsymbol{K}$ with respect to g_{ij}. Then a global Finsler tensor field a^{jk} is uniquely determined from $(g_{ij}, \ a_{ij})$ by

$$a_{ij}a^{jk}=m^k{}_i, \qquad l^i{}_ja^{jk}=0. \tag{3.6}$$

A Lagrange connection of a generalized Lagrange space $(M, \ g_{ij})$ with an additional structure a_{ij} is called *compatible* with the pair $(g_{ij}, \ a_{ij})$, if it satisfies (2.1). Then the condition that a Lagrange connection $L\Gamma$ is compatible with the pair $(g_{ij}, \ a_{ij})$ is given by Theorem 2.1, if we define $V_j{}^i{}_k$, $\tilde{V}_j{}^i{}_k$ by

$$\begin{aligned} V_j{}^i{}_k&=-(a^{ir}a_{rj\overset{o}{|}k}+3l^i{}_sl^s{}_{j\overset{o}{|}k}-l^i{}_{j\overset{o}{|}k})/2, \\ \tilde{V}_j{}^i{}_k&=-(a^{ir}a_{rj}\overset{o}{\Big|}_k+3l^i{}_sl^s{}_j\overset{o}{\Big|}_k-l^i{}_j\overset{o}{\Big|}_k)/2, \end{aligned} \tag{3.7}$$

and Obata's operators $\underset{\alpha}{O}{}^{ir}_{sj}$ $(\alpha=1, \ 2)$ by

$$\begin{aligned} \underset{1}{O}{}^{ir}_{sj}&=(\delta^i_s\delta^r_j-\delta^i_sl^r{}_j-l^i{}_s\delta^r_j+3l^i{}_sl^r{}_j-a_{sj}a^{ir})/2, \\ \underset{2}{O}{}^{ir}_{sj}&=(\delta^i_s\delta^r_j+\delta^i_sl^r{}_j+l^i{}_s\delta^r_j-3l^i{}_sl^r{}_j+a_{sj}a^{ir})/2, \end{aligned} \tag{3.8}$$

and impose on the $B_j{}^i{}_k$ and $D_j{}^i{}_k$ the additional conditions:

$$\begin{aligned} l^r{}_ia_{sj}B_r{}^s{}_k&=-l^r{}_ia_{rj\overset{o}{|}k}, \qquad & l^r{}_ia_{sj}D_r{}^s{}_k&=-l^r{}_ia_{rj}\overset{o}{\Big|}_k, \\ l^i{}_sm^r{}_jB_r{}^s{}_k&=-l^i{}_sl^s{}_{j\overset{o}{|}k}, \qquad & l^i{}_sm^r{}_jD_r{}^s{}_k&=-l^i{}_sl^s{}_j\overset{o}{\Big|}_k. \end{aligned} \tag{3.9}$$

If we define the naturality of a pair $(g_{ij}, \ a_{ij})$ by (2.5), or equivalently (2.6) where $\underset{\alpha}{O}{}^{ir}_{sj}$ are defined by (3.8), then Propositions 2.1 and 2.2 still hold. Corresponding to Proposition 2.3, the condition that a generalized Lagrange space $(M, \ g_{ij})$ admits an additional structure a_{ij} of index k such that the pair $(g_{ij},\ a_{ij})$ is natural by a constant $\mu=\varepsilon c^2$ $(\varepsilon=\pm 1,\ c>0)$ is given by the existence of a Finsler tensor field $F^i{}_j$ of type $(1,1)$, k Finsler vector fields ξ^i_a $(a=1,\ \cdots, k)$ and k Finsler covector fields η^a_i $(a=1,\ \cdots, k)$ satisfying

$$\begin{aligned} &\varepsilon F^i{}_r F^r{}_j=\delta^i_j-\xi^i_a\eta^a_j, \quad \tau\varepsilon g_{rs}F^r{}_i F^s{}_j=g_{ij}-\sum_a \eta^a_i\eta^a_j, \\ &\eta^a_i F^i{}_j=0, \qquad F^i{}_j\xi^j_a=0, \quad \eta^a_i\xi^i_b=\delta^a_b. \end{aligned} \tag{3.10}$$

The existence and arbitrariness of Lagrange connections compatible with a natural pair $(g_{ij},\ a_{ij})$ with a constant $\mu\neq 0$ is given by Theorem 2.2, if we replace the respective terms $\underset{1}{O}{}^{sm}_{nr}$ and $a^{st}a_{tr\overset{\circ}{|}k}$, $a^{st}a_{tr}\overset{\circ}{|}_k$ in (2.9) by $\underset{1}{O}{}^{sm}_{nr}$ of (3.8) and $a^{st}a_{tr\overset{\circ}{|}k}+3l^s{}_t l^t{}_{r\overset{\circ}{|}k}-l^s{}_{r\overset{\circ}{|}k}$, $a^{st}a_{tr}\overset{\circ}{|}_k+3l^s{}_t l^t{}_r\overset{\circ}{|}_k-l^s{}_r\overset{\circ}{|}_k$.

Lastly, it is noted whether the naturality is necessary in order that the system of equations (1.8), (2.4), (3.9) with unknowns $B_j{}^i{}_k$, $D_j{}^i{}_k$ has a solution is an open problem.

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