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ON COLLINEAR CHANGES OF FINSLER CONNECTIONS

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Abstract

A Finsler connection on a differentiable manifold M is regarded as a linear connection in the tangent bundle $T(M)$ satisfying some conditions. In the present paper we consider the case that two given Finsler connections $F\Gamma$, $F\bar{\Gamma}$ induce a common linear connection in $T(M)$.

Introduction

The concept of Finsler connection on an n-dimensional differentiable manifold M has been defined from various standpoints. Matsumoto [4] defined a Finsler connection by using the Finsler bundle $F(M)$, and showed that it induces a linear connection of Finsler type in $T(M)$. From the standpoint of the geometry of the tangent bundle $T(M)$, Miron [5] gave a definition of a Finsler connection based on a linear connection in $T(M)$. On the other hand, Ichijyō [3] defined a Finsler connection from the standpoint of G-structures, as a G-connection relative to a $D(GL(n,R))$-structure in $T(M)$. In each standpoint a Finsler connection $F\Gamma$ on M is regarded as a linear connection ∇ in $T(M)$ satisfying some conditions. So it seems interesting to consider the case that two Finsler connections $F\Gamma$, $F\bar{\Gamma}$ induce a common linear connection $\nabla(=\bar{\nabla})$ in $T(M)$. In this case we shall call the change $F\Gamma \to F\bar{\Gamma}$ *collinear*.

In the first section, we shall introduce the notion of collinear change of Finsler connections, and express the change in terms of the connection coefficients (Theorem 1.1). In the second section, from the standpoint of G-structures in $T(M)$ we shall consider a collinear change (Theorem 2.1), and a change satisfying some weaker conditions (Theorem 2.2). In the last section, we shall treat various transformation formulas by a collinear change (Theorem 3.1), and consider the case that a Finsler metric is given (Theorem 3.2).

Throughout the present paper the terminology and notations are referred to Matsumoto [4] and Ichijyō [3].

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1. On collinear and weakly collinear changes of Finsler connections

Let M be an n-dimensional differentiable manifold and $T(M)$ the tangent bundle. A coordinate system $x=(x^i)$ in M induces a canonical coordinate system $(x,y)=(x^i, y^i)$ in $T(M)$. We put $\partial_i=\partial/\partial x^i$, $\dot{\partial}_i=\partial/\partial y^i$. If a non-linear connection $N^i{}_j$ is given in $T(M)$, we have the $2n$-frame $\{X_A\}=\{X_i, X_{(i)}\}$, where

$$(1.1) \qquad X_i=\partial_i-N^r{}_i\dot{\partial}_r, \quad X_{(i)}=\dot{\partial}_i.$$

This frame $\{X_A\}$ is called the *N-frame* with respect to $N^i{}_j$.

Given a Finsler connection $F\Gamma=(N^i{}_j, F_j{}^i{}_k, C_j{}^i{}_k)$ on M in the sense of Matsumoto [4], we have a linear connection ∇ in $T(M)$ as follows:

$$(1.2) \qquad \begin{aligned} &\nabla_{X_k}X_j=F_j{}^i{}_kX_i, \quad \nabla_{X_k}X_{(j)}=F_j{}^i{}_kX_{(i)},\\ &\nabla_{X_{(k)}}X_j=C_j{}^i{}_kX_i, \quad \nabla_{X_{(k)}}X_{(j)}=C_j{}^i{}_kX_{(i)}. \end{aligned}$$

This ∇ is called the linear connection *induced* from $F\Gamma$.

Conversely given a non-linear connection and a linear connection satisfying (1.2), we have a Finsler connection $F\Gamma=(N^i{}_j, F_j{}^i{}_k, C_j{}^i{}_k)$. So we shall denote a Finsler connection by $F\Gamma=(N,\nabla)$, too.

Let $F\Gamma=(N,\nabla)$, $F\bar{\Gamma}=(\bar{N},\bar{\nabla})$ be Finsler connections on M. We shall consider the case of $\nabla=\bar{\nabla}$.

Definition 1.1. Let $F\Gamma$, $F\bar{\Gamma}$ be Finsler connections on M. If $F\Gamma$, $F\bar{\Gamma}$ induce a common linear connection ∇ in $T(M)$, then $F\Gamma$, $F\bar{\Gamma}$ are said to be *collinearly related*, and the change $F\Gamma\to F\bar{\Gamma}$ is called *collinear*.

Then we have

Theorem 1.1. *Let $F\Gamma=(N^i{}_j, F_j{}^i{}_k, C_j{}^i{}_k)$, $F\bar{\Gamma}=(\bar{N}^i{}_j, \bar{F}_j{}^i{}_k, \bar{C}_j{}^i{}_k)$ be Finsler connections on M. Then $F\Gamma$, $F\bar{\Gamma}$ are collinearly related, if and only if the connection coefficients of $F\Gamma$, $F\bar{\Gamma}$ are related in the following form:*

$$(1.3) \qquad \bar{N}^i{}_j=N^i{}_j-B^i{}_j,$$

$$(1.4) \qquad \bar{F}_j{}^i{}_k=F_j{}^i{}_k+C_j{}^i{}_rB^r{}_k,$$

$$(1.5) \qquad \bar{C}_j{}^i{}_k=C_j{}^i{}_k,$$

where $B^i{}_j$ is a Finsler tensor field satisfying the conditions

$$(1.6h) \qquad B^i{}_{j|k}=0, \qquad (1.6v) \qquad B^i{}_j|_k=0,$$

with respect to $F\Gamma$ (or equivalently with respect to $F\bar{\Gamma}$).

Proof. Suppose that $F\Gamma$, $F\bar{\Gamma}$ are collinearly related. If we define $B^i{}_j$ by (1.3), then the respective N-frames $\{X_A\}$, $\{\bar{X}_A\}$ with respect to $N^i{}_j$, $\bar{N}^i{}_j$ satisfy the relations

$$(1.7) \qquad \bar{X}_i=X_i+B^r{}_iX_{(r)}, \quad \bar{X}_{(i)}=X_{(i)}.$$

Since $F\Gamma$, $F\bar{\Gamma}$ induce a common linear connection ∇, we have

(1.8) $$\nabla_{\bar{X}_k}\bar{X}_j=\bar{F}_j{}^i{}_k\bar{X}_i,\ \nabla_{\bar{X}_k}\bar{X}_{(j)}=\bar{F}_j{}^i{}_k\bar{X}_{(i)},$$

(1.9) $$\nabla_{\bar{X}_{(k)}}\bar{X}_j=\bar{C}_j{}^i{}_k\bar{X}_i,\ \nabla_{\bar{X}_{(k)}}\bar{X}_{(j)}=\bar{C}_j{}^i{}_k\bar{X}_{(i)}.$$

From (1.9) we have (1.5), (1.6v). From (1.8) we have (1.4) and $B^i{}_{j|k}+B^i{}_j|_rB^r{}_k=0$, which is reduced to (1.6h) owing to (1.6v).

The converse is also true. Q. E. D.

Suggested by Theorem 1.1, we shall extend the notion of collinear change as follows.

Definition 1.2. The change of Finsler connections $F\Gamma=(N^i{}_j,\ F_j{}^i{}_k,\ C_j{}^i{}_k)\rightarrow F\bar{\Gamma}=(\bar{N}^i{}_j,\ \bar{F}_j{}^i{}_k,\ \bar{C}_j{}^i{}_k)$ is called *weakly collinear* if the difference tensor field $B^i{}_j=N^i{}_j-\bar{N}^i{}_j$ satisfies (1.6h), (1.6v) with respect to $F\Gamma$.

The geometrical meaning of weakly collinear changes will be made clear in the next section from the standpoint of G-structures.

2. Considerations from the standpoint of G-structures

We shall consider collinear changes of Finsler connections from the standpoint of G-structures. According to Ichijyō [3], if a non-linear connection $N^i{}_j$ is given in $T(M)$, the tangent bundle $T(M)$ admits a $D(GL(n,R))$-structure $\boldsymbol{P}$ as a reduction of the standard integrable almost tangent structure, and the converse is also true. Then the N-frame $\{X_A\}$ is an adapted frame of $\boldsymbol{P}$, and a linear connection ∇ in $T(M)$ is a G-connection relative to $\boldsymbol{P}$ if and only if

(2.1) $$\nabla Q=0,\ \nabla P=0,$$

where $Q=\begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}$, $P=\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$ with respect to the N-frame $\{X_A\}$. From (2.1), it is shown that the linear connection ∇ is expressed in the form (1.2). Thus a G-connection ∇ relative to $\boldsymbol{P}$ is a linear connection induced from a Finsler connection $F\Gamma$ in the sense of Matsumoto [4].

Conversely, if a Finsler connection $F\Gamma=(N,\nabla)$ on M is given, the linear connection ∇ satisfies (2.1). So ∇ is a G-connection relative to $\boldsymbol{P}$ determined by the given N.

Thus we have

Theorem 2.1. *If Finsler connections $F\Gamma=(N,\nabla)$, $F\bar{\Gamma}=(\bar{N},\nabla)$ are collinearly related, then the linear connection ∇ is a common G-connection relative to $\boldsymbol{P}$, $\bar{\boldsymbol{P}}$ determined by N, $\bar{N}$.*

Conversely let N, $\bar{N}$ be non-linear connections. If a linear connection ∇ is a common G-connection relative to $\boldsymbol{P}$, $\bar{\boldsymbol{P}}$, then the Finsler connections $F\Gamma=(N,\nabla)$, $F\bar{\Gamma}=(\bar{N},\nabla)$ are collinearly related.

Now, given non-linear connections N, $\bar{N}$, owing to (1.7) the corresponding Q, P and $\bar{Q}$, $\bar{P}$ are related in the form:

(2.2) $$\bar{Q}=Q,\ \bar{P}=P+\begin{pmatrix} 0 & 0 \\ 2B & 0 \end{pmatrix},$$

where $B^i{}_j=N^i{}_j-\bar{N}^i{}_j$,

We shall consider a weakly collinear change $F\Gamma=(N,\nabla)\to F\bar{\Gamma}=(\bar{N},\bar{\nabla})$. Since ∇ satisfies (2.1), we have $\nabla\bar{Q}=0$, $\nabla\bar{P}=0$ from (2.2) and (1.6h), (1.6v). Therefore ∇ is also a G-conncetion relative to $\bar{\boldsymbol{P}}$.

The converse is also true and we have

Theorem 2.2. *The change of Finsler connections* $F\Gamma=(N,\nabla)\to F\bar{\Gamma}=(\bar{N},\bar{\nabla})$ *is weakly collinear if and only if the linear connection* ∇ *is a common* G*-connection relative to* $\boldsymbol{P}$, $\bar{\boldsymbol{P}}$ *determined by* N, $\bar{N}$.

3. Transformation formulas by a collinear change

We shall give the transformation formulas of various geometrical objects by a collinear change $F\Gamma\to F\bar{\Gamma}$ (cf. [2]).

The torsion tensor fields are transformed as follows:

(3.1) $$\bar{R}^i{}_{jk}=R^i{}_{jk}+S_r{}^i{}_sB^r{}_jB^s{}_k-T_j{}^r{}_kB^i{}_r+A_{jk}\{P^i{}_{jr}B^r{}_k-B^i{}_rC_j{}^r{}_sB^s{}_k\},$$

(3.2) $$\bar{P}^i{}_{jk}=P^i{}_{jk}+S_r{}^i{}_kB^r{}_j-C_j{}^r{}_kB^i{}_r,$$

(3.3) $$\bar{T}_j{}^i{}_k=T_j{}^i{}_k+A_{jk}\{C_j{}^i{}_rB^r{}_k\},$$

(3.4) $$\bar{S}_j{}^i{}_k=S_j{}^i{}_k,$$

(3.5) $$\bar{C}_j{}^i{}_k=C_j{}^i{}_k,$$

where $A_{jk}\{\cdots\}$ denotes the alternate summation.

The curvature tensor fields are transformed as follows:

(3.6) $$\bar{R}_j{}^i{}_{kl}=R_j{}^i{}_{kl}+S_j{}^i{}_{rs}B^r{}_kB^s{}_l+A_{kl}\{P_j{}^i{}_{kr}B^r{}_l\},$$

(3.7) $$\bar{P}_j{}^i{}_{kl}=P_j{}^i{}_{kl}+S_j{}^i{}_{rl}B^r{}_k,$$

(3.8) $$\bar{S}_j{}^i{}_{kl}=S_j{}^i{}_{kl}.$$

The deflection tensor field is transformed as follows:

(3.9) $$\bar{D}^i{}_j=D^i{}_j+y^i|_rB^r{}_j.$$

If $\det(y^i|_j)\neq 0$, then $\bar{D}^i{}_j=D^i{}_j$ is equivalent to $B^i{}_j=0$. Thus we have

Theorem 3.1. *Let* $F\Gamma$ *be a Finsler connection satisfying* $\det(y^i|_j)\neq 0$. *If a collinear change* $F\Gamma\to F\bar{\Gamma}$ *of Finsler connections preserves the deflection tensor field, then the change is the identity.*

Since $y^i|_j=\delta^i_j+y^rC_r{}^i{}_j$, the so-called C_1-condition $y^rC_r{}^i{}_j=0$ yields $\det(y^i|_j)\neq 0$. Thus we have

Corollary. *In a collinear change $F\Gamma \to F\bar{\Gamma}$, if $F\Gamma$ satisfies the C_1-condition and the deflection tensor fields of $F\Gamma$, $F\bar{\Gamma}$ vanish, then the change is the identity.*

Further it is noted that the following quantities appearing in Matsumoto [4] are invariant by a collinear change.

(3.10) $$\Gamma_j{}^i{}_k = F_j{}^i{}_k + C_j{}^i{}_r N^r{}_k,$$

(3.11) $$N_j{}^i{}_k = \partial_k N^i{}_j + N^r{}_j \Gamma_r{}^i{}_k - N^i{}_r \Gamma_j{}^r{}_k,$$

(3.12) $$N_j{}^i{}_{(k)} = \dot{\partial}_k N^i{}_j + N^r{}_j C_r{}^i{}_k - N^i{}_r C_j{}^r{}_k.$$

Finally, we consider the case of a Finsler space. In a collinear change $F\Gamma \to F\bar{\Gamma}$, if we take the Berwald connection as $F\Gamma$, the change preserves all the torsion tensor fields and the curvature tensor fields $P_j{}^i{}_{kl}$, $S_j{}^i{}_{kl}$.

By Miron-Hashiguchi [6], in a change $F\Gamma \to F\bar{\Gamma}$ of Finsler connections given by (1.3), (1.4), (1.5), if $F\Gamma$ is metrical, then $F\bar{\Gamma}$ is also metrical. Thus we have

Theorem 3.2. *By a collinear change of Finsler connections the metrical property is preserved.*

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