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ON MATSUMOTO'S FINSLER SPACE WITH TIME MEASURE

Dedicated to Professor Dr. Makoto Matsumoto on the occasion of his seventieth birthday

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Abstract

On his recent paper [14], M. Matsumoto showed that a slope of a mountain is a Finsler surface with respect to a time measure. Suggested by this result, we discuss a Finsler space with an (α, β)-metric of type $\alpha^2/(\alpha-\beta)$.

1. Matsumoto spaces

A slope of a mountain is represented as the graph S of a differentiable function $x^3 = f(x^1, x^2)$, where (x^1, x^2, x^3) is a rectangular coordinate system in a three-dimensional Euclidean space. We put $y^i = \dot{x}^i$, and $\partial_i = \partial/\partial x^i$. Then a Riemannian metric α is induced on S by

$$(1.1) \qquad \alpha(x,y) = \{(y^1)^2 + (y^2)^2 + (b_1 y^1 + b_2 y^2)^2\}^{1/2},$$

where $x=(x^i)$, $y=(y^i)$, and $b_i = \partial_i f$. We put

$$(1.2) \qquad \beta(x,y) = b_1 y^1 + b_2 y^2.$$

When a man can walk v meters par a minute on a horizontal plane, how many minutes does it take him to walk along a road on S?

Recently, Matsumoto [14] showed that the man will walk in $s = \int_0^t L(x(t), y(t))\, dt$ minutes along a road $x(t)$ on S, by taking L as

$$(1.3) \qquad L = \alpha^2/(v\alpha - w\beta),$$

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where $2w$ is the gravitational constant, and thus a slope of a mountain is regarded as a Finsler surface with such a time measure L.

As was also pointed out by P. Finsler himself in his letter to Matsumoto (cf. [13, 14]), a time measure is thought to be a typical model of a Finsler metric. Moreover, it is noted that (1.3) is an (α, β)-metric. The notion of (α, β)-metric was introduced by Matsumoto [12] and has been studied in detail. As well-known examples, there are a Randers metric $\alpha+\beta$ [20], a Kropina metric α^2/β [9, 10] and a generalized Kropina metric α^{m+1}/β^m [3], whose studies have greatly contributed to the growth of Finsler geometry, so the metric of type (1.3) seems to be interesting as a new example of (α, β)-metrics.

Since $L=\alpha^2/(v\alpha-w\beta)=(\alpha/v)^2/\{(\alpha/v)-(w\beta/v^2)\}$, we shall normalize (1.3) as

$$L=\alpha^2/(\alpha-\beta), \tag{1.4}$$

and taking a general Riemannian metric α and a general non-zero 1-form β on a general differentiable manifold M, we shall define as follows.

Definition 1.1. On an n-dimensional differentiable manifold M, an (α, β)-metric L of type (1.4) is called a *slope metric* or a *Matsumoto metric*, and then a Finsler space (M, L) is called a *Matsumoto space*.

In the present paper dedicated to Prof. Dr. Makoto Matsumoto, treating the above space we shall introduce some of his great achievement in Finsler geometry. In §2 and §3 sequent to this introductory §1 we shall give respective conditions that a Matsumoto space be conformally flat (Theorem 2.2) and be projectively flat (Theorem 3.1). In §4 we shall treat the case of two dimensions and give a condition that a Matsumoto space be a Landsberg space (Theorm 4.3). In order to obtain the condition, we shall reform the expression given in [3] for the derivative I_s of the main scalar I, and give a condition that a Finsler space with general (α, β)-metric be a Landsberg space, in a more convenient form (Theorem 4.1).

A Matsumoto space may be thought to have an intermediate position between a Randers space and a Kropina space. But, the conditions obtained in Theorem 2.2 and Theorem 4.3 are the same as in the case of Randers space (Remark 2.1, Remark 4.1), while the one in Theorem 3.1 is much stronger than in each case of Randers space and Kropina space (Remark 3.1).

The terminology and notation are referred to Matsumoto [13], and also to Ichijyō-Hashiguchi [7] (in §2), Matsumoto [17] (in §3), and Hashiguchi-Hōjō-Matsumoto [3] (in §4, where A_0, A_0^* and b_{12} are modified).

Throughout the present paper we shall effectively use the following Propositions.

Proposition 1.1. *The derivatives of Matsumoto metric L with respect to α and β are given by*

$$(1.5)\qquad \alpha(\alpha-\beta)L_\alpha=(\alpha-2\beta)L,\ (\alpha-\beta)L_\beta=L,$$

$$(1.6)\qquad (\alpha-\beta)^2L_{\beta\beta}=2L,\ (\alpha-\beta)^3L_{\beta\beta\beta}=6L,$$

where $L_\alpha=\partial L/\partial\alpha$, $L_\beta=\partial L/\partial\beta$, $L_{\beta\beta}=\partial L_\beta/\partial\beta$, $L_{\beta\beta\beta}=\partial L_{\beta\beta}/\partial\beta$.

Proposition 1.2. *Let $P(x,y)$, $Q(x,y)$, $R(x,y)$ be functions of x^i and y^i satisfying $PR+Q=0$. If P and Q are rational functions with respect to y^i, and R is an irrational function with respect to y^i, such as α and $\alpha-2\beta$, then we have $P=0$, $Q=0$.*

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2. Conformally flat Matsumoto spaces

A Finsler space (M, L) is called *conformally flat* if for any point p of M there exist a local coordinate neighbourhood (U, x) of p and a differentiable function $\sigma(x)$ on U such that $e^\sigma L$ is locally Minkowski. In order to get a condition that a Matsumoto space be conformally flat, we shall find a condition that a Matsumoto space be locally Minkowski, by Kikuchi's method [8] in the case of Randers space.

In a Matsumoto space (M, L), where $L=\alpha^2/(\alpha-\beta)$, we put

$$(2.1)\qquad \alpha=(a_{ij}(x)y^iy^j)^{1/2},\ \beta=b_i(x)y^i.$$

Let $B\Gamma=(G_j{}^i{}_k, G^i{}_k, 0)$ be the Berwald connection of (M, L) and $\Gamma=(\gamma_j{}^i{}_k)$ the Riemannian connection of the associated Riemannian space (M, α). The h-covariant differentiation with respect to $B\Gamma$ is denoted by ";" and the covariant differentiation with respect to Γ by "∇". Since $B\Gamma$ satisfies $L_{;k}=0$ and $y^i{}_{;k}=0$, we have

$$L_{;k}=\alpha_{;k}L_\alpha+\beta_{;k}L_\beta=(2\alpha^2)^{-1}\{(a_{ij;k}y^iy^j)\,\alpha L_\alpha+2(b_{i;k}y^i)\,\alpha^2L_\beta\}=0,$$

so using (1.5) we have from $(\alpha-\beta)L_{;k}=0$

$$(a_{ij;k}y^iy^j)(\alpha-2\beta)+2(b_{i;k}y^i)\,\alpha^2=0.$$

If (M, L) is a Berwald space, $G_j{}^i{}_k$ are independent of y^i, so $a_{ij;k}y^iy^j$ and $b_{i;k}y^i$ are polynomials of y^i. Thus from Proposition 1.2 we have $a_{ij;k}y^iy^j=0$ and $b_{i;k}y^i=0$, that is, $a_{ij;k}=0$ and $b_{i;k}=0$, which yield $G_j{}^i{}_k=\gamma_j{}^i{}_k$ and $\nabla_kb_i=0$. Then the h-curvature tensor

$K_{h\ jk}^{\ i}$ of $B\Gamma$ coincides with the curvature tensor $R_{h\ jk}^{\ i}$ of Γ. Therefore, if (M, L) is locally Minkowski, then $K_{h\ jk}^{\ i}$ vanishes, so we have $R_{h\ jk}^{\ i}=0$. As was shown in [4], the converse is true for general (α, β)-metrics. Thus we have the same result as Kikuchi's Theorem for a Randers space.

Theorem 2.1. *A Matsumoto space is a Berwald space if and only if* $\nabla_k b_i=0$ *is satisfied. A Matsumoto space is locally Minkowski if and only if* $R_{h\ jk}^{\ i}=0$ *and* $\nabla_k b_i=0$ *are satisfied.*

Recently, Ichijyō-Hashiguchi [7] showed that in a Finsler space with general (α, β)-metric there exists a conformally invariant symmetric linear connection $M\Gamma = (M_{j\ k}^{\ i})$, and gave a condition that a Randers space be conformally flat in terms of $M\Gamma$. We put $(a^{ij}) = (a_{ij})^{-1}$, $b^i=a^{ir}b_r$, and $b= (b_r b^r)^{1/2}$. $M_{j\ k}^{\ i}$ is defined by

$$M_{j\ k}^{\ i}= \gamma_{j\ k}^{\ i}+ \delta^i_j M_k+ \delta^i_k M_j-M^i a_{jk}, \tag{2.2}$$

where $M_j= (1/b^2)\ \{b^r\nabla_r b_j- (\nabla_r b^r)\, b_j\, /\, (n-1)\}$, $M^i=a^{ir}M_r$. We denote by $\overset{m}{\nabla}$ and $M_{h\ jk}^{\ i}$ the covariant differentiation with respect to $M\Gamma$ and the curvature tensor of $M\Gamma$ respectively. Then based on Kikuchi's conditions $R_{h\ jk}^{\ i}=0$ and $\nabla_k b_i=0$ it is shown that a condition that a Randers space be conformally flat is

$$M_{h\ jk}^{\ i}=0,\ \overset{m}{\nabla}_k M_j=\overset{m}{\nabla}_j M_k,\ \overset{m}{\nabla}_k b_j= -M_k b_j. \tag{2.3}$$

In the same way we have from Theorem 2.1

Theorem 2.2. *A Matsumoto space is conformally flat if and only if* (2.3) *is satisfied.*

Remark 2.1. It is remarkable that the condition (2.3) is given in the tensorial form expressed in terms of the given metric itself. Furthermore, the condition (2.3) is sufficient in order that a Finsler space with general (α, β)-metric be conformally flat (cf. [6]), but it is also necessary in Finsler spaces with (α, β)-metric of type that locally Minkowski spaces necessarily satisfy Kikuchi's conditions $R_{h\ jk}^{\ i}=0$ and $\nabla_k b_i=0$.

Recently, Matsumoto [18] called a locally Minkowski space satisfying $R_{h\ jk}^{\ i}=0$, $\nabla_k b_i =0$ *flat-parallel*, and based on his recent research [16] of the Berwald connection of a Finsler space with (α, β)-metric he gave a useful method to verify if a Finsler space with (α, β)-metric be flat-parallel. It is shown there that a locally Minkowski Matsumoto space is flat-parallel and contained in more general examples.

Locally Minkowski spaces constitute a single but quite wide class in Finsler spaces. It is interesting to find a special subclass closely related to the given metric, such as the class of Finsler spaces with flat-parallel (α, β)-metric. As another interesting example the notion of *T-Minkowski space* is discussed in Matsumoto's recent paper [15] in relation to the 1-from metric due to Matsumoto-Shimada [19].

3. Projectively flat Matsumoto spaces

A Finsler space (M, L) is called *projectively flat* or *with rectilinear geodesics* if for any point p of M there exists a local coordinate neighbourhood (U, x) of p in which the geodesics can be represented by n-1 linear equations of x^i. A condition that a Randers space be projectively flat was given by Hashiguchi-Ichijyō [5], where discussions were based on the behavior of the equations of geodesics under the change $\alpha \to \alpha+\beta$, but using a beautiful method developed in Matsumoto's recent paper [17] we shall here find a condition that a Matsumoto space be projectively flat.

In a Finsler space with (α, β)-metric we define further

$$r_{ij}=(\nabla_j b_i+\nabla_i b_j)/2,\ s_{ij}=(\nabla_j b_i-\nabla_i b_j)/2,$$

$$s^i_j=a^{ir}s_{rj},\ s_i=b^r s_{ri},\ \gamma_{jhk}=a_{hr}\gamma_j{}^r{}_k.$$

Then, by [17, Theorem 1] a Finsler space (M, L) with (α, β)-metric is projectively flat if and only if for any point p of M there exists a local coordinate neighbourhood of p in which $\gamma_j{}^i{}_k$ satisfies

$$(3.1)\qquad (\gamma_0{}^i{}_0-\gamma_{000}y^i/\alpha^2)/2+(\alpha L_\beta/L_\alpha)s^i_0 + (L_{\alpha\alpha}/L_\alpha)(C+\alpha r_{00}/2\beta)(\alpha^2 b^i/\beta-y^i)=0,$$

where a subscript 0 means a contraction by y^i and C is given by

$$(3.2)\qquad C+(\alpha^2 L_\beta/\beta L_\alpha)s_0+(\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2-\beta^2)(C+\alpha r_{00}/2\beta)=0.$$

Since $\alpha^2 L_{\alpha\alpha}=\beta^2 L_{\beta\beta}$, the formula (3.2) is rewritten in the form

$$(3.3)\qquad \{1+(L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2-\beta^2)\}(C+\alpha r_{00}/2\beta)=(\alpha/2\beta)\{r_{00}-(2\alpha L_\beta/L_\alpha)s_0\}.$$

Now, let (M, L) be a Matsumoto space. Then (3.3) becomes from Proposition 1.1

$$(3.4)\qquad 2\beta\{(1+2b^2)\alpha-3\beta\}(C+\alpha r_{00}/2\beta)=(\alpha-\beta)\{(\alpha-2\beta)r_{00}-2\alpha^2 s_0\}.$$

Substituting in (3.1) from (3.4) we have from $\alpha^2 L_{\alpha\alpha}=\beta^2 L_{\beta\beta}$ and Proposition 1.1

$$(3.5)\qquad \{(1+2b^2)\alpha-3\beta\}\{(\alpha-2\beta)(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)+2\alpha^4 s^i_0\} +2\alpha\{(\alpha-2\beta)r_{00}-2\alpha^2 s_0\}(\alpha^2 b^i-\beta y^i)=0,$$

which is written in the form $P\alpha+Q=0$, where

$$P=-(5+4b^2)\beta(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)+2(1+2b^2)\alpha^4 s^i{}_0-4(\alpha^2 s_0+\beta r_{00})(\alpha^2 b^i-\beta y^i),$$

$$Q=\{(1+2b^2)\alpha^2+6\beta^2\}(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)-6\alpha^4\beta s^i{}_0+2\alpha^2 r_{00}(\alpha^2 b^i-\beta y^i).$$

Since P and Q are polynomials of y^i, if (3.5) is satisfied, we have $P=0$, $Q=0$ from Proposition 1.2.

First, it follows from $Q=0$ that $\beta^2\gamma_{000}y^i$ has a factor α^2, so we can put

(3.6) $$\gamma_{000}=\nu_0\alpha^2 \quad (\nu_0=\nu_i(x)y^i).$$

Substituting in P from (3.6) it follows from $P=0$ that $\beta^2 r_{00}y^i$ has a factor α^2, so we can put

(3.7) $$r_{00}=\rho(x)\alpha^2.$$

Substituting in Q from (3.6) and (3.7) we have from $Q=0$

(3.8) $$\{(1+2b^2)\alpha^2+6\beta^2\}(\gamma_0{}^i{}_0-\nu_0 y^i)-6\alpha^2\beta s^i{}_0+2\alpha^2\rho(\alpha^2 b^i-\beta y^i)=0,$$

from which it follows that $\beta^2(\gamma_0{}^i{}_0-\nu_0 y^i)$ has a factor α^2. We can put $\gamma_0{}^i{}_0-\nu_0 y^i=\lambda^i(x)\alpha^2$, but contracting this by $y_i=a_{ir}y^r$ we have from (3.6)

(3.9) $$\gamma_0{}^i{}_0=\nu_0 y^i,$$

that is,

(3.10) $$\gamma_j{}^i{}_k=\delta^i{}_j\nu_k+\delta^i{}_k\nu_j,$$

which shows that the associated Riemannian space (M,α) is projectively flat.

Next, from (3.8), (3.9) we have $\rho\alpha^2 b^i=\beta(3s^i{}_0+\rho y^i)$. Since α^2 is positive definite, we have $\rho=0$, and so $s^i{}_0=0$. Hence, we have $r_{ij}=0$, $s_{ij}=0$, from which $\nabla_j b_i=0$ follows.

Conversely, if $\nabla_j b_i=0$, then we have $r_{00}=0$, $s^i{}_0=0$, and $s_0=0$, so (3.5) follows from (3.9) and $\nabla_j b_i=0$. Thus we have proved

Theorem 3.1. *A Matsumoto space (M, L) is projectively flat if and only if the associated Riemannian space (M,α) is projectively flat and $\nabla_j b_i=0$ is satisfied.*

Remark 3.1. A Randers space $(M,\alpha+\beta)$ is projectively flat if and only if the associated Riemannian space (M,α) is projectively flat and $s_{ij}=0$ is satisfied (Hashiguchi-Ichijyō [5], Matsumoto [17]). On the other hand, a Kropina space $(M,$

α^2/β) is projectively flat if and only if for any point of p of M there exists a local coordinate neighbourhood of p in which $\gamma_j{}^i{}_k$ is written in the form

$$\gamma_j{}^i{}_k = \delta^i_j \nu_k + \delta^i_k \nu_j - (s^i a_{jk} + b^i r_{jk})/b^2, \tag{3.11}$$

where $s^i = a^{ir} s_r$, and the condition

$$b^2 s_{ij} = b_i s_j - b_j s_i \tag{3.12}$$

is satisfied ([17]). It is noted that the condition of Theorem 3.1 is stronger than the one for a Randers space and is also stronger than the one for a Kropina space.

4. Two-dimensional Landsberg Matsumoto spaces

A Finsler space (M, L) is called a *Landsberg space* if the second curvature tensor $P_h{}^i{}_{jk}$ of the Cartan connection $C\Gamma$ of (M, L) vanishes. Such a space of two dimensions was first considered by Landsberg [11], in the process of a trial to generalize the Gauss-Bonnet theorem in the surface theory of Gauss to a general two-dimensional variation problem. In this last section, by the method treated in Hashiguchi-Hōjō-Matsumoto [3] we shall find a condition that a two-dimensional Matsumoto space be a Landsberg space.

A condition that a two-dimensional Finsler space (M, L) be a Landsberg space is generally given by $I_s = 0$, where I_s is the derivative of the main scalar I with respect to the are-length s of a geodesic (Berwald [1, 2]). We shall first give a convenient expression for I_s in the case of general (α, β)-metric L. Around any point of M we refer to an isothermal coordinate system, with respect to which α is written in the form

$$\alpha = a(x)\mu, \text{ where } \mu = \{(y^1)^2 + (y^2)^2\}^{1/2}. \tag{4.1}$$

Let z^i *be* $z^1 = y^2$, $z^2 = -y^1$. Putting $\gamma = b_i z^i$, and

$$E = \alpha L_\alpha + \gamma^2 L_{\beta\beta}, \tag{4.2}$$

the main scalar I is given in [3, Proposition 1] by

$$I = -\{3EL_{(y)} + LE_{(y)}\} / \{2(LE^3)^{1/2}\}, \tag{4.3}$$

where $L_{(y)} = \gamma L_\beta$, $E_{(y)} = \gamma E_\beta - \beta E_\gamma$, so we have

$$I = -\gamma\{3(\alpha L_\alpha L_\beta - \beta L L_{\beta\beta}) + \gamma^2(3L_\beta L_{\beta\beta} + L L_{\beta\beta\beta})\} / \{2(LE^3)^{1/2}\}. \tag{4.4}$$

It is noted that I is a (0) p-homogeneous function of α, β and γ, and the relation $\alpha I_\alpha+\beta I_\beta+\gamma I_\gamma=0$ is satisfied.

Denoting by "$,_j$" the differentiation by x^j, we put

$$A=\alpha_{,j}y^j=A_0\alpha, \text{ where } A_0=(a_{,j}/a)y^j,$$

$$A^*=\alpha_{,j}z^j=A^*_0\alpha, \text{ where } A^*_0=(a_{,j}/a)z^j,$$

$$B=\beta_{,j}y^j=b_{i,j}y^iy^j,\ C=\gamma_{,j}y^j=b_{i,j}z^iy^j,\ X=B-A_0\beta,$$

$$b=\{(b_1)^2+(b_2)^2\}^{1/2},\ b_{12}=(b_{1,2}-b_{2,1})/a^2,$$

$$c=b/a,\ C_0=cc_{,j}y^j,$$

where the notations A_0, A^*_0, and b_{12} differ with the definition $A_0=a_{,j}y^j$, $A^*_0=a_{,j}z^j$, $b_{12}=b_{1,2}-b_{2,1}$ in [3]. Then the derivative I_s of I is given in [3, Proposition 4] by

$$(4.5)\qquad (LE\gamma)I_s=\{E(A\gamma-C\alpha)+H\alpha\beta\}I_\alpha+\{E(B\gamma-C\beta)+Hc^2\alpha^2\}I_\beta,$$

where H is given by

$$(4.6)\qquad H=A^*_0\alpha L_\alpha-b_{12}\alpha^2L_\beta-X\gamma L_{\beta\beta}.$$

We shall reform the expression (4.5) for I_s in a more convenient form. It is noted that α, β and γ satisfy

$$(4.7)\qquad \beta^2+\gamma^2=c^2\alpha^2.$$

Differentiating the both sides of (4.7) by x^i and contracting by y^i, we have $B\beta+C\gamma=Ac^2\alpha+C_0\alpha^2$. Multiplying this by α and β respectively, and paying attention to (4.7) we have

$$(4.8)\qquad \gamma(A\gamma-C\alpha)=\alpha(X\beta-C_0\alpha^2),\ \gamma(B\gamma-C\beta)=\alpha^2(c^2X-C_0\beta).$$

Substituting in (4.5) from (4.8) we have

$$(4.9)\qquad (LE\gamma^2/\alpha)I_s=(EX+H\gamma)I_*+C_0E\alpha\gamma I_\gamma,$$

where we put

$$(4.10)\qquad I_*=\beta I_\alpha+c^2\alpha I_\beta.$$

From (4.2) and (4.6) we have $EX+H\gamma = \alpha\ (YL_\alpha - b_{12}\alpha\gamma L_\beta)$, where

$$(4.11) \qquad Y=X+A^*_0\gamma = \{(ab_{i,j}-a_{,j}b_i)y^iy^j+a_{,j}b_iz^iz^j\}/a.$$

Thus we have obtained a new expression for I_s

$$(4.12) \qquad (LE\gamma^2/\alpha^2)I_s=(YL_\alpha - b_{12}\alpha\gamma L_\beta)I_* + C_0E\gamma I_\gamma.$$

Thus we have

Theorem 4.1. *A two-dimensional Finsler space (M, L) with (α, β)-metric is a Landsberg space if and only if, with respect to the referred isothermal coordinate system, the following condition is saisfied:*

$$(4.13) \qquad (YL_\alpha - b_{12}\alpha\gamma L_\beta)I_* + C_0E\gamma I_\gamma = 0.$$

It is noted that (4.13) is satisfied by $Y=0$, $b_{12}=0$, $C_0=0$. But we can show that $C_0=0$ follows from $Y=0$, $b_{12}=0$. In fact, evaluating Y as a formula of y^i, we have from $Y=0$, $b_{12}=0$

$$(4.14) \qquad \begin{cases} a_{,1}b_1 - a_{,2}b_2 = ab_{1,1} = -ab_{2,2}, \\ a_{,1}b_2 + a_{,2}b_1 = ab_{1,2} = ab_{2,1}. \end{cases}$$

If we solve (4.14) with respect to $a_{,1}, a_{,2}$, we have $a_{,j}=ab_{,j}/b$, from which we have $c_{,j}=0$ and so $C_0=0$. Thus we have proved

Poposition 4.1. (4.13) *is satisfied if $Y=0$, $b_{12}=0$.*

From (4.14) it is shown that if $Y=0$, $b_{12}=0$ then besides $C_0=0$ we have $b_{1,1}+b_{2,2}=0$. Conversely, if $C_0=0$, $b_{12}=0$, $b_{1,1}+b_{2,2}=0$ are satisfied, then we have (4.14) and so $Y=0$. Thus we have

Proposition 4.2. $Y=0$, $b_{12}=0$ *is equivalent to* $C_0=0$, $b_{12}=0$, $b_{1,1}+b_{2,2}=0$, *which means locally that c is constant and there exists a differentiable function f satisfying $b_i=\partial_i f$,* $\partial_1\partial_1 f+\partial_2\partial_2 f=0$.

Thus, we have a sufficient condition that a two-dimensional Finsler space with general (α, β)-metric be a Landsberg space.

Theorem 4.2. *A two-dimensional Finsler space* (M, L) with (α, β)*-metric is a Landsberg space if, with respect to the referred isothermal coordinate system* (x^i), b/a *is locally constant, and* b_i *is locally a gradient vector of a harmonic function of* x^i.

Now, let (M, L) be a Matsumoto space. Using Proposition 1.1 we have from (4.4)

$$(4.15)\quad I=-(3\gamma/2)\{\alpha^2-5\alpha\beta+4(\beta^2+\gamma^2)\}/\{\alpha^2-3\alpha\beta+2(\beta^2+\gamma^2)\}^{3/2}.$$

Calculating from (4.15) and using (4.7) we have

$$(4.16)\quad UI_\alpha=\alpha(\alpha-\beta)I_1,\ UI_\beta=\alpha(\alpha-\beta)I_2,\ U\gamma I_\gamma=-2\alpha(\alpha-\beta)^2I_3,$$

where we put

$$U=(4/3\gamma)\{\alpha^2-3\alpha\beta+2(\beta^2+\gamma^2)\}^{5/2},$$

$$I_1=2(1+8c^2)\alpha-15\beta,\ I_2=(1-16c^2)\alpha+12\beta,\ I_3=(1+8c^2)\alpha-6\beta.$$

Then we have

$$(4.17)\qquad UI_*=\alpha(\alpha-\beta)(P_1\alpha\beta+P_2),$$

where $P_1=2(1+14c^2)$, $P_2=c^2(1-16c^2)\alpha^2-15\beta^2$.

Since $(\alpha-\beta)^2E=\alpha\{(1+2c^2)\alpha-3\beta\}L$, if (M, L) is a Landsberg space, we have from (4.13) and (1.5)

$$(4.18)\quad \{Y\alpha-(2Y\beta+b_{12}\alpha^2\gamma)\}(P_1\alpha\beta+P_2)-2C_0\alpha^2(Q_1\alpha\beta+Q_2)=0,$$

where $Q_1=-9(1+4c^2)$, $Q_2=(1+2c^2)(1+8c^2)\alpha^2+18\beta^2$. The condition (4.18) is written in the form $R_1\alpha+R_2=0$, where

$$(4.19)\quad \begin{cases} R_1=YP_2-(2Y\beta+b_{12}\alpha^2\gamma)P_1\beta-2C_0Q_1\alpha^2\beta, \\ R_2=YP_1\alpha^2\beta-(2Y\beta+b_{12}\alpha^2\gamma)P_2-2C_0Q_2\alpha^2. \end{cases}$$

From Proposition 1.2 we have $R_1=0$, $R_2=0$. On the domain $D=\{(x, y);\ c\neq 1/4\}$, we have $Y=0$, because YP_2 has the factor β from $R_1=0$, and has the factor α^2 from $R_2=0$. Then from (4.19) we have

$$\begin{cases} b_{12}P_1\gamma + 2C_0Q_1 = 0, \\ b_{12}P_2\gamma + 2C_0Q_2 = 0. \end{cases} \tag{4.20}$$

Since $P_1Q_2 - P_2Q_1 \neq 0$, we have $b_{12}=0$, $C_0=0$. By the continuity the conditions $Y=0$, $b_{12}=0$, $C_0=0$ are also satisfied on the boundary of D. On the exterior of D we have $P_2 = -15\beta^2$, $C_0=0$. Then from $R_1=0$ we have $(15+2P_1)Y\beta + b_{12}P_1\alpha^2\gamma = 0$. Since $b_{12}P_1\alpha^2\gamma$ has the factor β, we have $b_{12}=0$, which yields $Y=0$.

The converse is true from Proposition 4.1. Thus we have proved

Theorem 4.3. *A two-dimensional Matsumoto space (M, L) is a Landsberg space if and only if, with respect to the referred isothermal coordinate system (x^i), b/a is locally constant, and b_i is locally a gradient vector of a harmonic function of x^i.*

Since $a=b/c$, we can express L of a two-dimensional Landsberg Matsumoto space (M, L) as $L=(b/c)^2\mu^2/\{(b/c)\mu-\beta\}$. Thus by the transformation $x^i \to x^i/c$ of the isothermal coordinates, on a domain where c is constant and b_i is a gradient vector, we have

Theorem 4.4. *A two-dimensional Matsumoto space (M, L) is a Landsberg space if and only if around any point of M there exists a coordinate system (x^i), with respect to which L is written in the form*

$$L=\frac{\{(b_1)^2+(b_2)^2\}\{(y^1)^2+(y^2)^2\}}{\{(b_1)^2+(b_2)^2\}^{1/2}\{(y^1)^2+(y^2)^2\}^{1/2}-c(b_1y^1+b_2y^2)}, \tag{4.21}$$

where c is a constant and b_i is a gradient vector of a harmanic function of x^i.

Remark 4.1. It is noted that Y is staged in (4.13) instead of H in (4.5), and then (4.13) is expressed as a linear equation with respect to Y, b_{12}, C_0, whose vanishment characterizes Landsberg spaces. By Theorem 4.2 Finsler spaces with (α, β)-metric satisfying the condition given in Theorem 4.2 constitute a special class of Landsberg spaces. Landsberg Matsumoto spaces belong to this class together with Landsberg Randers spaces.

References

[1] L. Berwald, Über zweidimensionale allgemeine metrische Räume. I, II. J. Reine Angew. Math. **156** (1927), 191-210, 211-222.

[2] L. Berwald, On Finsler and Cartan geometries. III. Two-dimensional Finsler spaces with rectilinear extremals, Ann. of Math. (2) **42** (1941), 84-112.

[3] M. Hashiguchi, S. Hōjō and M. Matsumoto, On Landsberg spaces of two dimensions with (α, β)-metric, J. Korean Math. Soc. **10** (1973), 17-26.

[4] M. Hashiguchi and Y. Ichijyō, On some special (α, β)-metrics, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) **8** (1975), 39-46.

[5] M. Hashiguchi and Y. Ichijyō, Randers spaces with rectilinear geodesics, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) **13** (1980), 33-40.

[6] Y. Ichijyō and M. Hashiguchi, On conformally flat Randers spaces, Symp. Finsler Geom., Awara, Japan, 1989.

[7] Y. Ichijyō and M. Hashiguchi, On the condition that a Randers space be conformally flat, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) **22** (1989), 7-14.

[8] S. Kikuchi, On the condition that a space with (α, β)-metric be locally Minkowskian, Tensor, N. S. **33** (1979), 242-246.

[9] V. K. Kropina, On projective Finsler spaces with a metric of some special form, Naučn. Doklady Vysš. Skoly, Fiz.-Mat. Nauki **1959**, no. 2 (1960), 38-42 (Russian).

[10] V. K. Kropina, Projective two-dimensional Finsler spaces with special metric, Trudy Sem. Vektor. Tenzor. Anal. **11** (1961), 277-292 (Russian).

[11] G. Landsberg, Über die Krümmung in der Variationsrechnung, Math. Ann. **65** (1908), 313-349.

[12] M. Matsumoto, On C-reducible Finsler spaces, Tensor, N. S. **24** (1972), 29-37.

[13] M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, Kaiseisha Press, Otsu, Japan, 1986.

[14] M. Matsumoto, A slope of a mountain is a Finsler surface with respect to a time measure, J. Math. Kyoto Univ. **29** (1989), 17-25.

[15] M. Matsumoto, Conformal change of Finsler space with 1-form metric, to appear in An. Ştiinţ. Univ. "A1. I. Cuza" Iaşi.

[16] M. Matsumoto, The Berwald connection of a Finsler space with an (α, β)-metric, to appear in Tensor, N. S.

[17] M. Matsumoto, Projectively flat Finsler spaces with (α, β)-metric, to appear in Rep. Math. Phys.

[18] M. Matsumoto, A special class of locally Minkowski spaces with (α, β)-metric and conformally flat Kropina spaces, to appear in Tensor, N. S.

[19] M. Matsumoto and H. Shimada, On Finsler spaces with 1-form metric, Tensor, N. S. **32** (1978), 161-169.

[20] G. Randers, On an asymmetrical metric in the four-space of general relativity, Phys. Rev. (2) **59** (1941), 195-199.