ON INFINITESIMAL AUTOMORPHISMS OF D (GL(n, R))-STRUCTURES ON TANGENT BUNDLES

Rep. Fac. Sci., Kagoshima Univ. (Math., Phys. & Chem.),
No. 21, p. 13-24, 1988.

ON INFINITESIMAL AUTOMORPHISMS OF $D(GL(n,R))$-STRUCTURES ON TANGENT BUNDLES

Tadashi AIKOU*

(Received September 10, 1988)

Abstract

A tangent bundle with a non-linear connection admits a $D(GL(n,R))$-structure as a reduction of the standard almost tangent structure. The purpose of the present paper is to investigate infinitesimal automorphisms of the $D(GL(n,R))$-structure.

Introduction

Let M be an n-dimensional differentiable manifold and TM the tangent bundle over M. The geometry of tangent bundles has been studied by many authors. Especially, if a non-linear connection is given in TM, there are defined various important geometrical structures on TM (Kandatsu [4], Yano-Ishihara [8,9]).

In his recent papers [2,3], Ichijyō has studied G-structures on tangent bundles and obtained many remarkable results. The tangent bundle TM admits the *standard almost tangent structure* $\boldsymbol{P_0}$ whose structure group is given by

$$\left\{ \begin{pmatrix} A & O \\ B & A \end{pmatrix} ;\ A \in GL(n,R),\ B \in gl(n,R) \right\},$$

and the natural frame is an adapted frame to $\boldsymbol{P_0}$ (cf. Fujimoto [1]). If a non-linear connection is given in TM, then TM admits a $D(GL(n,R))$*-structure* $\boldsymbol{P_1}$ as a reduction of $\boldsymbol{P_0}$, whose structure group is given by

$$D(GL(n,R)) = \left\{ \begin{pmatrix} A & O \\ O & A \end{pmatrix} ;\ A \in GL(n,R) \right\}$$

* Department of Mathematics, Faculty of Science, Kagoshima University, Kagoshima, 890 Japan.

(Ichijyō [2]). Furthermore, if M is a generalized metric space, TM admits a $D(O(n))$*-structure* $\boldsymbol{P_2}$ as a reduction of $\boldsymbol{P_0}$ whose structure group is given by

$$D(O(n)) = \left\{ \begin{pmatrix} A & O \\ O & A \end{pmatrix} ; A \in O(n) \right\}$$

(Ichijyō [3]). These G-structures play an important role in Ichijyō's theory. In fact, the properties of these G-structures reflect some geometrical structures on the base manifold M.

On the other hand, the concept of infinitesimal automorphisms of G-structures is very important (cf. Fujimoto [1]). Let V be a vector field in TM, and let $\{f_t\}$ be the local 1-parameter group of local transformations f_t generated by V. Then we can consider the *natural lift* $\{\tilde{f}_t\}$of $\{f_t\}$ to the frame bundle $L(TM)$ over TM. V is called an *infinitesimal automorphism of a G-structure* $\boldsymbol{P}$ if for any adapted frame $\{Z_\alpha\}$ to $\boldsymbol{P}$ the local frame $\{\tilde{f}_t(Z_\alpha)\}$ is also adapted to $\boldsymbol{P}$.

The purpose of the present paper is to investigate infinitesimal automorphisms of the G-structures $\boldsymbol{P_1}, \boldsymbol{P_2}$ on TM and study some relations to the other geometrical structures on TM. In Section 1 we shall consider infinitesimal automorphisms of the $D(GL(n,R))$-structure $\boldsymbol{P_1}$ (Theorem 1.1, Theorem 1.4), and in Section 2 some relations to some G-structures defined by the given non-linear connection (Theorem 2.1, Theorem 2.2). In Section 3 we shall consider infinitesimal automorphisms of the $D(O(n))$-structure $\boldsymbol{P_2}$ (Theorem 3.1), and in the last section some relations to almost Hamilton vector fields (Theorem 4.3).

Throughout the present paper, the terminology and notation are referred to Ichijyō [2,3] and Matsumoto [5]. As to the indices, we assume that Greek indices take the values 1, 2,$\cdots$, $2n$ and Latin 1, 2,$\cdots$, n, and (i), (j),$\cdots$ stand for respective values $i+n$, $j+n$,$\cdots$.

The author wishes to express here his sincere gratitude to Professor Dr. Y. Ichijyō for the helpful comments and criticism. The author's attention was drawn by him to the subject of the present paper. The author is also grateful to Professor Dr. M. Matsumoto and Professor Dr. M. Hashiguchi for the invaluable suggestions and encouragement.

1. Infinitesimal automorphisms of $D(GL(n,R))$-structures

Let $\{U, (x^i)\}$ be a coordinate system on an n-dimensional differentiable manifold M and $\{\pi^{-1}(U), (x^i, y^i)\}$ the induced canonical coordinate system on TM, where $\pi: TM \longrightarrow M$ is the natural projection.

Suppose that a non-linear connection $N^i{}_j(x,y)$ is given in TM. Then there exists the $D(GL(n,R))$-structure $\boldsymbol{P_1}$ on TM and, putting $\partial_i = \partial/\partial x^i$, $\dot{\partial}_i = \partial/\partial y^i$ *and* $\delta_i = \partial_i - N^r{}_i\dot{\partial}_r$, the $2n$-frame $\{X_\alpha\}$ on $\pi^{-1}(U)$ given by $X_i = \delta_i$ and $X_{(i)} = \dot{\partial}_i$ is an adapted frame to the $D(GL(n,R))$-structure $\boldsymbol{P_1}$. This frame is called the *N-frame*. The *horizontal* and *vertical distributions* are defined as assignments to each point of TM of n-dimensional vector spaces spanned by $\{X_i\}$ and $\{X_{(i)}\}$ respectively.

Let V be an infinitesimal automorphism of the $D(GL(n,R))$-structure $\boldsymbol{P_I}$ and $\{f_t\}$ the local 1-parameter group of local transformations f_t generated by V. If we denote by $\{\tilde{f}_t\}$ the natural lift of $\{f_t\}$ to the frame bundle $L(TM)$ over TM, then the local frame $\{\tilde{f}_t(X_\alpha)\}$ is also adapted to $\boldsymbol{P_I}$. Hence we see

$$(\tilde{f}_{-t}(X_\alpha(f_t(x,y)))) = (X_\alpha(x,y)) \cdot \begin{pmatrix} A(x,y,t) & O \\ O & A(x,y,t) \end{pmatrix},$$

where $(A(x,y,t)) \in GL(n,R)$. So the Lie derivative of X_α with respective to V is written in the form

$$(\mathcal{L}_V X_\alpha(x,y)) = (X_\alpha(x,y)) \cdot \lim_{t\to o} \left\{ \begin{pmatrix} A(x,y,t) & O \\ O & A(x,y,t) \end{pmatrix} - \begin{pmatrix} E & O \\ O & E \end{pmatrix} \right\} / t.$$

Consequently, if we put $\mathcal{L}_V X_\alpha = T^\beta{}_\alpha X_\beta$, we have $(T^\beta{}_\alpha) \in D(gl(n,R))$, that is, $T^i{}_j = T^{(i)}{}_{(j)}, T^i{}_{(j)} = T^{(i)}{}_j = 0$.

Putting $V = V^i X_i + V^{(i)} X_{(i)}$ with respect to the N-frame X_α, because of $\mathcal{L}_V X_\alpha = [V, X_\alpha]$ we have

$$(\mathcal{L}_V X_\alpha) = (X_\alpha) \begin{pmatrix} -\delta_j V^i & -\dot{\partial}_j V_i \\ V^r R^i{}_{rj} - V^{(r)} \dot{\partial}_r N^i{}_j - \delta_j V^i & V^r \dot{\partial}_j N^i{}_r - \dot{\partial}_j V^{(i)} \end{pmatrix},$$

where $R^i{}_{rj} = \delta_j N^i{}_r - \delta_r N^i{}_j$ is the curvature tensor of the non-linear connection N. From this we have

Proposition 1.1. *A vector field* $V = V^i X_i + V^{(i)} X_{(i)}$ *in TM is an infinitesimal automorphism of the* $D(GL(n,R))$*-structure* $\boldsymbol{P_I}$ *on TM if and only if the following conditions are satisfied:*

$$\delta_j V^i = \dot{\partial}_j V^{(i)} - V^r \dot{\partial}_j N^i{}_r, \tag{1.1}$$

$$\dot{\partial}_j V^i = 0, \tag{1.2}$$

$$V^r R^i{}_{rj} - V^{(r)} \dot{\partial}_r N^i{}_j - \delta_j V^{(i)} = 0. \tag{1.3}$$

From (1.2) we see $V^i = V^i(x)$, that is, V is *fibre-preserving*. So the condition (1.1) is written in the form $\partial_j V^i = \dot{\partial}_j (V^{(i)} - V^r N^i{}_r)$. From this we have $V^{(i)} - V^r N^i{}_r = (\partial_j V^i) y^j + B^i(x)$ with arbitary vector field $B^i(x)$. Hence V is written in the form

(1.4) $$V = (V^i(x)\cdot\partial/\partial x^i)^c + (B^i(x)\cdot\partial/\partial x^i)^v,$$

where the first (resp. second) term of the right-hand side is the *complete* (resp. *vertical*) *lift* of a vector field $V^i(x)\cdot\partial/\partial x^i$ (*resp.* $B^i(x)\cdot\partial/\partial x^i$) in the base manifold M. Conversely, a vector field V written in the form (1.4) satisfies the conditions (1.1), (1.2).

Given a non-linear connection $N^i{}_j$ in TM, we can define a linear connection on TM whose coefficients $F_j{}^i{}_k$, $C_j{}^i{}_k$ with respect to the N-frame are given by $F_j{}^i{}_k = \dot\partial_j N^i{}_k$, $C_j{}^i{}_k = 0$. If we denote a covariant derivative of a vector field T^i in TM by

(1.5) $$\overset{N}{\nabla}_j T^i = \delta_j T^i + T^r \dot\partial_r N^i{}_j,$$

then the condition (1.3) is written in the form

(1.6) $$\overset{N}{\nabla}_j V^{(i)} = V^r R^i{}_{rj}.$$

Thus we have

Theorem 1.1. *In a tangent bundle TM with a non-linear connection* $N^i{}_j$, *a vector field* $V = V^i X_i + V^{(i)} X_{(i)}$ *in TM is an infinitesimal automorphism of the* $D(GL(n,R))$*-structure* $\boldsymbol{P}_1$ *if and only if the following conditions are satisfied:*
(1) *V is written in the form* (1.4),
(2) *V satisfisfies* (1.6).

Instead of a matrix of $D(GL(n,R))$ and the N-frame $\{X_\alpha\}$, if we use a matrix of type $\begin{pmatrix} A & O \\ B & A \end{pmatrix}$ and the natural frame, in the same way we can derive the follwing theorem obtained by Ichijyō [2].

Theorem 1.2. *A vector field V in a tangent bundle is an infinitesimal automorphism of the standard almost tangent structure* $\boldsymbol{P}_0$ *if and only if V is written in the form* (1.4).

On the other hand, since $\mathcal{L}_V X_i = -(\delta_i V^h) X_h - (\overset{N}{\nabla}_i V^{(h)} - V^r R^h{}_{rj}) X_{(h)}$, the condition (1.6) has a geometrical meaning that V preserves the horizontal distribution. Thus Theorem 1.1 is restated as follows.

Theorem 1.3. *In a tangent bundle with a non-linear connection, a vector field is an infinitesimal automorphism of the* $D(GL(n,R))$*-structure* $\boldsymbol{P}_1$ *if and only if the following conditions are satisfied:*
(1) *V is an infinitesimal automorphism of the standard almost tangent structure* $\boldsymbol{P}_0$,

(2) *V preserves the horizontal distribution.*

By virtue of the expression (1.4), we shall consider the two cases where V are the complete and vertical lifts of a vector field in M.

In the case of $V = (v^i(x) \cdot \partial/\partial x^i)^c$, the components of V with respect to the N-frame $\{X_\alpha\}$ are given by $V^i = v^i(x)$, $V^{(i)} = y^m \partial_m v^i + N^i{}_r v^r$. So we have

$$\overset{N}{\nabla}_j V^{(i)} - V^r R^i{}_{rj} = y^k \partial_j \partial_k v^i + v^m \partial_m N^i{}_j + y^r \partial_r v^m \dot{\partial}_m N^i{}_j - (\partial_m v^i) N^m{}_j + (\partial_j v^m) N^i{}_m.$$

By the definition, the right-hand side is the Lie derivative of the non-linear connection $N^i{}_j$ with respect to the vector field $v = v^i(x) \cdot \partial/\partial x^i$ in M (cf. Yano [7]). So we see that the condition (1.6) is equivalent to

$$\mathcal{L}_v N^i{}_j = 0 . \tag{1.7}$$

So we have the following characterization of (1.7).

Theorem 1.4. *Let v be a vector field in M and $N^i{}_j$ a non-linear connection in TM. The complete lift of v is an infinitesimal automorphism of the $D(GL(n,R))$-structure $\boldsymbol{P_1}$, if and only if v satisfies* (1.7).

In the case of $V = (v^i(x) \cdot \partial/\partial x^i)^v$, the condition (1.6) is reduced to

$$\overset{N}{\nabla}_j V^i = \partial_j v^i + v^r \dot{\partial}_r N^i{}_j = 0 . \tag{1.8}$$

Thus we have

Theorm 1.5. *Let $v = v^i(x) \cdot \partial/\partial x^i$ be a vector field in M and $N^i{}_j$ a non-linear connection in TM. The vertical lift of v is an infinitesimal automorphism of the $D(GL(n,R))$-structure $\boldsymbol{P_1}$, if and only if v satisfies* (1.8).

2. Almost product *N*-structures and almost complex *N*-structures

The (1,1)-tensor field P on TM, given by

$$P = \begin{pmatrix} E & O \\ O & -E \end{pmatrix} \tag{2.1}$$

with respect to the N-frame $\{X_\alpha\}$, defines an almost product structure on TM. We shall call P the *almost product N-structure.* A vector field V in TM satisfying $\mathcal{L}_V P = 0$ is said to be an *infinitesimal automorphism of P.*

Putting $V = V^i X_i + V^{(i)} X_{(i)}$, V satisfies $\mathcal{L}_V P = 0$ if and only if

(2.2) $$V^i = V^i(x),\ \overset{N}{\nabla}_j V^{(i)} = V^r R^i{}_{rj}.$$

So Theorem 1.1 is also restated as follows.

Theorem 2.1. *In a tangent bundle with a non-linear connection, a vector field V is an infinitesimal automorphism of the $D(GL(n,R))$-structure $\boldsymbol{P_1}$ if the only if the following conditions are satisfied:*
(1) V is an infinitesimal automorphism of the standard almost tangent structure $\boldsymbol{P_0}$,
(2) V is an infinitesimal automorphism of the almost product N-structure P.

On the other hand, the (1,1)-tensor field F on TM, given by

(2.3) $$F = \begin{pmatrix} O & -E \\ E & O \end{pmatrix}$$

with respect to the N-frame $\{X_\alpha\}$, defines an almost complex structure on TM called the *almost complex N-structure* (Matsumoto [5, §23]). In general, a vector field V satisfying $\mathcal{L}_V F = 0$ for an almost complex structure F is said to be *almost analytic.* Kandatsu [4] obtained the conditions that the horizontal and vertical vector fields are almost analytic. We shall study the general case.

Putting $V \doteq V^i X_i + V^{(i)} X_{(i)}$, we have

$$\mathcal{L}_V F = \begin{pmatrix} V^r R^i{}_{rj} - \overset{N}{\nabla}_j V^{(i)} - \dot{\partial}_j V^i & \delta_j V^i + V^r \dot{\partial}_j N^i{}_r - \dot{\partial}_j V^{(i)} \\ \delta_j V^i + V^r \dot{\partial}_j N^i{}_r - \dot{\partial}_j V^{(i)} & \dot{\partial}_j V^i + \overset{N}{\nabla}_j V^{(i)} - V^r R^i{}_{rj} \end{pmatrix}.$$

So we get

Proposition 2.1. *Let F be the almost complex N-structure on TM. A vector field $V = V^i X_i + V^{(i)} X_{(i)}$ in TM is almost analytic if and only if the following conditions are satisfied:*

(2.4) $$V^r R^i{}_{rj} - \overset{N}{\nabla}_j V^{(i)} - \dot{\partial}_j V^i = 0,$$

(2.5) $$\delta_j V^i + V^r \dot{\partial}_j N^i{}_j - \dot{\partial}_j V^{(i)} = 0.$$

The condition (2.5) coincides with (1.1), and also the condition (2.4) coincides with (1.6) under the assumption $V^i = V^i(x)$. Hence we have

Theorem 2.2. *In a tangent bundle TM with a non-linear connection, let V be a fibre-preserving vector field in TM: $V^i = V^i(x)$. V is an infinitesimal automor-*

phism of the $D(GL(n,R))$-structure $\boldsymbol{P_1}$ if and only if V is almost analytic with respect to the almost complex N-structure F.

3. Infinitesimal automorphisms of $D(O(n))$-structures

We shall consider the case where M is a generalized metric space with a generalized metric $g_{ij}(x,y)$ in the sence of Miron [6], and suppose that a non-linear connection $N^i{}_j(x,y)$ is given in TM. If we put

$$G = \begin{pmatrix} g_{ij} & O \\ O & g_{ij} \end{pmatrix} \tag{3.1}$$

with respect to the N-frame $\{X_\alpha\}$, the $(0,2)$-tensor field G is a Riemannian metric on TM and defines an $O(2n)$-structure. Then the $D(O(n))$-structure $\boldsymbol{P_2}$ on TM is defined as the intersection of this $O(2n)$-structure and the $D(GL(n,R))$-structure $\boldsymbol{P_1}$ (Ichijyō [3]).

A vector field V in TM is an infinitesimal automorphism of $\boldsymbol{P_2}$ if and only if V is an infinitesimal automorphism of $\boldsymbol{P_1}$ and V satisfies

$$\mathcal{L}_V G = 0 . \tag{3.2}$$

In order to calculate $\mathcal{L}_V G$, we shall use the linear connection ∇ on TM whose coefficients $F_j{}^i{}_k$, $C_j{}^i{}_k$ with respect to the N-frame $\{X_\alpha\}$ are given by

$$F_j{}^i{}_k = g^{ir}(\delta_j g_{rk} + \delta_k g_{rj} - \delta_r g_{jk})/2,$$
$$C_j{}^i{}_k = g^{ir}(\dot{\partial}_j g_{rk} + \dot{\partial}_k g_{rj} - \dot{\partial}_r g_{jk})/2.$$

It is known that ∇ is a G-connection with respect to $\boldsymbol{P_2}$, and metrical, that is,

$$g_{ij|k} = \delta_k g_{ij} - g_{ir}F_j{}^r{}_k - g_{rj}F_i{}^r{}_k = 0 ,$$
$$g_{ij}|_k = \dot{\partial}_k g_{ij} - g_{ir}C_j{}^r{}_k - g_{rj}C_i{}^r{}_k = 0 ,$$

where the short and long bars denote the two kinds of covariant differentiations.

Putting $V = V^i X_i + V^{(i)} X_{(i)}$, we have with respect to ∇

$$\mathcal{L}_V G = \begin{pmatrix} V_{i|j} + V_{j|i} + V^{(r)}\dot{\partial}_r g_{ij} & V_i|_j - V^r C_{rij} - V^r R_{irj} + V_{(i)|j} + V^{(r)} P_{ijr} \\ V_i|_j - V^r C_{rij} - V^r R_{irj} + V_{(i)|j} + V^{(r)} P_{ijr} & V_{(i)}|_j + V_{(j)}|_i - V^r P_{irj} - V^r P_{jri} \end{pmatrix}$$

where we put $P^i{}_{jk} = \dot{\partial}_k N^i{}_j - F_k{}^i{}_j$, $V_i = g_{ir} V^r$, $V_{(i)} = g_{ir} V^{(r)}$. So we get

Proposition 3.1. *A vector field* $V = V^iX_i + V^{(i)}X_{(i)}$ *in TM satisfies* (3.2) *if and only if the following conditions are satisfied:*

$$V_{i|j} + V_{j|i} + V^{(r)}\dot{\partial}_r g_{ij} = 0, \tag{3.3}$$

$$V_{(i)}|_j + V_{(j)}|_i - V^rP_{irj} - V^rP_{jri} = 0, \tag{3.4}$$

$$V_i|_j - V^rC_{rij} - V^rR_{irj} + V_{(i)|j} + V^{(r)}P_{ijr} = 0. \tag{3.5}$$

In the case of $V^i = V^i(x)$, it is easily seen that

$$V_i|_j - V^rC_{rij} - V^rR_{irj} + V_{(i)|j} + V^{(r)}P_{ijr} = g_{ir}(\overset{N}{\nabla}_j V^{(r)} - V^sR^r{}_{sj}). \tag{3.6}$$

Hence the condition (3.5) is equivalent to (1.6) if $V^i = V^i(x)$. So we have

Theorem 3.1. *Let* (M, g_{ij}) *be an n-dimensional generalized metric space. In the tangent bundle TM with a non-linear connection, a vector field* $V = V^iX_i + V^{(i)}\cdot X_{(i)}$ *is an infinitesimal automorphsim of the* $D(O(n))$*-structure* $\boldsymbol{P_2}$ *if and only if the following conditions are satisfied:*
(1) *V is an infintitesimal automorphism of the* $D(GL(n,R))$*-structure* $\boldsymbol{P_1}$,
(2) *V satisfies* (3.3) *and* (3.4).

We shall consider the case where $V = V^iX_i + V^{(i)}X_{(i)}$ is the complete lift of a vector field $v = v^i(x)\cdot\partial/\partial x^i$ in M. In this case, since the components of V are given by $V^i = v^i(x)$, $V^{(i)} = y^m\partial_m v^i + N^i{}_r v^r$, the condition (3.3) is equivalent to

$$\pounds_v g_{ij} = 0, \tag{3.7}$$

that is, v is a *Killing vector field* in (M, g_{ij}), and because of $V_{(i)}|_j = v_{i|j} + v^rP_{irj} + (y^m\partial_m v^r + N^r{}_s v^s)C_{rij}$, the condition (3.4) is equivalent to

$$v_{i|j} + v_{j|i} + 2(y^m\partial_m v^r + N^r{}_s v^s)C_{rij} = 0. \tag{3.8}$$

Consequently we have

Theorem 3.2. *Let* (M, g_{ij}) *be a generalized metric space. In the tangent bundle TM with a non-linear connection* $N^i{}_j$, *the complete lift of a vector field* $v = v^i(x)\cdot\partial/\partial x^i$ in M is an infinitesimal automorphism of the $D(O(n))$-structure $\boldsymbol{P_2}$ *if and only if the following conditions are satisfied:*
(1) *v is a Killing vector field in* (M, g_{ij}) ,
(2) *v satisfies* (1.7) *and* (3.8).

If g_{ij} is a Finsler metric, the condition (3.8) is equivalent to (3.3).

Cororally 3.1. *In the condition* (2) *of Theorem 3.2, the condition* (3.8) *is omitted if* $g_{ij}(x,y)$ *is a Finsler metric.*

Next we shall consider the case where V is the vertical lift of a vector field $v = v^i(x)\cdot\partial/\partial x^i$ in M. In this case, the conditions in Proposition 3.1 are written in the forms $v^r\dot{\partial}_r g_{ij} = 0$, $v_{i|j} + v^r P_{irj} = 0$, $v_i|_j + v_j|_i = 0$. Because of $v^i = v^i(x)$, these conditions are written in the forms

$$(3.9) \qquad v^r\dot{\partial}_r g_{ij} = 0\,,\ \overset{N}{\nabla}_j v^i = 0\,,\ v^r C_{rij} = 0\,.$$

The second condition in (3.9) expresses the condition (1.6). So we have

Theorm 3.3. *Let* (M,g_{ij}) *be a generalized metric space. In the tangent bundle* TM *with a non-linear connection, the vertical lift of a vector field* $v = v^i(x)\cdot\partial/\partial x^i$ *in* M *is an infinitesimal automorphism of the* $D(O(n))$*-structure* $\boldsymbol{P}_2$ *if and only if* v *satisfies* (3.9).

Cororally 3.2. *In* (3.9) *of Theorem 3.3, the first condition coincides with the last if* $g_{ij}(x,y)$ *is a Finsler metric.*

4. Almost Hamilton vector field

Let (M,g_{ij}) be a generalized metric space, and suppose that a non-linear connection $N^i{}_j$ is given in TM. There exists an almost complex structure F defined by (2.3) and a Riemannian metric G defined by (3.1), and the pair $\{F,G\}$ defines an almost Hermitian structure on TM. If we put

$$(4.1) \qquad \omega = GF,$$

the (0,2)-tensor field ω on TM is skew-symmetric, and it defines an almost symplectic form on TM. A vector field V in TM satisfying

$$(4.2) \qquad \mathcal{L}_V\omega = 0$$

is called an *almost Hamilton vector field of* ω (Ichijyō [2]).

Putting $V = V^iX_i + V^{(i)}X_{(i)}$, the left-hand side of (4.2) is written in the form

$$\begin{pmatrix} g_{ir}(V^hR^r{}_{hj}-\overset{N}{\nabla}_jV^{(r)})-g_{rj}(V^hR^r{}_{hi}-\overset{N}{\nabla}_iV^{(r)}) & -V_{j|i}+P_{irj}V^r-V^{(r)}\dot{\partial}_r g_{ij}-(\dot{\partial}_jV^{(r)})g_{ir} \\ V_{i|j}-P_{jri}V^r+V^{(r)}\dot{\partial}_r g_{ij}+(\dot{\partial}_iV^{(r)})g_{jr} & (\dot{\partial}_jV^r)g_{rj}-(\dot{\partial}_iV^r)g_{rj} \end{pmatrix},$$

where the covariant differentiations are those with respect to the G-connection defined in the previous section. So we get

Proposition 4.1. *A vector field* $V = V^iX_i + V^{(i)}X_{(i)}$ in TM satisfies (4.2) *if and only if the following conditions are satisfied:*

$$g_{ir}(V^hR^r{}_{hi} - \overset{N}{\nabla}_jV^{(r)})-g_{rj}(V^hR^r{}_{hi}-\overset{N}{\nabla}_iV^{(r)}) = 0, \tag{4.3}$$

$$V_{j|i} - P_{irj}V^r+V^{(r)}\dot{\partial}_r g_{ij}+(\dot{\partial}_jV^{(r)})g_{ir} = 0, \tag{4.4}$$

$$(\dot{\partial}_jV^r)g_{ir}-(\dot{\partial}_iV^r)g_{rj} = 0. \tag{4.5}$$

Now we shall investigate some relations between the almost Hamilton vector fields and the infinitesimal automorphisms of the $D(O(n))$-structure $\boldsymbol{P_2}$. We shall restrict our considerations to the two cases where V are the complete and vertical lifts of a vector field in M.

First we shall consider the case where V is the complete lift of a vector field $v = v^i(x)\cdot\partial/\partial x^i$ in M. Then the condition (4.5) is trivial, and we can easily show that the condition (4.4) is equivalent to (3.3), that is, v is a Killing vector field in (M,g_{ij}).

On the other hand, because of the equation $V^rR^i{}_{rj}-\overset{N}{\nabla}_jV^{(i)}=\mathcal{L}_vN^i{}_j$ obtained in Section 1, the condition (4.3) is rewritten in the form

$$g_{rj}(\mathcal{L}_vN^r{}_i)-g_{ir}(\mathcal{L}_vN^r{}_j)=0. \tag{4.3'}$$

It is noted that (4.3) or (4.3′) is equivalent to the condition (2) in Theorem 9 of Ichijyō [2]. Thus we have the following theorem obtained by Ichijyō [2].

Theorem 4.1. *Let (M,g_{ij}) be a generalized metric space, and suppose that a non-linear connection $N^i{}_j$ is given in TM. The complete lift of a vector field v in M is an almost Hamilton vector field if and only if the following conditions are satisfied:*
(1) *v is a Killing vector field in (M,g_{ij}),*
(2) *v satisfies (4.3′).*

Next we shall consider the case where V is the vertical lift of $v=v^i(x)\cdot\partial/\partial x^i$

in M. Then the conditions in Proposition 4.1 are written as follows:

$$(4.6) \qquad g_{ir}\overset{N}{\nabla}_j v^r = g_{rj}\overset{N}{\nabla}_i v^r, \quad v^r \dot{\partial}_r g_{ij} = 0 .$$

Thus we have

Theorem 4.2. *Let* (M, g_{ij}) *be a generalized metric space, and suppose that a non-linear connection is given in TM. The vertical lift of* $v = v^i(x) \cdot \partial/\partial x^i$ *in M is an almost Hamilton vector field if and only if the conditions* (4.6) *are satisfied.*

Let g_{ij} be a Finsler metric. If we use the Cartan connection ∇^* determined by g_{ij}, the conditions (4.6) become the following conditions in Theorem 13 of Ichijyō [2]:

$$\nabla^*{}_j v_i = \nabla^*{}_i v_j, \quad v^r \dot{\partial}_r g_{ij} = 0 .$$

Lastly, we shall mention a relation between the almost Hamilton vector fields and the infinitesimal antomorphisms of the $D(O(n))$-structure P_2. For any vector field V in TM, we get from (4.1)

$$\mathcal{L}_V W = (\mathcal{L}_V G) F + G(\mathcal{L}_V F) .$$

So by virtue of Theorem 2.2 and Theorem 3.1, we see that any infinitesimal automorphism of $\boldsymbol{P_2}$ satisfies (4.2).

Conversely, if an infinitesimal automorphism of the $D(GL(n,R))$-structure $\boldsymbol{P_1}$ satisfies (4,2), it satisfies (3.2). Thus we have

Theorem 4.3. *Let* (M, g_{ij}) *be a generalized metric space, and suppose that a non-linear connection is given in TM. Then any infinitesimal automorphism of the* $D(O(n))$*-structure* $\boldsymbol{P_2}$ *is an almost Hamilton vector field of* ω.

Conversely, if an almost Hamilton vector field of ω *is an infinitesimal automorphism of the* $D(GL(n,R))$*-structure* $\boldsymbol{P_1}$, *then it is an infinitesimal automorphism of the* $D(O(n))$*-structure* $\boldsymbol{P_2}$.

References

[1] A. Fujimoto, Theory of G-structures, Publ. of the Study Group of Geom. (Japan), Vol. **1** (1972).

[2] Y. Ichijyō, On the Finsler group and an almost symplectic structure on a tangent bundle, J. Math. Kyoto Univ. **28** (1988), 153-163.

[3] Y. Ichijyō, The D(O(n))-structures in tangent bundles, to appear.

[4] A. Kandatsu, Tangent bundle of a manifold with a non-linear connection, Kōdai Math. Sem. Rep. **18** (1966), 259-270.

[5] M. Matsumoto, Foundations of Finsler Geometry and Special Finsler Spaces, Kaiseisha

Press, Otsu, Japan, 1986.
[6] R. Miron, Metrical Finsler structures and metrical Finsler connections, J. Math. Kyoto Univ. **23** (1983), 219-224.
[7] K. Yano, The theory of Lie Derivatives and Its Applications, North-Holland, Amsterdam and New York, 1957.
[8] K. Yano and S. Ishihara, Differential geometry in tangent bundle, Kōdai Math. Sem. Rep. **18** (1966), 271-291.
[9] K. Yano and S. Ishihara, Tangent and Cotangent Bundles, Marcel Dekker, New York, 1973.