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COMBINATORIAL IDENTITIES VIA DEFINITE INTEGRALS

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Abstract

The purpose of the present note is to derive certain combinatorial identities from the evaluation of definite integrals. As an advantage of this method, we can obtain simultaneously *two* different identities from *a* definite integral as shown in the Theorem.

1. Evaluation of the integral $I_p(n)$.

Let

$$I_p(n)=\int_0^{\frac{\pi}{2}}(1+\cos x)^n \cos^p x dx \qquad (n,\ p=0,\ 1,\ 2,\ \cdots).$$

In this section we are exclusively concerned with obtaining the expressions of $I_p(n)$ $(p=1,\ 2,\ 3,\ \cdots)$ in terms of $I_0(n)$. First of all we shall get explicit expressions for $I_0(n)$ itself in three forms.

Lemma 1. *For any positive integer* n,

$$I_0(n)=\frac{1}{2^{n+1}}\binom{2n}{n}\pi+\frac{1}{2^n}\binom{2n}{n}\sum_{k=0}^{n-1}\frac{2^k(k!)^2}{(2k+1)!}, \tag{1}$$

$$I_0(n)=\frac{1}{2^n}\binom{2n}{n}\left(\frac{\pi}{2}+2\right)-\frac{1}{n}-\frac{1}{2^n}\binom{2n}{n}\sum_{k=1}^{n-1}\frac{2^k(k-1)!k!}{(2k+1)!}, \tag{2}$$

$$I_0(n)=\frac{1}{2^{n+1}}\binom{2n}{n}\pi+\frac{(n!)^2}{2^{n-1}}\binom{2n}{n}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{(n-2k-1)!(n+2k+1)!(2k+1)}, \tag{3}$$

where in (3) *the square bracket denotes the integral part function.*

Proof. In the case $n=1$, the validity of (2) is easily verified if we agree with the usual convention that the empty sum means 0, and in the same case it is easy to see that (1) and (3) also hold. Hence, in the sequel, we may suppose $n\geqq 2$.

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By the integration by parts we have

$$I_1(n)=\int_0^{\frac{\pi}{2}}(1+\cos x)^n\cos x dx$$

$$=\Big[(1+\cos x)^n\sin x\Big]_0^{\frac{\pi}{2}}+n\int_0^{\frac{\pi}{2}}(1+\cos x)^{n-1}\sin^2 x dx=1+n\{I_0(n)-I_1(n)\}.$$

Thus

(4) $$I_1(n)=\frac{n}{n+1}I_0(n)+\frac{1}{n+1}.$$

On the other hand, it is clear that

(5) $$I_0(n+1)-I_0(n)=I_1(n).$$

From (4) and (5) we get

(6) $$I_0(n+1)=\frac{2n+1}{n+1}I_0(n)+\frac{1}{n+1},$$

which implies

$$\frac{2^n n!(n+1)!}{(2n+1)!}I_0(n+1)=\frac{2^{n-1}(n-1)!n!}{(2n-1)!}I_0(n)+\frac{2^n(n!)^2}{(2n+1)!},$$

which, in turn, implies

$$\frac{2^{n-1}(n-1)!n!}{(2n-1)!}I_0(n)-I_0(1)=\sum_{k=1}^{n-1}\left\{\frac{2^k k!(k+1)!}{(2k+1)!}I_0(k+1)-\frac{2^{k-1}(k-1)!k!}{(2k-1)!}I_0(k)\right\}=\sum_{k=1}^{n-1}\frac{2^k(k!)^2}{(2k+1)!},$$

that is,

$$\frac{2^n(n!)^2}{(2n)!}I_0(n)=\frac{\pi}{2}+1+\sum_{k=1}^{n-1}\frac{2^k(k!)^2}{(2k+1)!}=\frac{\pi}{2}+\sum_{k=0}^{n-1}\frac{2^k(k!)^2}{(2k+1)!}.$$

This completes the proof of (1).

In the meantime, from (4) it follows that

(7) $$I_0(n)=\frac{n+1}{n}I_1(n)-\frac{1}{n}.$$

Thus, substituting (7) into (5), we have

$$I_1(n)=\frac{n+2}{n+1}I_1(n+1)-\frac{n+1}{n}I_1(n)+\frac{1}{n(n+1)},$$

namely,

$$I_1(n+1)=\frac{(n+1)(2n+1)}{n(n+2)}I_1(n)-\frac{1}{n(n+2)}.$$

Multiplying the both members of this equality by $\frac{2^n(n!)^2(n+2)}{(2n+1)!}$, we have

$$\frac{2^n(n!)^2(n+2)}{(2n+1)!}I_1(n+1)=\frac{2^{n-1}\{(n-1)!\}^2(n+1)}{(2n-1)!}I_1(n)-\frac{2^n(n-1)!n!}{(2n+1)!}.$$

Hence

$$\frac{2^{n-1}\{(n-1)!\}^2(n+1)}{(2n-1)!}I_1(n)-2I_1(1)$$

$$=\sum_{k=1}^{n-1}\left\{\frac{2^k(k!)^2(k+2)}{(2k+1)!}I_1(k+1)-\frac{2^{k-1}\{(k-1)!\}^2(k+1)}{(2k-1)!}I_1(k)\right\}=-\sum_{k=1}^{n-1}\frac{2^k(k-1)!k!}{(2k+1)!}.$$

Noting $I_1(1)=1+\frac{\pi}{4}$, we get

$$\frac{2^n n!(n+1)!}{n(2n)!}I_1(n)=\left(\frac{\pi}{2}+2\right)-\sum_{k=1}^{n-1}\frac{2^k(k-1)!k!}{(2k+1)!},$$

that is,

$$\frac{n+1}{n}I_1(n)=\frac{1}{2^n}\binom{2n}{n}\left(\frac{\pi}{2}+2\right)-\frac{1}{2^n}\binom{2n}{n}\sum_{k=1}^{n-1}\frac{2^k(k-1)!k!}{(2k+1)!}.$$

Substituting this result into (7), we conclude (2).

Finally we shall prove (3). It is known that ([1], 222)

$$(1+\cos x)^n=\frac{1}{2^n}\binom{2n}{n}\left\{1+2\sum_{k=1}^{n}\frac{(n!)^2}{(n-k)!(n+k)!}\cos kx\right\}.$$

Therefore

$$I_0(n)=\int_0^{\frac{\pi}{2}}(1+\cos x)^n dx=\frac{1}{2^n}\binom{2n}{n}\left\{\frac{\pi}{2}+2\sum_{k=1}^{n}\frac{(n!)^2}{(n-k)!(n+k)!k}\sin\frac{k\pi}{2}\right\}$$

$$=\frac{1}{2^n}\binom{2n}{n}\left\{\frac{\pi}{2}+2\sum_{\substack{k=1\\ k:\,odd}}^{n}\frac{(n!)^2}{(n-k)!(n+k)!k}\sin\frac{k\pi}{2}\right\}$$

$$=\frac{1}{2^n}\binom{2n}{n}\frac{\pi}{2}+\frac{(n!)^2}{2^{n-1}}\binom{2n}{n}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k}{(n-2k-1)!(n+2k+1)!(2k+1)}.$$

Thus we show the validity of (3).

Remark 1. In addition to the above three results (1), (2) and (3) we have one more expression for $I_0(n)$ which is obtained through the expansion of $(1+\cos x)^n$ by the binomial theorem. In this manner we have (see Lemma 3 on p.25)

$$I_0(n)=\sum_{k=0}^{n}\binom{n}{k}\int_0^{\frac{\pi}{2}}\cos^k x dx$$

$$=\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{2k}\int_0^{\frac{\pi}{2}}\cos^{2k}xdx+\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}\int_0^{\frac{\pi}{2}}\cos^{2k+1}xdx$$

$$=\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{2k}\binom{2k}{k}\frac{\pi}{2^{2k+1}}+\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}\frac{2^{2k}(k!)^2}{(2k+1)!}.$$

The comparison of (1), say, and the fourth result thus obtained yields two identities

(8)
$$\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{2k}\binom{2k}{k}\frac{1}{2^{2k}}=\frac{1}{2^n}\binom{2n}{n},$$

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n}{2k+1}\frac{2^{2k}(k!)^2}{(2k+1)!}=\frac{1}{2^n}\binom{2n}{n}\sum_{k=0}^{n-1}\frac{2^k(k!)^2}{(2k+1)!}.$$

However, this method to obtain two identities from the evaluation of $I_0(n)$ in two ways is what we wish to describe in this note from a general point of view. Thus we omit the fourth form for $I_0(n)$ in Lemma 1, and (8) is a special case of the results which will be obtained later.

Remark 2. As a by-product of Lemma 1, we obtain the following identity :

(9)
$$\sum_{k=1}^{n-1}\frac{2^k(k+1)!(k-1)!}{(2k+1)!}=1-\frac{2^n}{n\binom{2n}{n}} \quad (n\geqq 1).$$

Because from (1) and (2) it follows that

$$\frac{1}{2^n}\binom{2n}{n}+\frac{1}{2^n}\binom{2n}{n}\sum_{k=1}^{n-1}\frac{2^k(k!)^2}{(2k+1)!}=\frac{1}{2^{n-1}}\binom{2n}{n}-\frac{1}{n}-\frac{1}{2^n}\binom{2n}{n}\sum_{k=1}^{n-1}\frac{2^k(k-1)!k!}{(2k+1)!},$$

which implies

$$\frac{1}{2^n}\binom{2n}{n}\sum_{k=1}^{n-1}\frac{2^k(k+1)!(k-1)!}{(2k+1)!}=\frac{1}{2^n}\binom{2n}{n}-\frac{1}{n}.$$

From this result we obtain (9) at once.

Remark 3. By making use of the Stirling's formula, we observe

$$\frac{2^n}{n\binom{2n}{n}}\cong\frac{\sqrt{\pi}}{\sqrt{n}\,2^n}\quad(\text{as } n\to\infty),$$

where $a_n\cong b_n$ means $\lim\frac{a_n}{b_n}=1$. The combination of (9) and the last asymptotic relation yields

$$\sum_{n=1}^{\infty}\frac{2^n(n+1)!(n-1)!}{(2n+1)!}=1.$$

Remark 4. Similarly, the combination of (1) and (3), also that of (2) and (3) imply

$$(n!)^2\sum_{k=0}^{[\frac{n-1}{2}]}\frac{(-1)^k}{(n-2k-1)!(n+2k+1)!(2k+1)}$$

$$=\sum_{k=0}^{n-1}\frac{2^{k-1}(k!)^2}{(2k+1)!}=1-\frac{2^{n-1}}{n\binom{2n}{n}}-\frac{1}{2}\sum_{k=1}^{n-1}\frac{2^k(k-1)!k!}{(2k+1)!}.$$

Lemma 2. *For any positive integer p, there holds the following relation :*

$$I_p(n)=\frac{A_p(n)}{(n+1)(n+2)\cdots(n+p)}I_0(n)+\frac{B_p(n)}{(n+1)(n+2)\cdots(n+p)},\tag{10}$$

where $A_p(n)$ and $B_p(n)$ are polynomials in n and

$$\deg A_p(n)=p,\qquad \deg B_p(n)=p-1.\tag{11}$$

Moreover, the sequences of polynomials $\{A_p(n)\}$ and $\{B_p(n)\}$ satisfy the following recurrence relations :

$$\begin{cases} A_{p+1}(n)=(2n+1)A_p(n+1)-(n+p+1)A_p(n), \\ B_{p+1}(n)=A_p(n+1)+(n+1)B_p(n+1)-(n+p+1)B_p(n), \end{cases}\tag{12}$$

with

$$A_1(n)=n,\qquad B_1(n)=1.\tag{13}$$

Proof. The proof of (10) proceeds by induction in p. When $p=1$, (10) is clear from (4) and (13). Suppose (10) be true for some positive integer p. Then,

$$I_{p+1}(n)=\int_0^{\frac{\pi}{2}}(1+\cos x)^n\cos^{p+1}x\,dx=I_p(n+1)-I_p(n)$$

$$=\frac{A_p(n+1)}{(n+2)(n+3)\cdots(n+p+1)}I_0(n+1)+\frac{B_p(n+1)}{(n+2)(n+3)\cdots(n+p+1)}$$

$$-\frac{A_p(n)}{(n+1)(n+2)\cdots(n+p)}I_0(n)-\frac{B_p(n)}{(n+1)(n+2)\cdots(n+p)}.$$

Therefore in view of (6) and (12) we have

$$(n+1)(n+2)\cdots(n+p+1)I_{p+1}(n)$$

$$=(n+1)A_p(n+1)I_0(n+1)+(n+1)B_p(n+1)-(n+p+1)A_p(n)I_0(n)-(n+p+1)B_p(n)$$

$$=\{(2n+1)A_p(n+1)-(n+p+1)A_p(n)\}I_0(n)$$

$$+A_p(n+1)+(n+1)B_p(n+1)-(n+p+1)B_p(n)$$

$$=A_{p+1}I_0(n)+B_{p+1}(n).$$

Hence (10) is true for $p+1$. This completes the proof of (10).

Next we shall prove (11) also by induction in p. When $p=1$ (11) is clear from (13). Suppose (11) be true for some positive integer p. Then we may put

$$A_p(n)=a_0n^p+(\text{ lower terms }),\quad a_0\neq 0,$$

$$B_p(n)=b_0n^{p-1}+(\text{ lower terms }),\ b_0\neq 0.$$

These relations together with (12) imply

$$A_{p+1}(n)=(2n+1)(a_0n^p+\text{ lower terms })-(n+p+1)(a_0n^p+\text{ lower terms })$$

$$=a_0n^{p+1}+(\text{ lower terms }),$$

and

$$B_{p+1}(n)=(a_0n^p+\text{ lower terms })+(n+1)(b_0n^{p-1}+\text{ lower terms })$$

$$-(n+p+1)(b_0n^{p-1}+\text{ lower terms })$$

$$=a_0n^p+(\text{ lower terms }).$$

Thus deg $A_{p+1}(n)=p+1$ and deg $B_{p+1}(n)=p$. Therefore (11) is proved and this completes the proof of Lemma 2.

Remark. We may find the polynomials $A_p(n)$ and $B_p(n)$ $(p=1,\ 2,\ 3,\ \cdots)$ on the basis of (12) and (13). The first six polynomials $A_p(n)$ and those of $B_p(n)$ are as follows:

$$(14)\qquad \begin{cases} A_1(n)=n, \quad A_2(n)=n^2+n+1, \quad A_3(n)=n^3+3n^2+5n, \\ A_4(n)=n^4+6n^3+17n^2+12n+9, \quad A_5(n)=n^5+10n^4+45n^3+80n^2+89n, \\ A_6(n)=n^6+15n^5+100n^4+315n^3+574n^2+345n+225, \end{cases}$$

$$(15)\quad \begin{cases} B_1(n)=1, \quad B_2(n)=n, \quad B_3(n)=n^2+2n+4, \quad B_4(n)=n^3+5n^2+13n, \\ B_5(n)=n^4+9n^3+37n^2+48n+64, \quad B_6(n)=n^5+14n^4+87n^3+238n^2+389n. \end{cases}$$

2. Derivation of the combinatorial identities.

In this section we derive two combinatorial identities from a definite integral dealt with in section 1. We require two simple lemmas.

Lemma 3. *For any non-negative integer* n

$$\int_0^{\frac{\pi}{2}} \cos^{2n} x dx = \binom{2n}{n} \frac{\pi}{2^{2n+1}},$$

$$\int_0^{\frac{\pi}{2}} \cos^{2n+1} x dx = \frac{2^{2n}}{(2n+1)\binom{2n}{n}}.$$

Lemma 4. *Let* a, b, c *and* d *be rational numbers. If* $a\pi+b=c\pi+d$, *then* $a=c$ *and* $b=d$.

Lemma 4 gives us a clue to conclude two identities from an identity.

Theorem. *Let* p *be any even positive integer. Then*

$$(16)\quad \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \binom{2k+p}{k+\frac{p}{2}} \frac{1}{2^{2k+p}} = \frac{1}{2^n} \binom{2n}{n} \frac{A_p(n)}{(n+1)(n+2)\cdots(n+p)},$$

$$(17)\quad \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} \frac{2^{2k+p}}{(2k+p+1)\binom{2k+p}{k+\frac{p}{2}}} = \frac{1}{2^n} \binom{2n}{n} \frac{c_n A_p(n)}{(n+1)(n+2)\cdots(n+p)} + \frac{B_p(n)}{(n+1)(n+2)\cdots(n+p)}.$$

Furthermore, let p *be any odd positive integer. Then*

$$(18)\quad \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} \binom{2k+p+1}{k+\frac{p+1}{2}} \frac{1}{2^{2k+p}} = \frac{1}{2^{n-1}} \binom{2n}{n} \frac{A_p(n)}{(n+1)(n+2)\cdots(n+p)},$$

$$(19)\quad \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \frac{2^{2k+p-1}}{(2k+p)\binom{2k+p-1}{k+\frac{p-1}{2}}} = \frac{1}{2^n} \binom{2n}{n} \frac{c_n A_p(n)}{(n+1)(n+2)\cdots(n+p)} + \frac{B_p(n)}{(n+1)(n+2)\cdots(n+p)},$$

where $A_p(n)$ *and* $B_p(n)$ *are given by* (12) *and* (13) *in Lemma* 2, *and* c_n *is given by*

$$c_n=\sum_{k=0}^{n-1}\frac{2^k(k!)^2}{(2k+1)!}.$$

Proof. We shall prove the first part only of the theorem, since the proof of the second part can be carried out in a similar fashion.

Let p be an even positive integer, so we may put $p=2m$, say, where m is a positive integer. Then, in view of Lemma 3, we have

$$I_p(n)=\int_0^{\frac{\pi}{2}}(1+\cos x)^n\cos^p x dx=\sum_{k=0}^{n}\binom{n}{k}\int_0^{\frac{\pi}{2}}\cos^{k+2m}x dx$$

$$=\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}\int_0^{\frac{\pi}{2}}\cos^{2k+2m}x dx+\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}\int_0^{\frac{\pi}{2}}\cos^{2k+2m+1}x dx$$

$$=\pi\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}\binom{2k+2m}{k+m}\frac{1}{2^{2k+2m+1}}+\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}\frac{2^{2k+2m}}{(2k+2m+1)\binom{2k+2m}{k+m}}$$

$$=\pi\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}\binom{2k+p}{k+\frac{p}{2}}\frac{1}{2^{2k+p+1}}+\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}\frac{2^{2k+p}}{(2k+p+1)\binom{2k+p}{k+\frac{p}{2}}}.$$

On the other hand, by (1) and Lemma 2

$$I_p(n) = \frac{A_p(n)}{(n+1)(n+2)\cdots(n+p)2^{n+1}}\binom{2n}{n}\pi$$

$$+\frac{c_nA_p(n)}{(n+1)(n+2)\cdots(n+p)}\frac{1}{2^n}\binom{2n}{n}+\frac{B_p(n)}{(n+1)(n+2)\cdots(n+p)}.$$

Taking into account of Lemma 4 and the last two equalities, we get (16) and (17) immediately.

Remark 1. We mention the following results which are easily obtained from (14), (15) and the Theorem.

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}\binom{2k+2}{k+1}\frac{1}{2^{2k+2}} = \frac{1}{2^n}\binom{2n}{n}\frac{n^2+n+1}{(n+1)(n+2)}, \qquad (p=2 \text{ in } (16))$$

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}\binom{2k+4}{k+2}\frac{1}{2^{2k+4}} = \frac{1}{2^n}\binom{2n}{n}\frac{n^4+6n^3+17n^2+12n+9}{(n+1)(n+2)(n+3)(n+4)}, \qquad (p=4 \text{ in } (16))$$

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} \frac{2^{2k+2}}{(2k+3)\binom{2k+2}{k+1}} = \frac{c_n}{2^n}\binom{2n}{n}\frac{n^2+n+1}{(n+1)(n+2)} + \frac{n}{(n+1)(n+2)}, \qquad (p=2 \text{ in } (17))$$

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} \frac{2^{2k+4}}{(2k+5)\binom{2k+4}{k+2}} = \frac{c_n}{2^n}\binom{2n}{n}\frac{n^4+6n^3+17n^2+12n+9}{(n+1)(n+2)(n+3)(n+4)}$$
$$+\frac{n^3+5n^2+13n}{(n+1)(n+2)(n+3)(n+4)}, \qquad (p=4 \text{ in } (17))$$

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1}\binom{2k+2}{k+1}\frac{1}{2^{2k+1}} = \frac{1}{2^{n-1}}\binom{2n}{n}\frac{n}{n+1}, \qquad (p=1 \text{ in } (18))$$

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1}\binom{2k+4}{k+2}\frac{1}{2^{2k+3}} = \frac{1}{2^{n-1}}\binom{2n}{n}\frac{n^3+3n^2+5n}{(n+1)(n+2)(n+3)}, \qquad (p=3 \text{ in } (18))$$

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} \frac{2^{2k}}{(2k+1)\binom{2k}{k}} = \frac{c_n}{2^n}\binom{2n}{n}\frac{n}{n+1} + \frac{1}{n+1}, \qquad (p=1 \text{ in } (19))$$

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} \frac{2^{2k+2}}{(2k+3)\binom{2k+2}{k+1}} = \frac{c_n}{2^n}\binom{2n}{n}\frac{n^3+3n^2+5n}{(n+1)(n+2)(n+3)}$$
$$+\frac{n^2+2n+4}{(n+1)(n+2)(n+3)} \qquad (p=3 \text{ in } (19)).$$

Remark 2. When $p=0$, taking into account of (10), it is natural to interpret

$$(20) \qquad \frac{A_p(n)}{(n+1)(n+2)\cdots(n+p)}=1 \quad \text{and} \quad \frac{B_p(n)}{(n+1)(n+2)\cdots(n+p)}=0,$$

in other words,

$$A_p(n)=1 \qquad \text{and} \qquad B_p(n)=0.$$

Under these conventions (20), (16) and (17) in the Theorem still hold when $p=0$, since in this case (16) and (17) reduce to (8), which were established earlier. Thus, as already mentioned in section 1, (8) is a special case of the Theorem.

Remark 3. In the proof of the Theorem, if we use (2) or (3) instead of (1), we obtain similar results. But we omit the details here.

Remark 4. Along the same line of arguments, we are also able to derive simultaneously *four* identities from *a* definite integral. The details will be published elsewhere.

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