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FINSLER-WEYL STRUCTURES AND CONFORMAL FLATNESS

Dedicated to Professor Dr. Makoto Matsumoto
on the occasion of his seventieth birthday

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Abstract

In the present paper we shall introduce the notion of Finsler-Weyl structure $[L, N, \theta]$ and investigate the conformal flatness of (L, N)-structures. Especially, as an application we shall consider the conformal flatness of Finsler manifolds with (α, β)-metric.

Introduction

The notion of Weyl structure on a differentiable manifold, introduced in Weyl [10] from a physical viewpoint, has also been studied geometrically and various interesting results have been obtained (cf. Folland [1] or Higa [2]).

The purpose of the present paper is to generalize the notion of Weyl structure to the case of (L, N)-structure on a Finsler manifold, and to give a condition that an (L, N)-structure be conformally flat in terms of such a generalized Weyl struture (Theorem 3.1).

As to a Finsler manifold with (α, β)-metric, a condition that it be locally conformal to a locally Minkowski space is known for some distinguished case (cf. Ichijyō-Hashiguchi [6], Matsumoto [9]). In the last section, we shall consider Finsler manifolds with (α, β)-metric, and express the above results in terms of a generalized Weyl structure (Theorem 4.1, Theorem 4.2).

Throughout the present paper, the terminology and notation are referred to Ichijyō [4, 5] and Matsumoto [8].

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1. Weyl structures

First we shall here define a Weyl structure on a differentiable manifold M admit-

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ting a Riemannian metric a and a global 1-form θ as follows, where $a=a_{ij}dx^i\otimes dx^j$ and $\theta=\theta_i dx^i$ on any coordinate neighbourhood $\{U,(x^i)\}$ of M. Let $\sigma(x)$ be an arbitrary function on M. With the Riemannian metric $\tilde{a}=e^{2\sigma}a$ on M we associate a 1-form $\tilde{\theta}$ such that

$$\tilde{\theta}=\theta-d\sigma \tag{1.1}$$

is satisfied. The family $[a,\theta]$ of the pairs $(\tilde{a},\tilde{\theta})$ is called a *Weyl structure* on M.

Now, let $[a,\theta]$ be a Weyl structure on M. Then there exists a unique symmetric linear connection ∇ such that

$$\nabla_k a_{ij}+2\theta_k a_{ij}=0 \tag{1.2}$$

is satisfied. This connection ∇ is called a *Weyl connection* of $[a,\theta]$. The coefficients $\Gamma_j{}^i{}_k(x)$ of ∇ are given by

$$\Gamma_j{}^i{}_k=\left\{{}_j{}^i{}_k\right\}+\delta_j^i\theta_k+\delta_k^i\theta_j-\theta^i a_{jk}, \tag{1.3}$$

where $\left\{{}_j{}^i{}_k\right\}$ are the coefficents of the Riemannian connection $\overset{r}{\nabla}$ of a and $\theta^i=a^{ir}\theta_r$, $(a^{ij})=(a_{ij})^{-1}$. It is clear that ∇ is compatible with $[a,\theta]$.

The curvature tensor field $W_j{}^i{}_{kl}$ of ∇ is given by

$$W_j{}^i{}_{kl}=\overset{r}{R}_j{}^i{}_{kl}+@_{(kl)}\{\delta^i_k B_{jl}+\delta^i_j B_{kl}-a_{jk}B^i_l\}, \tag{1.4}$$

where $\overset{r}{R}_j{}^i{}_{kl}$ is the curvature tensor field of $\overset{r}{\nabla}$ and

$$B_{ij}=\overset{r}{\nabla}_j\theta_i-\theta_i\theta_j+\frac{1}{2}\theta^r\theta_r a_{ij},\quad B^i_j=a^{ir}B_{rj}.$$

Here and in the following the notation $@_{(kl)}$ means the alternative summation with respect to k and l.

Then we have a sufficient condition that a Riemannian manifold (M,a) be conformally flat as follows.

Theorem 1.1. *Let M be a differentiable manifold admitting a Weyl structure $[a,\theta]$. The Riemannin manifold (M,a) is conformally flat if the following conditions are satisfied:*
(1) *The 1-form θ associated with a is closed,*
(2) *The Weyl connection ∇ of $[a,\theta]$ is flat : $W_j{}^i{}_{kl}=0$.*

In fact, if the condition (1) is satisfied, then θ is written as $\theta=d\sigma$ for a function $\sigma(x)$ defined on a suitable neighbourhood U of each point of M. Now we consider the conformal change $a\longrightarrow\tilde{a}=e^{2\sigma}a$ on U. Since we can consider a function σ^* on M such that $\sigma^* = \sigma$ on U, the condition (1.1) leads us to $\tilde{\theta} = 0$ on U. Thus, from (1.4) the curvature tensor field of ∇ is given, on U, by

$$W_j{}^i{}_{kl} = \overset{\tilde{r}}{R}_j{}^i{}_{kl},$$

where $\overset{\tilde{r}}{R}_j{}^i{}_{kl}$ is the curvature tensor field of the Riemannian connection $\overset{\tilde{r}}{\nabla}$ of $\tilde{a}$. So, if ∇ is flat, then $\overset{\tilde{r}}{\nabla}$ is locally flat, that is, the given Riemannian manifold (M, a) is conformally flat.

We shall call a Weyl structure $[a, \theta]$ to be *flat* if it satisfies the conditions (1) and (2) in Theorem 1.1. We shall here give an example of flat Weyl structures.

Example 1.1. Let H^n be the upper-half space of R^n, that is,

$$H^n = \{(x^1, \cdots, x^n) \in R^n;\ x^n > 0\}.$$

We define a Riemannian metric a and a 1-form θ on H^n as

$$a_{ij}(x) = \frac{\delta_{ij}}{(x^n)},\quad \theta_i = \frac{\delta_{in}}{x^n} = \frac{\partial \log(x^n)}{\partial x^i},$$

and further define $\tilde{\theta}_i = \theta_i - \partial\sigma/\partial x^i$ for $\tilde{a} = e^{2\sigma(x)}a$. Since $\tilde{a} = (x^n)^2 a$ is a Euclidian metric and $\tilde{\theta} = 0$, the Weyl structure $[a, \theta]$ is flat.

2. (L, N)-structures

Let L be a Finsler metric on a differentiable manifold M and N a non-linear connection on the tangent bundle TM over M. Then the pair (L, N) is called an (L, N)*-structure* (Ichijyō [4, 5]).

We denote by $\{\tilde{U}, (x^i, y^i)\}$ the canonical coordinate system of TM induced from a coordinate system $\{U, (x^i)\}$ of M, where $\tilde{U} = \pi_T{}^{-1}(U)$ (π_T: the projection of TM). For the vertical distribution on TM, we take the local basis $Y = \{Y_i\}$ defined by

$$Y_i = \frac{\partial}{\partial y^i}\ (i = 1, \cdots, n).$$

Then, for the horizontal distribution determined by N, we can take the local basis $X = \{X_i\}$ defined by

$$X_i = \frac{\partial}{\partial x^i} - N^m{}_i \frac{\partial}{\partial y^m}\ (i = 1, \cdots, n),$$

where $N^i{}_j(x, y)$ are the coefficients of N.

If a non-linear connection N is given on TM, then we know that TM admits a $D(GL(n, R))$-structure $\boldsymbol{P}_1$ as a reduction of the standard almost tangent structure

(Ichijyō [3]). We shall call a *Finsler connection* a $D(GL(n, R))$-connection ∇ of $\boldsymbol{P}_1$. A Finsler connection ∇ satisfies

$$\nabla_{X_j} X_i = F_i{}^k{}_j X_k,\ \nabla_{X_j} Y_i = F_i{}^k{}_j Y_k,$$

$$\nabla_{Y_j} X_i = C_i{}^k{}_j X_k,\ \nabla_{Y_j} Y_i = C_i{}^k{}_j Y_k,$$

where $N^i{}_j$, $F_j{}^i{}_k$, $C_j{}^i{}_k$ are called the *coefficients* of ∇.

If an (L, N)-structure is given on TM, then a Riemannian metric G on TM is defined by

$$G = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{ij} \end{pmatrix}$$

with respect to $\{X, Y\}$, where $g_{ij} = (Y_i Y_j L^2)/2$. In the following we shall denote by ∇_k briefly the covariant derivation with respect to X_k. Then we have

Proposition 2.1. *For a given (L, N)-structure, there exists a unique Finsler connection $\nabla = (N^i{}_j, F_j{}^i{}_k, C_j{}^i{}_k)$ satisfying the following conditions:*

(1) ∇ *is h-metrical, that is,* $\nabla_k g_{ij} = 0$,

(2) $F_j{}^i{}_k = F_k{}^i{}_j$,

(3) $C_j{}^i{}_k = 0$.

In fact, from (1) and (2) we get

$$F_j{}^i{}_k = g^{ir}(X_j g_{rk} + X_k g_{jr} - X_r g_{jk})/2,$$

where $(g^{ij}) = (g_{ij})^{-1}$. We call this Finsler connection the *Rund-type connection* of (L, N) and denote by $\overset{R}{\nabla}$.

Remark 2.1. If an (L, N)-structure is given on TM, it is noted that the $D(GL(n, R))$-structure $\boldsymbol{P}_1$ is reduced to a $D(O(n))$-structure $\boldsymbol{P}_2$ (Ichijyō [3]). The Rund-type connection $\overset{R}{\nabla}$ of an (L, N)-structure is a $D(GL(n, R))$-connection of $\boldsymbol{P}_1$, but it is not a $D(O(n))$-connection of $\boldsymbol{P}_2$. Thus, $\overset{R}{\nabla}$ is not the (L, N)-connection in Ichijyō [5].

With respect to $\overset{R}{\nabla}$, we have two surviving curvature tensor fields $R_j{}^i{}_{kl}$, $P_j{}^i{}_{kl}$:

$$R_j{}^i{}_{kl} = @_{kl}\{X_l F_j{}^i{}_k + F_j{}^m{}_k F_m{}^i{}_l\},\ P_j{}^i{}_{kl} = Y_l F_j{}^i{}_k.$$

If the all curvature tensor fields of a Finsler connection ∇ vanish identically, we say that ∇ is of *zero-curvature.*

An (L, N)-structure is said to be *flat* if for any point p of M there exists a local coordinate neighbourhood $\{U, (x^i)\}$ of p such that the condition

$$X_i g_{jk}=0$$

is satisfied on $\tilde{U}=\pi_T^{-1}(U)$. Then we have easily

Proposition 2.2. *An (L, N)-structure is flat if and only if the Rund-type connection $\overset{R}{\nabla}$ of (L, N) is of zero-curvature.*

Also the following theorem is obtained in Ichijyō [5].

Theorem 2.1. *An (L, N)-structure is flat if and only if its Finsler metric L is locally Minkowski and the following equation is satisfied:*

$$(2.1) \qquad (Y_m g_{jk}) P^m{}_{ir} y^r=0,$$

where we put $P^i{}_{jk}=Y_k N^i{}_j - F_k{}^i{}_j$ for $\overset{R}{\nabla}$.

3. Finsler-Weyl structures and conformal flatness

In this section, we shall generalize the notion of Weyl structure to the case of (L, N)-structure on a Finsler manifold (M, L), and from the standpoint we shall investigate the conformal flatness of (L, N)-structures. We shall define

Definition 3.1. Let an (L, N)-structure (L, N) and a global 1-form $\theta=\theta_i(x, y)\,dx^i$ be given on TM. Let $\sigma(x)$ be an arbitrary function on TM, depending on (x^i) alone. With the Finsler metric $\tilde{L}=e^{\sigma(x)}L$ and the non-linear connection $\tilde{N}=N$, we associate a 1-form $\tilde{\theta}=\tilde{\theta}_i(x, y)\,dx^i$ on TM such that

$$(3.1) \qquad \tilde{\theta}=\theta-d\sigma$$

is satisfied. The family $[L, N, \theta]$ of the triads $(\tilde{L}, \tilde{N}, \tilde{\theta})$ is called a *Finsler-Weyl structure* on TM.

Then we have

Proposition 3.1. *Let $[L, N, \theta]$ be a Finsler-Weyl structure on TM. Then there exists a unique Finsler connection $\overset{W}{\nabla}=(N^i{}_j, W_j{}^i{}_k, C_j{}^i{}_k)$ such that*

$$\text{(i)}\ \overset{W}{\nabla}_k g_{ij}+2\theta_k g_{ij}=0,\ \text{(ii)}\ W_j{}^i{}_k=W_k{}^i{}_j,\ \text{(iii)}\ C_j{}^i{}_k=0$$

are satisfied.

In fact, the condition (i) is equivalent to

$$X_k g_{ij} - g_{ir} W_j{}^r{}_k - g_{rj} W_i{}^r{}_k + 2\,\theta_k g_{ij} = 0,$$

from which and the condition (ii) we have

$$W_j{}^i{}_k = F_j{}^i{}_k + \delta^i{}_j\,\theta_k + \delta^i{}_k\,\theta_j - \theta^i g_{jk}, \tag{3.2}$$

where $F_j{}^i{}_k$ are the coefficents of the Rund-type connection $\overset{R}{\nabla}$ of (L, N) and $\theta^i = g^{ir}\,\theta_r$. It is clear that $\overset{W}{\nabla}$ is compatible with $[L, N, \theta]$.

We shall call this connection $\overset{W}{\nabla}$ the *Finsler-Weyl connection* of $[L, N, \theta]$. The surviving curvature tensor fields of $\overset{W}{\nabla}$ are given by

$$\text{(3.3)}\qquad \begin{aligned} K_j{}^i{}_{kl} &= R_j{}^i{}_{kl} + @_{kl}\{\delta^i{}_k B_{jl} + \delta^i{}_j B_{kl} - g_{jk} B^i{}_l\}, \\ F_j{}^i{}_{kl} &= P_j{}^i{}_{kl} + Y_l(\delta^i{}_j\,\theta_k + \delta^i{}_k\,\theta_{kj} - \theta^i g_{jk}), \end{aligned}$$

where $R_j{}^i{}_{kl}$, $P_j{}^i{}_{kl}$ are the curvature tensor fields of the Rund-type connection $\overset{R}{\nabla}$ of (L, N), and

$$B_{ij} = \overset{R}{\nabla}_j\,\theta_i - \theta_i\,\theta_j + \frac{1}{2}\,\theta^r\,\theta_r g_{ij},\ B^i{}_j = g^{ir} B_{rj}.$$

Now we shall define

Definition 3.2. An (L, N)-structure is said to be *conformally flat* if, for any point p of M there exists a local coordinate nighbourhood $\{U, (x^i)\}$ of p and a function $\sigma(x)$ on U such that the structure $(\tilde{L}, \tilde{N})$ is flat, where $\tilde{L} = e^{\sigma(x)} L$ and $\tilde{N} = N$.

Now we shall characterize the conformal flatness of an (L, N)-structure, where L is a non-Riemannian. For a conformal change $L \longrightarrow \tilde{L} = e^{\sigma} L$, the tensor field $(Y_j g_{im}) P^m{}_{kr} y^r$ occured in Theorem 2.1 is changed as follows.

$$(Y_j \tilde{g}_{im})\,\tilde{P}^m{}_{kr} y^r = e^{2\sigma}\,(Y_j g_{im})\,(P^m{}_{kr} - \sigma_r\,\delta^m{}_k - \sigma_k\,\delta^m{}_r + \sigma^m g_{kr}) y^r,$$

where we put $\sigma_i = \partial\sigma / \partial x^i$ and $\sigma^i = g^{ir}\sigma_r$. Moreover, if we put $C_m = g^{ij}(Y_j g_{im})\ /2$ and $C^k = g^{kr} C_r$, we have

$$\tilde{C}^k \tilde{C}_m \tilde{P}^m{}_{kr} y^r = e^{-2\sigma}\,(C^k C_m P^m{}_{kr} y^r - \sigma_r\, y^r C^k C_k).$$

So, if we put $B=C_mP^m{}_{rs}C^ry^s/C^2$ and $C^2=C_mC^m$, the 1-form $\theta=\theta_idx^i$ defined by

$$\theta_i(x,y)=Y_iB \tag{3.4}$$

satisfies (3.1) (cf. Ichijyō [5]). Thus the given (L, N) and the 1-form θ define a Finsler-Weyl structure $[L, N, \theta]$ on TM. Then we have

Theorem 3.1. *Let L be a non-Riemannian Finsler metric on M and N a non-linear connection on TM. With respect to the Finsler-Weyl structure $[L, N, \theta]$ defined by (3.4) for (L, N), the (L, N)-structure is conformally flat if and only if the following conditions are satisfied:*

(1) *θ is reduced to a closed 1-form on M,*

(2) *The Finsler-Weyl connection $\overset{W}{\nabla}$ of $[L, N, \theta]$ is of zero-curvature.*

Proof. We suppose that the (L, N)-structure is conformally flat. By definition 3.2, for each point of M there exists a neighbourhood U and a function $\sigma(x)$ on U such that $(\tilde{L}, \tilde{N})$ is flat. Then we have $\tilde{B}=0$ from Theorem 2.1, that is, $\tilde{\theta}=0$, and hence θ is closed. Moreover, from Proposition 2.2, we see that $\tilde{R}_j{}^i{}_{kl}=0$, $\tilde{P}_j{}^i{}_{kl}=0$ and $\tilde{B}_{ij}=0$ on U. Hence, by (3.3) we see that $\overset{W}{\nabla}$ is of zero-curvature.

Conversely, if the conditions (1) and (2) is satisfied we can show that (L, N) is conformally flat in the same way as in Theorem 1.1.

Q. E. D.

We say a Finsler-Weyl structure to be *flat* if it satisfies the conditions (1) and (2) in Theorem 3.1.

4. Applications to Finsler spaces with (α, β)-metric

Let $L(\alpha, \beta)$ be an (α, β)-metric, that is, $L(\alpha, \beta)$ be a (1)p-homogeneous function of two variables

$$\alpha(x,y)=\{a_{ij}(x)y^iy^j\}^{1/2},\quad \beta(x,y)=b_i(x)y^i,$$

where a_{ij} is a Riemannian metric and $b_i(x)$ is a covariant vector field on M (Matsumoto [8]). As to a condition that a Finsler manifold be locally conformal to a locally Minkowski space, we know some results in Ichijyō-Hashiguchi [6] and Matsumoto [9]. In this section we shall express these results in terms of a Finsler-Weyl structure.

For any (α, β)-metric $L(\alpha, \beta)$, where $\beta\neq 0$, we define a 1-form $\theta=\theta_i(x)dx^i$ by

$$\theta_k=\frac{1}{\|b\|^2}\left(b^m\overset{r}{\nabla}_mb_k-\frac{\overset{r}{\nabla}_mb^m}{n-1}b_k\right),\quad \|b\|^2=a^{ij}b_ib_j. \tag{4.1}$$

Given an (α, β)-metric $L(\alpha, \beta)$, it is shown that $[a, \theta]$ is a Weyl structure (cf. Ichijyō-Hashiguchi [6]). The coefficients $\Gamma_j{}^i{}_k(x)$ of the Weyl connection ∇ of $[a, \theta]$ are given by (1.3) for θ_i defined by (4.1).

If we put

$$N^i_j = y^k \Gamma_k{}^i{}_j, \tag{4.2}$$

then N^i_j give a non-linear connection N on TM, independent on the choice of $(\tilde{a}, \tilde{\theta}) \in [a, \theta]$. For the (L, N)-structure such that the non-linear connection N is given by (4.2), we can define a Finsler Weyl structure $[L, N, \theta]$ by (4.1). We shall call $[L, N, \theta]$ the *induced Finsler-Weyl structure.*

Now we shall seek the condition that the connection $\nabla = (N^i_j, \Gamma_j{}^i{}_k, 0)$ given by (1.3) and (4.2) be the Finsler-Weyl connection of $[L, N, \theta]$. In this case, the h-covariant derivation ∇_k with respect to ∇ and the usual derivation by Y_i are commutable. Then we have

$$\nabla_k g_{ij} = Y_i Y_j (L \nabla_k L) = Y_i Y_j \left(L \frac{\partial L}{\partial \alpha} \nabla_k \alpha + \frac{\partial L}{\partial \beta} \nabla_k \beta\right).$$

Since $\Gamma_j{}^i{}_k(x)$ is the Weyl connection of $[a, \theta]$, we have $\nabla_k \alpha = -\theta_k \alpha$. So, if we assume

$$\nabla_k b_i = -\theta_k b_i, \tag{4.3}$$

that is, $\nabla_k \beta = -\theta_k \beta$, we have

$$\begin{aligned} \nabla_k g_{ij} &= Y_i Y_j \left(L\left(\frac{\partial L}{\partial \alpha}(-\theta_k \alpha) + \frac{\partial L}{\partial \beta}(-\theta_k \beta)\right)\right) \\ &= Y_i Y_j (-\theta_k L^2) = -2\theta_k g_{ij}. \end{aligned}$$

Thus ∇ is the Finsler-Weyl connection of $[L, N, \theta]$. The converse is also true, and we have

Proposition 4.1. *The Finsler connection* $(N^i_j, \Gamma_j{}^i{}_k, 0)$ *given by* (1.3) *and* (4.2) *is the Finsler-Weyl connection of* $[L, N, \theta]$ *if and only if the condition* (4.3) *is satisfied.*

Thus, in the same way as in Theorem 1.1 we have

Proposition 4.2. *In a Finsler manifold* (M, L) *with* (α, β)*-metric such that* $\|b\|^2 \neq 0$, *if the condition* (4.3) *is satisfied and the induced Finsler-Weyl structure* $[L, N, \theta]$ *is flat, then the* (L, N)*-structure is conformally flat.*

In genaral, the converse of Proposition 4.2 is not true. In the case of Randers metric $L=\alpha+\beta$, however, by the well-known theorem of Kikuchi [7], we have the following theorem due to Ichijyō-Hashiguchi [6].

Theorem 4.1. *A Randers space* (M, L) *is locally conformal to a locally Minkowski space if and only if the condition* (4.3) *is satisfied and the induced Finsler-Weyl structure* $[L, N, \theta]$ *is flat.*

In Matsumoto [9], a Finsler manifold (M, L) with (α, β)-metric such that Kikuchi's theorem holds good is said to be *flat-parallel*. Theorem 4.1 is generalized as follows.

Theorem 4.2. *A Finsler manifold* (M, L) *with flat-parallel* (α, β)*-metric is locally conformal to a locally Minkowski space if and only if the condition* (4.3) *is satisfied and the induced Finsler-Weyl structure* $[L, N, \theta]$ *is flat.*

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