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DIFFERENTIAL GEOMETRY OF FINSLER VECTOR BUNDLES

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Abstracts

The present paper is a comprehensive report on Finsler geometry. In this paper, we shall discuss on some topics in the differential geometry of Finsler vector bundles according to Aikou [4].
Key Words: Finsler vector bundles, Finsler connections, Finsler-Weyl stuctures, G-structures, infinitesimal automorphisms.

Introduction

The theory of Finsler spaces and generalized Finsler spaces have been studied by many authors, and various important results have been obtained.

In Matsumoto [49-52], the Finsler connection is defined as a connection of so-called ***Finsler bundle*** and the geometry of Finsler spaces is developed by using it.

On the other hand, any Finsler metric or generalized Finsler metric is naturally lifted to its tangent bundle TM by using an arbitary non-linear connection on TM, and the lifted metric G becomes a Riemannian metric on TM. From this point of view, in Miron [53-56], a special linear connection on TM satisfying some conditions is introduced as a Finsler connection and the differential geometry on $\{TM, G\}$ is developed with respect to this connection.

Also, in Ichijyō [31-36], the geometry of tangent bundles over generalized Finsler spaces is developed from the standpoint of G-structure on TM, and various important and interesting notions are introduced and fruitful results are obtained. The connection on TTM treated in Ichijyō's or Miron's theory is a special linear connection so-called ***linear connection of Finsler type*** in Matsumoto [51], whose torsion are surviving.

In any point of view, it may be regarded as the geometry of some special vector bundles over TM, and many of recent researches in this field are considered as the geometry from this point of view (cf. Aikou [1-4], Aikou-Hashiguchi [5-9], Aikou-Hashiguchi-Yamauchi [10], Aikou-Ichijyō [11], Akbar-Zadeh [12], Anastasiei [13, 14], Asanov [15],

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Atanasiu-Hashiguchi-Miron [16], Atanasiu-Klepp [17], Cartan [20], Hashiguchi [26-29], Klepp [43], Nagano-Aikou [60], Oproiu [61], Rund [62], etc.).

The main purpose of the present paper is to study the geometry of Finsler vector bundles, which are defined in the first section as some special vector bundles over TM, and state some applications of it to conformal flatness of Finsler structures and infinitesimal automorphisms of some G-structures on TM.

The notion of Weyl structures on Riemannian manifolds has been studied by many authors (cf. Folland [23], Higa [30]). As the analogy of it, we shall introduce the notion of ***Finsler-Weyl structures***, and characterize the conformal flatness of non-Riemannian Finsler structures in terms of it. Also we shall introduce a decomposition of TTM into the Whitney-sum of Finsler vector bundles, which induce a natural G-structure on TM. Then we shall consider infinitesimal automorphisms of some G-structures which are obtained as the reductions of it, and characterlize them in terms of Finsler connection. Furthermore, in the last section, we shall consider the Lie algebras of them and state some relations between the Lie algebras. The proofs of almost results are omitted.

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1. Finsler vector bundles and Finsler connections

First in the present paper, we shall review the theory of connections on Finsler vector bundles, and state some basic notions in Finsler geometry. As to the general theory of connections, we refer to Kobayshi [45, 46], Kobayashi-Nomizu [47].

Let M be a differential manifold and $\pi: TM \to M$ its tangent bundle. For an arbitrary vector bundle $\boldsymbol{E}$ over M, the pull-back $\pi^*\boldsymbol{E}$ is determined uniquely up to the isomorphic classes. Then we shall define as follows:

Definition 1.1. A vector bundle $\boldsymbol{F}$ over TM is said to be a ***Finsler vector bundle*** if it is isomorphic to the pull-back $\pi^*\boldsymbol{E}$ of a vector bundle $\boldsymbol{E}$ over M, that is, $\boldsymbol{F} \cong \pi^*\boldsymbol{E}$.

We denote by $\{g_{UV}\}$ the transition functions of $\boldsymbol{E}$ with respect to an open cover $\{U, s_U\}$ with local frame fields $s_U = (s_1, \cdots, s_n)$. We may consider any local frame field s_U of $\boldsymbol{E}$ as the one of $\boldsymbol{F}$ on $\pi^{-1}(U)$. Then the transition functions of $\boldsymbol{F}$ with respect to s_U are given by $\{g_{UV} \circ \pi\}$.

Definition 1.2. A connection D on a Finsler vector bundle $\boldsymbol{F}$ is called a ***Finsler connection*** on E.

We denote by $\Gamma(\boldsymbol{F})$ the space of smooth sections of $\boldsymbol{F}$. By definition, a Finsler connection D on $\boldsymbol{F}$ is a linear mapping $D: \Gamma(\boldsymbol{F}) \to \Gamma(\boldsymbol{F} \otimes TTM^*)$ satisfying the Leipnitz rule:

$$D(f\xi) = df \otimes \xi + f(D\xi) \tag{1.1}$$

for any function f on TM and for any $\xi\in\Gamma(\boldsymbol{F})$. The connection form $\omega_U=(\omega^\alpha_\beta)$ with respect to s_U is defined by $Ds_\alpha=s_\beta\omega^\beta_\alpha$. Here and in the following, we use the Einstein summation. Then, for any section $\xi\in\Gamma(\boldsymbol{F})$, the covariant derivative $D\xi$ is given by

$$D\xi=(d\xi^\alpha+\xi^\beta\omega^\alpha_\beta)\otimes s_\alpha.$$

From (1.1), the connection form $\omega_U=(\omega^\alpha_\beta)$ of D satisfies the following transformation law:

$$\omega_V=g_{UV}{}^{-1}dg_{UV}+g_{UV}{}^{-1}\omega_U g_{UV} \tag{1.2}$$

Where $\{g_{UV}\}$ is the transition functions of $\boldsymbol{E}$.

Conversely, a Finsler connection D on $\boldsymbol{F}$ is defined by the family $\{\omega_U\}$ of local 1-forms $\omega_U=(\omega^\alpha_\beta)$ on TM satisfying (1.2).

The covariant derivation D can be extended to the exterior covariant differential $D:\Gamma(F\otimes\wedge^k TTM^*)\to\Gamma(F\otimes\wedge^{k+1}TTM^*)$ by

$$D(\xi\otimes\varphi)=(D\xi)\wedge\varphi+\xi\otimes d\varphi$$

for $\xi\in\Gamma(\boldsymbol{F})$ and $\varphi\in\Gamma(\wedge^k TTM^*)$. Then the curvature form $\Omega_U=(\Omega^\alpha_\beta)$ with respect to s_U, which is a $\boldsymbol{End}(\boldsymbol{F})$-valued 2-form on TM, is defined by $D^2 s_\alpha=s_\beta\Omega^\beta_\alpha$, and given by

$$\Omega^\alpha_\beta=d\omega^\alpha_\beta-\omega^\sigma_\beta\wedge\omega^\alpha_\sigma. \tag{1.3}$$

From (1.2), the curvature form satisfies the following transformation law:

$$\Omega_V=g_{UV}{}^{-1}\Omega_U g_{UV}.$$

If the curvature form Ω_U of D vanishes identically, we say that D is *flat*.

A *flat structure* in $\boldsymbol{F}$ is given by an open cover $\{\pi^{-1}(U),\ s_U\}$ such that the all transition functions $\{g_{UV}\circ\pi\}$ with respect to s_U are constant matrixes. A Finsler vector bundle $\boldsymbol{F}$ with a flat structure is said to be *flat*. Then the following propositions are obvious.

Proposition1.1. *A Finsler vector bundle* $\boldsymbol{F}$ *is flat if and only if it admits a flat Finsler connection* D.

Proposition1.2. *A Finsler vector bundle* $\boldsymbol{F}=\pi^*\boldsymbol{E}$ *is flat if and only if the vector bundle* $\boldsymbol{E}$ *over* M *is flat.*

Definition 1.3. *A Finsler structure* g on $\boldsymbol{F}$ is a smooth field of inner products in the fibres of $\boldsymbol{F}$, that is, it satisfies the following conditions:

(1) g is a positive-definite and bi-linear form on each fibres,

(2) $g(\xi,\ \eta)$ is differentiable function on TM for arbitrary $\xi,\ \eta\in\Gamma(\boldsymbol{F})$.

Let $(\boldsymbol{F},\ g)$ be a Finsler vector bundle with a Finsler structure g. A Finsler connection D on $(\boldsymbol{F},\ g)$ is said to be *metrical* or *compatible* with g if it satisfies

$$dg(\xi,\ \eta)=g(D\xi,\ \eta)+g(\xi,\ D\eta) \tag{1.4}$$

for arbitrary $\xi,\ \eta\in\Gamma(\boldsymbol{F})$.

Remark 1.1 (1) The principal bundle FM over TM associated to $\boldsymbol{F}$ is called a ***Finsler bundle.*** As to the geometry of FM, we refere to Matsumoto [51].

(2) By our definition, Finsler geometry is considered as the geometry on the pull-back $\pi^*\boldsymbol{E}$ of $\boldsymbol{E}$ to TM. Of course, there are many discussions on Finsler geometry. For example, in Kobayashi [44], for a holomorphic vector bundle over a complex manifold M, the geometry of the pull-back $p^*\boldsymbol{E}$ is considered, where $p:P(\boldsymbol{E})\rightarrow M$ is the projective bundle associated $\boldsymbol{E}$.

2. Non-linear connections

In the present section, we shall state the notion of non-linear connection and give two usefull examples of Finsler vector bundles which give a decomposition of the tangent bundle TTM into a Whitney-sum of them.

Let $\{U,(x^i)\}$ be a coordinate system on M and $\{\pi^{-1}(U),\ (x^i, y^i)\}$ the coordinate system on TM induced from an open cover $\{U, s_U\}$ on TM. For the differential $d\pi$ of the projection π, we put

$$V=Ker\ d\pi=\{\xi\in TTM;\ d\pi(\xi)=0\}.$$

We see that $\boldsymbol{V}$ is a vector sub-bundle of TTM, and that $\{\pi^{-1}(U),\ Y_U\}$ is an open cover of $\boldsymbol{V}$, where we put $Y_U=(Y_1,\cdots,Y_n)$, $Y_i=\partial/\partial y^i$. Then we see that $\boldsymbol{V}$ is a Finsler vector bundle of rank n. We call $\boldsymbol{V}$ the ***vertical sub-bundle*** of TTM. Then there exists a vector sub-bundle $\boldsymbol{H}$ of TTM such that

$$TTM\cong \boldsymbol{H}\oplus \boldsymbol{V}. \tag{2.1}$$

We call $\boldsymbol{H}$, which is uniquely determined up to the isomorphic class, the ***horizontal subbundle*** of TTM. Then we can see easily the following:

Theorem 2.1. *The horizontal and verical bundle* $\boldsymbol{H}$ *and* $\boldsymbol{V}$ *are Finsler vector bundles, and for any Finsler vector bundle* $\boldsymbol{F}$, $\boldsymbol{F}\otimes TTM^*$ *is decomposed into the Whitney-sum of Finsler vector bundles.*

We have an obvious exact sequence of vector bundles:

$$0\longrightarrow V\stackrel{i}{\longrightarrow} TTM\longrightarrow \pi^*TM\longrightarrow 0 \tag{2.2}$$

Since $\boldsymbol{H}\cong\pi^*TM$, a splitting of the exact sequence (2.2) is equivalent to the existence of an isomorphim $TTM\cong \boldsymbol{H}\oplus \boldsymbol{V}$.

Definition 2.1. (cf. Miron-Anastasiei [58]) A ***non-linear connection*** N on TM is a splitting of the exact sequence (2.2).

We see easily that there exist some well-defined local functions $N_j^i(x,y)$ such that the following n-vector field $X_i (1\leq i\leq n)$ consist a local frame field $\boldsymbol{H}$ on $\pi^{-1}(U)$:

$$X_i=\partial/\partial x^i-N_i^m\partial/\partial y^m. \tag{2.3}$$

The functions N_j^i are called the ***coefficients*** of the non-linear connection N on TM, and it is known that if the base manifold M is para-compact, then there always exists a non-linear connection on TM.

Let $\boldsymbol{F}$ be a Finsler vector bundle with a Finsler connection $D:\Gamma(\boldsymbol{F})\longrightarrow\Gamma(\boldsymbol{F}\otimes TTM^*)$. According to the decomposition (2.1), the covariant derivation D is also decomposed as follws:

$$D=D^h+D^v,$$

where $D^h:\Gamma(\boldsymbol{F})\to\Gamma(\boldsymbol{F}\otimes\boldsymbol{H}^*)$, $D^v:\Gamma(\boldsymbol{F})\to\Gamma(\boldsymbol{F}\otimes\boldsymbol{V}^*)$ are called the h-and *v-covariant derivation* respectively.

For the local expressions of Finsler connections in the later, we shall give an open cover $\{\pi^{-1}(U), X_U\}$ of $\boldsymbol{H}$, where $X_U=(X_1,\cdots,X_n)$ is given by (2.3) for the given non-linear connection N. We denote by $\{dx^i, \delta y^i\}$ the dual frame of N-frame $\{X_i, Y_i\}$. Hence $\{dx^i\}$ and $\{\delta y^i\}$ are local frame fields of $\boldsymbol{H}^*$ and $\boldsymbol{V}^*$ respectively, where we put $\delta y^i=dy^i+N^i_m dx^m$. Putting $\omega^\alpha_\beta=F^\alpha_{\beta k}dx^k+C^\alpha_{\beta k}\delta y^k$, the h- and v-covariant derviative $D^h\xi$ and $D^v\xi$ are written as follows:

$$D^h\xi=(X_k\xi^\alpha+\xi^\beta F^\alpha_{\beta k})s_\alpha\otimes dx^k,\ D^v\xi=(Y_k\xi^\alpha+\xi^\beta C^\alpha_{\beta k})s_\alpha\otimes\delta y^k$$

respetively, where we put $\xi=\xi^\alpha s_\alpha$. The triplets $(N^i_j, F^\alpha_{\beta k}, C^\alpha_{\beta k})$ are called the *coefficients* of the Finsler connection D.

3. Finsler structures and Finsler metrics

Let $\boldsymbol{F}$ be a Finsler vector bundle with a Finsler connection D. In the present section, we assume that a fixed non-linear connection N, and so a fixed N-frame $\{X_U, Y_U\}$ is given on TM. We also denote by the same notation D the Finsler connection on any Finsler vector bundle associated to $\boldsymbol{F}$. Since any Finsler structure g on $\boldsymbol{F}$ is considered as a smooth section of $\boldsymbol{F}^*\otimes\boldsymbol{F}^*$, the condition (1.4) is equivalent to $Dg=0$, that is, $D^hg=0$, $D^vg=0$. These conditions are written locally as follows:

$$X_kg_{\alpha\beta}-g_{\delta\beta}F^\delta_{\alpha k}-g_{\alpha\delta}F^\delta_{\beta k}=0,\ Y_kg_{\alpha\beta}-g_{\delta\beta}C^\delta_{\alpha k}-g_{\alpha\delta}C^\delta_{\beta k}=0.$$

If a Finsler connection on $(\boldsymbol{F}, g)$ satisfies $D^hg=0$ (resp. $D^vg=0$), it is said that D is h- (resp. v-) ***metrical***.

In the present section, we shall investigate Finsler spaces from the standpoint of differnitial geometry of Finsler vector bundles. Hence we shall restrict our discussions to the case of $\boldsymbol{F}=\boldsymbol{H}(=\pi^*TM)$. In this case, we sometimes call the Finsler structure on $\boldsymbol{H}$ a ***generalized Finsler metric*** on M.

We assume that a Finsler structure g is given in $\boldsymbol{H}$ and put $g(X_i, X_j)=g_{ij}$ with respect to $X_U=(X_1,\cdots,X_n)$. Then the symmetric matrix (g_{ij}) is positive-definite. Then we introduce two typical Finsler connections on $(\boldsymbol{H}, g)$ which are usefull in the later discussions. In the following, we denote $(DX_i)(X_j)$ by $D_{X_j}X_i$.

We denote by $\omega^i_j=F^i_{jk}dx^k+C^i_{jk}\delta y^k$ the connection form of D. Then, from (1.3), the curvature Ω^i_j form of D is given by

$$\Omega^i_j=\frac{1}{2}R^i_{jkl}dx^k\wedge dx^l+P^i_{jkl}dx^k\wedge\delta y^l+\frac{1}{2}S^i_{jkl}\delta y^k\wedge\delta y^l,$$

where we put

$$\begin{aligned} R^i_{jkl}&=@_{(kl)}\{X_lF^i_{jk}+F^m_{jk}F^i_{ml}\}+C^i_{jm}R^m_{kl},\\ P^i_{jkl}&=Y_lF^i_{jk}-X_kC^i_{jl}-C^m_{jl}F^i_{mk}+C^i_{jm}Y_lN^m_k,\\ S^i_{jkl}&=@_{(kl)}\{Y_lC^i_{jk}+C^m_{jk}C^i_{ml}\}.\end{aligned}\tag{3.1}$$

Here and in the following, the notation $@_{(kl)}$ means the alternative summation with respect to k and l.

First we have

Proposition 1.1. *If a Finsler structure g is given in $\boldsymbol{H}$, then there exists a unique Finsler connection $\overset{M}{D}$ satisfying the following conditions*:

$$(1)\ \overset{M}{D}\text{ is metrical},\qquad (2)\ \overset{M}{D}_{X_j}X_i=\overset{M}{D}_{X_i}X_j,\qquad (3)\ \overset{M}{D}_{Y_j}X_i=\overset{M}{D}_{Y_i}X_j.$$

We call $\overset{M}{D}$ the *Miron-type connection* on $(\boldsymbol{H}, g)$ and also denote by $M\Gamma$. The connection form $\omega^i_j=F^i_{jk}dx^k+C^i_{jk}\delta y^k$ of $M\Gamma$ with respect to $\{X_U\}$ is given by

$$F^i_{jk}=g^{im}(X_jg_{mk}+X_kg_{jm}-X_mg_{jk})/2,\ C^i_{jk}=g^{im}(Y_jg_{mk}+Y_kg_{jm}-Y_mg_{jk})/2\tag{3.2}$$

The curvature form Ω^i_j of $M\Gamma$ with respect to $\{X_U\}$ is given by (3.1) and (3.2).

The following proposition is also easy.

Proposition 1.2. *If a Finsler structure g is given in $\boldsymbol{H}$, then there exists a unique Finsler connection $\overset{R}{D}$ satisfying the following conditions*:

$$(1)\ \overset{R}{D}\text{ is }h\text{-metrical},\qquad (2)\ \overset{R}{D}_{X_j}X_i=\overset{R}{D}_{X_i}X_j,\qquad (3)\ \overset{R}{D}_{Y_j}X_i=\overset{R}{D}_{Y_i}X_j=0.$$

We call $\overset{R}{D}$ the ***Rund-type connection*** and also denote by $R\Gamma$. The connection form $\omega^i_j=F^i_{jk}dx^k+C^i_{jk}\delta y^k$ of $R\Gamma$ with respect to $\{X_U\}$ is given by $F^i_{jk}=$the coefficients F^i_{jk} in (3.2) and $C^i_{jk}=0$. From (3.1) the curvature form Ω^i_j of $R\Gamma$ with respect to $\{X_U\}$ is given by

$$\Omega^i_j=\frac{1}{2}R^i_{jkl}dx^k\wedge dx^l+P^i_{jkl}dx^k\wedge\delta y^l,$$

where we put

$$R^i_{jkl}=@_{(kl)}\{X_lF^i_{jk}+F^m_{jk}F^i_{ml}\},\ P^i_{jkl}=Y_lF^i_{jk}.$$

Because of $dg_{ij}=X_kg_{ij}dx^k+Y_kg_{ij}\delta y^k$, we see that there exists an open cover $\{\pi^{-1}(U),\ X_U\}$ such that $dg_{ij}=0$ on each $\pi^{-1}(U)$ if and only if g is a flat Riemannian metric on M. So we put the following definition.

Definition 3.1. A Finsler structure g on $\boldsymbol{H}$ is said to be *N-flat* if there exists an open cover $\{\pi^{-1}(U),\ X_U\}$ of $\boldsymbol{H}$ such that $d^hg_{ij}=(X_kg_{ij})dx^k=0$ is satisfied on each $\pi^{-1}(U)$.

Then we have

Theorem 3.1. *Let g be a Finsler structure on* $\boldsymbol{H}$. *Then we have*

(1) *The connection* $\overset{M}{D}$ *on* $(\boldsymbol{H}, g)$ *is flat if and only if g is a flat Riemannian metric on M.*

(2) *The connection* $\overset{R}{D}$ *on* $(\boldsymbol{H}, g)$ *is flat if and only if g is a N-flat Finsler structure.*

Remark 3.1. (1) N-flatness depends on the choice of non-linear connection. For the change of non-linear connections, we have some formulas in Nagano-Aikou [61].

(2) Because of Theorem 3.1 and Proposition 1.2, we see that if $\boldsymbol{H}$ admits a N-flat Finsler structure, then the tangent bundle TM is flat, that is, the base manifold M is locally affine.

We assume that a positive function $L(x, y)$ on TM which is smooth on TM-$\{0\}$ and continuous at $y=0$ is given, and furthermore $L(x, y)$ satisfies the following conditions:

(1) $L(x, y)$ is (1)p-homogeneous in y, that is, $L(x, \lambda y)=\lambda L(x, y)$ for any $\lambda>0$,

(2) The following $n\times n$-matrix (g_{ij}) is positive-definite:

$$g_{ij}=(Y_j Y_i L^2)/2. \tag{3.3}$$

The function $L(x, y)$ is called a ***Finsler metric*** or ***fundamental function*** on M and the pair (M, L) is called a ***Finsler space*** (cf. Matsumoto [51]).

If a Finsler metric $L(x, y)$ is given on M, we can always define a Finsler structure g on $\boldsymbol{H}$ by $g(X_i, X_j)=g_{ij}$ for the matrix (g_{ij}) defined by (3.3) and an open cover $\{\pi^{-1}(U), X_U\}$ of $\boldsymbol{H}$.

A Finsler space (M, L) is said to be ***locally Minkowski*** if for each point p of M, there exists a coordinate system $\{U, (x^i)\}$ of p such that on each $\pi^{-1}(U)$ the fundamental function L depends only on y. The following theorem is usefull in the later discussions.

Theorem 3.2 (Ichijyō [36]) *Let g be a Finsler structure on* $\boldsymbol{H}$ *derived by* (3.3) *from a non-Riemannian Finsler metric L. Then g* is *N-flat if and only if* (M, L) *is locally Minkowski and the following condition is satisfied*:

$$(Y_m g_{hi})P^m_{jk}y^k=0 \tag{3.4}$$

where we put $P^m_{ki}=Y_l N^m_k - F^m_{lk}$ *for the coefficients* F^m_{lk} *of* $R\Gamma$.

Remark 3.2. If we take the non-linear connection N as the one defined by Cartan or Berwald (cf. Matsumoto [51]):

$$N^i_j=\partial G^i/\partial y^j,\ G^i=g^{ir}\{(\partial^2 G/\partial y^r\partial x^m)y^m-\partial G/\partial y^r\},\ G=L^2/2, \tag{3.5}$$

then the condition (3.4) is always satisfied.

If we give a non-linear connection N by (3.5), then the Miron (resp. Rund) -type connection of (M, L) is the so-called *Cartan* (resp. *Rund*) *connection* $\overset{C}{D}$ (resp. $\overset{R}{D}$). With respect to these connections, Theorem 3.1 can be written as follows:

Theorem 3.3. *Let g be a Finsler structure on* $\boldsymbol{H}$ *derived from a Finsler metric L* *by* (3.3), *and N the non-linear connection on TM defined by* (3.5). *Then we have*

(1) *The connection* $\overset{C}{D}$ *on* $(\boldsymbol{H}, g)$ *is flat if and only if g is a flat Riemannian metric on M.*

(2) *The connection* $\overset{R}{D}$ *on* $(\boldsymbol{H}, g)$ *is flat if and only if g is locally Minkowski.*

4. Finsler-Weyl structures and conformal flatness

The notion of Weyl structures on a differentiable manifold M was first introduced by Weyl [68] from a physical viewpoint and has been studied by many authors and various interesting results have been obtained (cf. Folland [23], Higa [30], etc.). In the present section, we shall generalize the notion to Finsler geometry and characterlize conformally flat Finsler structures in terms of it (cf. Aikou-Ichijyō [11]).

First we shall review the Weyl structures on M. We assume that a Rimannian metric $a=a_{ij}(x)dx^i\otimes dx^j$ on M and a global 1-form $\theta=\theta_i(x)dx^i$ on M be given, and denote by W the set of all the pairs (a, θ). Then we shall introduce an equivalent relation "$\sim$" in W as follows. For any (a, θ) and $(\hat{a}, \hat{\theta})$ in W, we define as $(a, \theta)\sim(\hat{a}, \hat{\theta})$ if there exists a function $\sigma(x)$ on M satisfying

$$\hat{a}=e^{2\sigma(x)}a, \ \hat{\theta}=\theta-d\sigma.$$

Then we denote by $[a, \theta]$ the equivalent class of $W/\sim$ admitting (a, θ) and call it a *Weyl structure* on M.

Let $[a, \theta]$ be a Weyl stucture on M. Then we see easily that there exists unique symmetric connection D on TM satisfying the following condition:

$$Da=-2\theta\otimes a$$

for an arbitrary representative (a, θ) of $[a, \theta]$. This connection D is called the *Weyl connection* of $[a, \theta]$. The connection form $\omega^i_j=\Gamma^i_{jk}(x)dx^k$ of D with respect to the natural frame $\{\partial/\partial x^i\}$ is given by

$$\Gamma^i_{jk}=\left\{ {}^{\,i}_{jk} \right\}+\delta^i_j\theta_k+\delta^i_k\theta_j-\theta^i a_{jk},$$

where we put $\theta^i=a^{ir}\theta_r$. It is clear that D is independent on the choice of a representative elemet (a, θ). Then the curvature form $\Omega^i_j=\frac{1}{2}W^i_{jkl}(x)dx^k\wedge dx^l$ of D with respect to the natural frame $\{\partial/\partial x^i\}$ is given by

$$W^i_{jkl}=R^i_{jkl}+@_{(kl)}\{\delta^i_k B_{jl}+\delta^i_j B_{kl}-a_{jk}B^i_l\},$$

where R^i_{jkl} is the curvature tensor field of $\left\{ {}^{\,i}_{jk} \right\}$ and we put

$$B_{ij}=\overset{r}{\nabla}_j\theta_i-\theta_i\theta_j+(a_{ij}\theta_r\theta^r)/2, \ B^i_j=a^{ir}B_{rj}.$$

Then we have

Theorem 4.1. *Let M be a differentiable manifold admitting a Weyl structure* $[a, \theta]$. *The Riemannian manifold* (M, a) *is conformally flat if the following conditions are satisfied*:

(1) *The* 1*-form* θ *is closed,*
(2) The *Weyl connection D of* $[a, \theta]$ *is flat.*

Noting that a Riemannian metric a on M is a inner product of the tangent bundle TM and the 1-form θ is a section of TM^*, we shall generalize the notion of Weyl structure to Finsler geometry, and we shall consider the conformal flatness of Finsler structures. From the above discussions, it is natural to consider the case of $\boldsymbol{H}=\pi^*TM$ with a Finsler structure g.

We denote by FW the set of all the pairs of a Finsler structure g on $\boldsymbol{H}$ and a global section θ of $\boldsymbol{H}^*$. Then we introduce an equivalent relation "$\sim$" in FW as follows. For (g, θ), $(\hat{g}, \hat{\theta}) \in FW$, we define as $(g, \theta) \sim (\hat{g}, \hat{\theta})$ if there exists a function $\sigma(x)$ on M such that

$$\hat{g}=e^{2\sigma(x)}a,\ \hat{\theta}=\theta-d\sigma. \tag{4.1}$$

where we consider the 1-form $d\sigma$ as a section of $\boldsymbol{H}^*$. Here we assume that the given nonlinear connection N is invariant by the change (4.1).

Definition 4.1. An equivalent class of $FW/\sim$ is called a *Finsler-Weyl structure* on $\boldsymbol{H}$.

We denote by $[g, \theta]$ the equivalent class of $FW/\sim$ admitting a pair (g, θ). Then we have

Proposition 4.1. *We assume that a Finsler-Weyl structure* $[g, \theta]$ *be given on* $\boldsymbol{H}$. *Then there exists a unique Finsler connection D on* $\boldsymbol{H}$ *which satisfies the following conditions for any representative* (g, θ).

$$(1)\, D^h g=-2\theta\otimes g, \qquad (2)\, D_{X_j}X_i=D_{X_i}X_j \qquad (3)\, D_{Y_j}X_i=D_{Y_i}X_j=0.$$

We see that the above connection D is invariant under the change (4.1), that is, it is independent on the choice of the representative (g, θ). We call it the *Finsler-Weyl connection* of $[g, \theta]$ and denote by $W\Gamma$. The connection form $\omega^i_j=W^i_{jk}(x, y)dx^k$ of $W\Gamma$ with respect to $\{X_U\}$ is given by

$$W^i_{jk}=F^i_{jk}+\delta^i_j\theta_k+\delta^i_k\theta_j-\theta^i g_{jk}$$

for the coeficients F^i_{jk} of $R\Gamma$ and $\theta^i=g^{ir}\theta_r$. The curvature form Ω^i_j of $W\Gamma$ with respect to $\{X_U\}$, which is a $End(\boldsymbol{H})$-valued 2-form on TM, is given by

$$\Omega^i_j=\frac{1}{2}K^i_{jkl}(x, y)dx^k\wedge dx^l+F^i_{jkl}(x, y)dx^k\wedge\delta y^l,$$

where we put

$$K^i_{jkl}=R^i_{jkl}+@_{(kl)}\{\delta^i_k B_{jl}+\delta^i_j B_{kl}-g_{jk}B^i_l\},\ F^i_{jkl}=P^i_{jkl}+Y_l(\delta^i_j\theta_k+\delta^i_k\theta_j-\theta^i g_{jk})$$

for the curvature tensor fields R^i_{jkl} and P^i_{jkl} of the Rund-type connection and

$$B_{ij}=\overset{R}{\nabla}_j\theta_i-\theta_i\theta_j+(g_{ij}\theta_m\theta^m)/2,\ B^i_j=g^{im}B_{mj}$$

for $\overset{R}{\nabla}_j\theta_i=X_j\theta_i-\theta_m F^m_{ij}$.

A Finsler structure g is said to be ***conformally N-flat*** if, for any point p of M, there exists a coordinate system $\{U, (x^i)\}$ of p and a function $\sigma(x)$ on U such that the Finsler structure $\hat{g} = e^{2\sigma(x)}g$ is N-flat. Then, from Theorem 3.1, as a generalization of Theorem 4.1, we have a sufficient condition that a Finsler structure g be conformally N-flat as follows:

Theorem 4.2. *A Finsler structure g on $\boldsymbol{H}$ is conformally N-flat if there exists a Finsler-Weyl structure $[g, \theta]$ satisfying the following conditions*:
(1) *θ is the pull-back of a closed 1-form on M,*
(2) *The Finsler-Weyl connection $W\Gamma$ of $[g, \theta]$ is flat.*

We say a Finsler-Weyl structure $[g, \theta]$ to be ***flat*** if it satifies the conditions (1) and (2) in Theorem 4.2. In the following, we shall characterlize the conformal N-flatness of non-Riemannian Finsler metric $L(x, y)$ in terms of a Finsler-Weyl structure on $\boldsymbol{H}$.

If a Finsler metric $L(x, y)$ is given on M, then a Finsler structure g is defined naturally on $\boldsymbol{H}$. We give an arbitrary non-linear connection N on TM. In the case of non-Riemannian Finsler metric, we constuct a Finsler-Weyl structure $[g, \theta]$ on $\boldsymbol{H}$ as follows.

Let $L(x, y)$ be a non-Riemanninan Finsler metric on M. We introduce the natural Finsler structure g on $\boldsymbol{H}$ by (3.3) from $L(x, y)$. Then the left-hand-side of (3.4) is changed by (4.1) as follows:

$$(Y_j\hat{g}_{im})\hat{P}^m_{kr}y^r = e^{2\sigma}(Y_jg_{im})(P^m_{kr} - \sigma_r\delta^m_k - \sigma_k\delta^m_r + \sigma^m g_{kr})y^r,$$

where we put $\sigma_k = \partial\sigma/\partial x^k$ and $\sigma^i = g^{ir}\sigma_r$. Then we define a global function B on TM by

$$B = C_m P^m_{rs} C^r y^s / C^2,$$

where we put $C_j = g^{rs}Y_jg_{rs}$, $C^i = g^{ir}C_r$ and $C^2 = C_mC^m$. For this function B, the 1-form $\theta = \theta_i(x, y)dx^i$ defined by $\theta_i = Y_iB$ is a global section of $\boldsymbol{H}^*$ and satisfies the condtion (4.1) (cf. Inchijyō [36]). Thus the Finsler structure g and the 1-form θ define a Finsler-Weyl stucture on $\boldsymbol{H}$. Then we have

Theorem 4.3. *Let $L(x, y)$ be a non-Riemannian Finsler metric on M and N a non-linear connectionon TM. With repspect to the Finsler-Weyl structure $[g, \theta]$ defined in the above, the Finsler structure is conformally N-flat if and only if the Finsler-Weyl structure $[g, \theta]$ is flat.*

Example 4.1. (Ichijyō-Hashiguchi [37], Aikou-Ichijyō [11]) We shall show an example of Finsler-Weyl structures. Let $L(x, y) = \alpha(x, y) + \beta(x, y)$ be a Randers space on M, where we put

$$\alpha(x, y) = a_{ij}(x)dx^i \otimes dx^j, \ \beta(x, y) = b_i(x)y^i$$

for a Riemannian metric $a = a_{ij}(x)dx^i \otimes dx_j$ and a 1-form $b = b_i(x)dx^i$ on M. It is known that a manifolld M admits a Randers metric if and only if M admits an $O(n\text{-}1) \times \{1\}$-structure (Ichijyō [32]).

For the Riemannian connection ∇ of (M, a), we define a 1-form $\theta = \theta_i(x)dx^i$ by

$$\theta_k=\frac{1}{\|b\|^2}(b^m\nabla_m b_k-\frac{\nabla_m b^m}{n\text{-}1}b_k),\ \|b\|^2=a^{ij}b_ib_j.$$

Then we see easily that the pair (a, θ) defines a Weyl-structure on M under the condition $\|b\|^2\neq 0$. For the Weyl connection $\Gamma^i_{jk}(x)$ of $[a, \theta]$, we define a non-linear connection N by $N^i_j=\Gamma^i_{jk}(x)y^k$ which is invariant under the change (4.1). If we define a Finsler connection D on $\boldsymbol{H}$ by $\omega^i_j=\pi^*(\Gamma^i_{jk}(x)dx^k)=\Gamma^i_{jk}(x)dx^k$, we see that D is the Finsler-Weyl connection of $[L, \theta]$. Then (M, L) is conformal to a locally Minkowski space if and only if θ is closed and D is flat.

5. G-structures on tangent bundles

In the present section, we state some G-structures on tangent bundles which play an important role in the theory of Finsler spaces. A G-structure on a manifold M is a reduction of the structure group of its linear frame bundle LM, that is, it is a principal bundle whose stucture group is G. First we state the G-structures defined by tensor fields (cf. Fujimoto [24]).

Let $\boldsymbol{V}$ be a finite dimensional vector space and $\rho:G\rightarrow GL(\boldsymbol{V})$ a representation of a linear Lie group G. Then a function T^* on a principal G-bundle P_GM over M is called an ***associated function*** of $(\rho, \boldsymbol{V})$-type if it satisfies the following conditions:

(1) T^* is a function on P_GM which values in $\boldsymbol{V}$,

(2) For any right action R_g of G on P_GM, we have $T^*\circ R_g=\rho(g^{-1})T^*$.

In the case of LM, an n-frame Z in LM is considered as a linear isomorphism $Z:v\in \boldsymbol{R}^n\rightarrow T_xM$. Thus for a tensor field T on M, there exists a unique associated tensor function T^*, and the converse is also true. For example, let $T=T^i_j(x)dx^j\otimes(\partial/\partial x^i)$ be a tensor field of $(1,1)$-type on M, and $\boldsymbol{V}=\mathrm{Hom}(\boldsymbol{R}^n, \boldsymbol{R}^n)$. Then the associated tensor function $T^*:LM\rightarrow \boldsymbol{V}$ is given by

$$T^*(Z)(v, v^*)=T_x(Zv, Z^{-1}v^*),$$

where we put $Zv=Z^i_mv^m(\partial/\partial x^i)$ and $Z^{-1}v^*=v_m(Z^{-1})^m_jdx^j$ for $v=v^ie_i\in\boldsymbol{R}^n$ and $v^*=v_me^m\in(\boldsymbol{R}^n)$. Hence T^* is given by

$$T^*(Z)=((Z^{-1}))^i_lT^l_mZ^m_j)e_i\otimes e^j.$$

Some G-structures on M are defined by tensor fields on M. For a G-structure $\boldsymbol{P}$ defined by a tensor field T, a linear connection D is a g-connection of $\boldsymbol{P}$ if and only if it satisfies $DT=0$ (cf. Fujimoto [24]).

As is well-known, the tangent bundle TM over M admits the standard almost tangent structure $\boldsymbol{P}_0$, and the natural frame is an adapted frame to $\boldsymbol{P}_0$. Furthermore we see that, if a non-linear connection N is given on TM, then TM admits a $D(GL(n, \boldsymbol{R}))$-structure $\boldsymbol{P}_1$ as a reduction of $\boldsymbol{P}_0$, and the N-frame $\{X, Y\}$ is an adapted frame to $\boldsymbol{P}_1$, where we put $D(GL(n, \boldsymbol{R}))=\{\begin{pmatrix} A & O \\ O & A\end{pmatrix};\ A\in GL(n, \boldsymbol{R})\}$. The most important fact is that these G-structures are defined by some tensor fields on TM. In fact, if we define the

(1, 1)-tensor fields Q and P_N on TM by $Q(X_i)=Y_i$, $Q(Y_i)=0$ and $P_N(X_i)=Y_i$, $P_N(Y_i)=-X_i$, then we have

$$\boldsymbol{P}_0=\{Z\in LTM;\ Q^*(Z)=Q_0\},$$

$$\boldsymbol{P}_1=\{Z\in LTM;\ Q^*(Z)=Q_0,\ P_N^*(Z)=P_0\},$$

where we put $Q_0=\begin{pmatrix} O & O \\ I_n & O \end{pmatrix}$, $P_0=\begin{pmatrix} I_n & O \\ O & -I_n \end{pmatrix}$. The G-structure defined by Q is the *standard almost tangent structure* and the tensor field P_N or the G-structure $\boldsymbol{P}_N$ defined by P_N is called the *almost product N-structure*. The structure group of $\boldsymbol{P}_N$ is given by $GL(n,\boldsymbol{R})\times GL(n,\boldsymbol{R})$. Hence we may express as $\boldsymbol{P}_1=\boldsymbol{P}_0\cap\boldsymbol{P}_N$.

Since $D(GL(n,\boldsymbol{R}))\subset GL(n,\boldsymbol{C})$, if a non-linear connection N is given on TM, then TM admits a G-structure $\boldsymbol{F}_N$ called *almost complex N-structure*, which is determined by the (1, 1)-tensor field F_N on TM defined by $F_N(X_i)=Y_i$, $F_N(Y_i)=-X_i$. Then we have

$$\boldsymbol{P}_1=\{Z\in\boldsymbol{P}_0;\ F_N(Z)=F_0\}=\{Z\in LTM;\ P_N^*(Z)=P_0,\ F_N^*(Z)=F_0\},$$

where we put $F_0=\begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix}$. Then we can express as $\boldsymbol{P}_1=\boldsymbol{P}_N\cap\boldsymbol{P}_F=\boldsymbol{P}_0\cap\boldsymbol{P}_F$.

Furthermore, if a Finsler structure g is given on $\boldsymbol{H}$ and $\boldsymbol{V}$, then a Riemannian metric G_N on TM is definefd by

$$G_N=g_{ij}dx^i\otimes dx^j+g_{ij}\delta y^i\otimes\delta y^j,$$

and it defines a $D(O(n))$-structure $\boldsymbol{P}_2$ as the reduction of $\boldsymbol{P}_1$:

$$\boldsymbol{P}_2=\{Z\in\boldsymbol{P}_1;\ G_N^*(Z)=G_0\},$$

where G_0 denotes the identity matrix of rank $2n$, and $D(O(n))=\{\begin{pmatrix} A & O \\ O & A \end{pmatrix};\ A\in O(n)\}$.

On the other hand, as we showed in the second section, if a non-linear connection N is given on TM, the tangent bundle TTM over TM is written in the form (2.1). Thus, if a Finsler connection D is given on $\boldsymbol{H}$ and $\boldsymbol{V}$, then a linear connection D^* on TTM is defined by the 1-form $\omega^*=\begin{pmatrix} \omega & O \\ O & \omega \end{pmatrix}$ with respect to the N-frame $\{X, Y\}$. It is obvious that D^* preserves the sub-bundle $\boldsymbol{H}$ and $\boldsymbol{V}$. Then we see easily that D^* satisfies $D^*Q=0$ and $D^*P_N=0$, that is, D^* is the g-connection of the G-structures $\boldsymbol{P}_1$ (cf. Proposition 5.1). The connection D^* is called a ***linear connection of Finsler-type*** (cf. Matsumoto [51]). Then, because of $\boldsymbol{P}_1=\boldsymbol{P}_0\cap\boldsymbol{P}_N=\boldsymbol{P}_N\cap\boldsymbol{P}_F$, we have

Proposition 5.1. *Let a non-linear connection N be given on TM. A linear connection D^* is a linear connction of Finsler-type if and only if the one of the following conditions is satisfied*:

(1)$D^*Q=0$, $D^*P_N=0$, (2)$D^*Q=0$, $D^*F_N=0$, (3)$D^*P_N=0$, $D^*F_N=0$.

The linear connection of Finsler type derived from the Miron-type connection $M\Gamma$ of g is a g-connection of the $D(O(n))$-structure $\boldsymbol{P}_2$. But, it is not the Riemannian connection of G_N. In fact, the torsion tensor fields of D^* are given as follows which are surviving in general. For any ξ, $\eta\in\Gamma(TTM)$, the torsion form T is defined as a $\Gamma(TTM)$-

valued 2-form T on TM by

$$T(\xi, \eta)=D^*_\xi\eta-D^*_\eta\xi-[\xi, \eta],$$

where we put $D^*_\xi\eta=(D^*\eta)(\xi)$. If we put $T=T^iX_i+T^{(i)}Y_i$, we have the follows:

$$T^i=-\frac{1}{2}T^i_{jk}dx^j\wedge dx^k-C^i_{jk}dx^j\wedge\delta y^k,$$

$$T^{(i)}=-\frac{1}{2}R^i_{jk}dx^j\wedge dx^k-P^i_{jk}dx^j\wedge\delta y^k-\frac{1}{2}S^i_{jk}\delta y^j\wedge\delta y^k,$$

where the five quantities T^i_{jk}, C^i_{jk}, R^i_{jk}, P^i_{jk}, S^i_{jk} are called the ***torsion tensor fields*** of D^* (or (D, N)), and given as follows:

$$T^i_{jk}=F^i_{jk}-F^i_{jk},\ C^i_{jk}=\text{the connection coefficients},$$

$$R^i_{jk}=X_kN^i_j-X_jN^i_k,\ P^i_{jk}=Y_kN^i_j-F^i_{kj},\ S^i_{jk}=C^i_{jk}-C^i_{jk}.$$

6. Infinitesimal automorphisms of some *G*-structures on tangent bundles

In the present section, we shall state some results on the infinitesimal automorphisms of some G-structures introduced in the previous section (cf. Aikou[2]).

Let X be a vector field on M, and $\{f_t\}$ the local 1-parameter group of local transformations f_t generated by X. Then we can consider the natural lift $\{\tilde{f}_t\}$ of $\{f_t\}$ to the frame bundle LM. A vector field X on M is an *infinitesimal automorphism* of a G-structure $\boldsymbol{P}$ on M if for any adapted frame $\{Z\}$ to $\boldsymbol{P}$ the local frame $\{\tilde{f}_t(Z)\}$ is also adapted to $\boldsymbol{P}$. The following proposition is usefull.

Proposition 6.1. *Let $\boldsymbol{P}$ be a G-structure on M defined by a tensor field T. Then a vector field X on M is an infinitesimal automorphism of $\boldsymbol{P}$ if and only if $L_XT=0$.*

In the present section, we also assume that the given non-linear connection N is (1) p-homogeneous in y and satisfies the following condition:

$$Y_jN^i_k=Y_kN^i_j.$$

First we shall state infinitesimal automorphisms of the G-structures $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$ defined in the previous section. The following proposition is fundamental in the present section.

Proposition 6.2. (Duc[22], Ichijyō[34]) *A vector field V on TM is an infinitesimal automorphism of the standard almost tangent structure $\boldsymbol{P}_0$ if and only if it is expressed as $V=A^c+B^v$, where the symbols "c" and "v" mean the complete and vertical lift of vector fields A and B on M respectively.*

From Proposition 6.1 and the relation $\boldsymbol{P}_1=\boldsymbol{P}_0\cap\boldsymbol{P}_N=\boldsymbol{P}_0\cap\boldsymbol{P}_F=\boldsymbol{P}_N\cap\boldsymbol{P}_F$, we have the following characterizations of infinitesimal automorphisms of the G-structure $\boldsymbol{P}_1$.

Theorem 6.1. *On the tangent bundle TM with a non-linear connection N, a* *vector field V on TM is an infinitesimal automorphism of the $D(GL(n, \boldsymbol{R}))$-structure $\boldsymbol{P}_1$ if and only if*

(1) *V is an infinitesmal automorphism of the standard almost tangent structure $\boldsymbol{P}_0$,*

(2) *V is an infinitesmal automorphism of the almost product (resp. almost complex) N-structure $\boldsymbol{P}_N$ (resp. $\boldsymbol{F}_N$),*

Theorem 6.2. *On the tangent bundle TM with a non-linear connection N, a vector field V on TM is an infinitesimal automorphism of the $D(GL(n, \boldsymbol{R}))$-structure $\boldsymbol{P}_1$ if and only if*

(1) *V is an infinitesimal automorphism of the almost product N-structure $\boldsymbol{P}_N$,*

(2) *V is an infinitesimal automorphism of the almost complex N-structure $\boldsymbol{F}_N$,*

Next we shall consider infinitesimal automorphisms of almost product (resp. almost complex) N-structure $\boldsymbol{P}_N$ (resp. $\boldsymbol{F}_N$). From Proposition 6.1, we consider the condition $L_V P_N = 0$ (resp. $L_V F_N = 0$). By direct calculations, we see that a vector field $V = V^i X_i + V^{(i)} Y_i$ on TM satisfies the condition $L_V P_N = 0$ if and only if it satisfies

$$Y_j V^i = 0, \quad \overset{B}{\nabla}_j V^{(i)} = V^m R^i_{mj}, \tag{6.1}$$

where the covariant derivation $\overset{B}{\nabla}_j$ is defined by $\overset{B}{\nabla}_j V^{(i)} = X_j V^{(i)} + V^{(m)} Y_m N^i_j$.

The first condition of (6.1) means that V preserves the vertical sub-bundle $\boldsymbol{V}$. Hence, if we consider the case of $\boldsymbol{P}_0$, it is always satisfied because of Proposition 6.2. Furthermore we see that the second condition of (6.1) means that V preserves the horizontal sub-bundle $\boldsymbol{H}$. Hence we have the following characterizations from Theorem 6.1 and 6.2.

Theorem 6.3. *Assume that a non-linear connection N be given on TM. Then a vector field V on TM is an infinitesimal automorphism of $D(GL(n, \boldsymbol{R}))$-structure $\boldsymbol{P}_1$ if and only if the following conditions are satisfied:*

(1) *V is an infinitesimal automorphism of $\boldsymbol{P}_0$,*

(2) *V preserves the horizontal vector bundle $\boldsymbol{H}$,*

Theorem 6.4. *Assume that a non-linear connection N be given on TM. Then a vector field V on TM is an infinitesimal automorphism of $D(GL(n, \boldsymbol{R}))$-structure $\boldsymbol{P}_1$ if and only if the following conditions are satisfied:*

(1) *V is an infinitesimal automorphism of the almost complex N-structure $\boldsymbol{F}_N$,*

(2) *V preserves the horizontal vector bundle $\boldsymbol{H}$,*

From Proposition 6.2, it is enough to consider the case where V is the completet lift or vertical lift of a vector field on M. We consider the only case of complete lift in the following. In this case, the components of $V = v^c$ are given as follows $V^i = v(x)$, $V^{(i)} = y^m \overset{B}{\nabla}_m v^i$. Then, the second condition of (6.1) is written as follows:

$$y^m \overset{B}{\nabla}_j \overset{B}{\nabla}_m v^i = v^m R^i_{mj}. \tag{6.2}$$

Example 6.1. Let M be a manifold with a symmetric linear connection $\Gamma^i_{jk}(x)$. Then we get natural non-linear connection $N^i_j = \Gamma^i_{jk}(x) y^k$. In this case, the connection $\overset{B}{\nabla}$ is given by $\omega^i_j = \pi^*(\Gamma^i_{jk}(x) dx^k)$, and the curvature tensor of N is given by $R^i_{jk} =$

$\underline{R}^i_{mjk}(x)y^m$ for the curvature tensor field $\underline{R}^i_{mjk}(x)$ of $\Gamma^i_{jk}(x)$. Also the complete lift of a vector field $v=v^i(x)(\partial/\partial x^i)$ is given by $v^c=v^iX_i+(y^m\overset{B}{\nabla}_m v^i)Y_i$. Then the condition (6.2) is written as

$$\nabla_k\nabla_j v^i+v^m\underline{R}^i_{jkm}=0$$

where ∇ is the covariant derivation with respect to $\Gamma^i_{jk}(x)$. Thus v^c is an infinitesimal automorphism of $\boldsymbol{P}_1$ if and only if v is an affine Killing vector field of the given symmetric affine connection $\Gamma^i_{jk}(x)$ on M.

Next we investigate infinitesimal automorphisms of the $D(O(n))$-structure $\boldsymbol{P}_2$. A vector field V on TM is an infinitesimal automorphism of $\boldsymbol{P}_2$ if and only if $V\in\boldsymbol{P}_1$ and it satisfies

$$L_VG_N=0. \tag{6.3}$$

For the calculations of this equation, we use the following notations:

$$D^h_{X_j}(V^iX_i)=(V^i{}_{|j})X_i,\ D^v_{Y_j}(V^{(i)}Y_i)=(V^{(i)}|_j)Y^i,$$

for the Miron-type connection $D=D^h+D^v$. Then we have

Theorem 6.5. *On the tangent bundle TM with a non-linear connection N, a vector field $V=V^iX_i+V^{(i)}Y_i$ on TM is an infinitesimal automorphism of the $D(O(n))$-structure $\boldsymbol{P}_2$ if and only if*

(1) *V is an infinitesimal automorphism of $D(GL(n,\boldsymbol{R}))$-structure $\boldsymbol{P}_1$,*

(2) *V satisfies the following equations:*

$$V_{i|j}+V_{j|i}+V^{(m)}Y_mg_{ij}=0, \tag{6.4}$$

$$V_{(i)}|_j+V_{(j)}|_i-V^mP_{imj}-V^mP_{jmi}=0, \tag{6.5}$$

The equations (6.4) and (6.5) are similar to the so-called *Killing Equation*. In fact, in the case of $V=v^c$, the condition (6.4) is written as

$$v^m(\partial g_{ij}/\partial x^m)+y^m(\partial v^t/\partial x_m)(\partial g_{ij}/\partial y^t)+(\partial v^m/\partial x^i)g_{mj}+g_{im}(\partial v^m/\partial x^j)=0.$$

According to Yano[64], this condition is written as $L_vg_{ij}=0$, that is, the vector field v is a *Killing vector filed* on the generalized metric space (M,g_{ij}). In the case where the given g is a Finsler metric: $g_{ij}=\partial^2L^2/\partial y^i\partial y^j$, we see that the condition (6.5) is equivalent to (6.4). So we have

Theorem 6.6. *Let (M,g_{ij}) be a generalized metric space and N a non-linear connection on its tangent bundle TM. Then the complete lift v^c of a vector field v on M is an infinitesimal automorphism of $D(O(n))$-structure $\boldsymbol{P}_2$, if and only if it satisfies the following conditions:*

(1) *v is a Killing vector field on (M,g_{ij}),*

(2) *v^c preserves the horizontal sub-bundle $\boldsymbol{H}$.*

We state about almost Hamilton vector fields. If TM admits a $D(O(n))$-structure $\boldsymbol{P}_2$, it also admits a Riemannian metric G_N, and an almost complex N-structure F_N. We see that the pair $\{G_N,F_N\}$ defines an almost Hermitian structure on TM. Then we define a 2-form Ψ on TM by

$$\Psi(V, W)=G_N(V, F_N(W)).$$

A vector field V on TM is said to be an *almost Hamilton vector field* of Ψ if it satisfies

$$L_V\Psi=0. \tag{6.6}$$

The left hand-side of (6.6) is written as $L_V\Psi=(L_VG_N)F_N+G_N(L_VF_N)$. Hence, if is an infinitesimal automorphism of $\boldsymbol{P}_2$, it also satisfies (6.6).

Conversely, if an almost Hamilton vector field V is an infinitesimal automorphism of $\boldsymbol{P}_1$, it satisfies (6.3). Thus we have

Theorem 6.7. *Let (M, g_{ij}) be a generalized metric space and N a non-linear connection on TM. Then any infinitesimal automorphism of $\boldsymbol{P}_2$ is an almost Hamilton vector field of Ψ.*

Conversely, if an almost Hamilton vector field of Ψ is an infinitesimal automorphism of $\boldsymbol{P}_1$, then it is an infinitesimal automorphism of $\boldsymbol{P}_2$.

Lastly we also consider the case of $V=v^c$. Then by direct calculations of (6.6), we get (6.4) and

$$@_{(ij)}\{g_{ir}(v^m R^r_{mj}-y^m \overset{B}{\nabla}_j \overset{B}{\nabla}_m v^r)\}=0. \tag{6.7}$$

Theorem 6.8. *Let (M, g_{ij}) be a generalized metric space and N a non-linear connection on TM. Then the complete lift of a vector field v on M is an almost Hamilton vector field if and only if*

(1) *v is a Killing vector field on (M, g_{ij}),*

(2) *v satisfies (6.7).*

7. Lie algebras of infinitesimal automorphisms of *G*-structures

In the present section, we shall consider some Lie algebras of infinitesimal automorphisms of some G-structures on TM investigated in the previous section. We denote the set of all infinitesimal automorphisms of $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_N$, $\boldsymbol{F}_N$ and Ψ by A_0, A_1, A_2, A_P, A_F and A_Ψ respectively. It is easily seen that these sets are Lie algebras under the usual Lie bracket. First, from Proposition 6.1, we see that $A_0=(\chi(M))^c+\chi((M))^v$ for the Lie algebra $\chi(M)$ of all vector fields on M. Furthermore, from the discussions in the previous section, we have

Proposition 7.1. *The Lie algebras of all infinitesimal automorphisms in the above satisfy the following relations*:

$$A_1=A_0\cap A_P=A_0\cap A_F=A_P\cap A_F, \quad A_2=A_1\cap A_\Psi$$

A vector field S on TM is called a *semi-spray* if it satisfies $d\pi(S(y))=y$ for $y=(x^i, y^i)\in T_xM$. Then S is expressed as

$$S(y)=y^i(\partial/\partial x^i)+F^i(x, y)(\partial/\partial y^i) \tag{7.1}$$

for a function $F^i(x, y)$ on TM. A semi-spray is called a ***spray*** if the function $F^i(x, y)$ is $(2)p$-homogeneous in y, or equivalently the following condition is satisfied:

$$L_C S = [C, S] = S \tag{7.2}$$

for the Liouville vector field $C = y^m Y_m$. For a given semi-spray S, a vector field V on TM is called an *infinitesimal automorphism* of S if V satisfies $L_V S = [V, S] = 0$. We denote by A_S the Lie algebra formed by all infinitesimal automorphisms of semi-spray S:

$$A_S = \{V \in A_0;\ L_V S = 0\}.$$

By the assumption for the given non-linear connection N, if we put

$$F^i(x, y) = N^i_m(x, y) y^m, \tag{7.3}$$

the semi-spray defined by (7.1) becomes a spray, and it is expressed as

$$S = y^m X_m \tag{7.4}$$

for the N-frame $\{X\}$ on $\boldsymbol{H}$. By direct calculation, we get the following proposition which is originally due to Grifone[25].

Proposition 7.2. *Let N be a non-linear connection on TM which is $(1)p$-homogeneous in y, and S the spray defined by* (7.4). *Then the almost product N-structure P_N is expressed as*

$$P_N = -L_S Q \tag{7.5}$$

for the standard almost tangent structure Q on TM.

By this proposition, we get the following proposition (cf. Klein[42])

Proposition 7.3. *Let S be the spray defined by* (7.4). *Then the Lie algebra A_s is a Lie sub-algebra of A_1.*

In the following, we shall consider the following Lie algebras:

$$\underline{A}_1 = \{v \in \chi(M);\ v^c \in A_1\},\ \underline{A}_P = \{v \in \chi(M);\ v^c \in A_P\},\ \underline{A}_S = \{v \in \chi(M);\ v^c \in A_S\},$$

If we put $V = v^c$ for a vector field v on M, then V is an element of A_0, and the condition (7.2) is written as

$$y^i y^j (\partial^2 v^k / \partial x^i \partial x^j) = v^m (\partial F^k / \partial x^m) - F^m (\partial v^k / \partial x^m) + y^m (\partial v^r / \partial x^m)(\partial F^k / \partial y^r) \tag{7.6}$$

Then, from the assumption on the non-linear connection N and Theorem 6.1, we have $\underline{A}_1 = \underline{A}_P$, and also by Proposition 7.3, we have $\underline{A}_S \subset \underline{A}_1$.

On the other hand, by direct calculation, we see that the condition (7.6) is equivalent to

$$y^j y^m \overset{B}{\nabla}_j \overset{B}{\nabla}_m v^i = y^j v^m R^i_{mj}. \tag{7.6'}$$

Hence, if $v \in \underline{A}_1$ or equivalently satisfies (6.1), the condition (7.6) or (7.6′) is also satisfied, that is, v is an element of $\underline{A}_S$. So we have $\underline{A}_1 \subset \underline{A}_S$, and hence we have $\underline{A}_1 = \underline{A}_S$.

Consequently we have

Proposition 7.4. *The three Lie algebras* $\underline{A}_1$, $\underline{A}_P$, $\underline{A}_S$ *in the above coincide with each other.*

$$\underline{A}_1=\underline{A}_P=\underline{A}_S.$$

In Loos [49], the Lie algebra $\underline{A}_S$ for an arbitrary semi-spray S is studied and showed that dim. $\underline{A}_S \leq n(n+1)$ under the condition that each element of $\underline{A}_S$ is complete. Especially, in the case of dim. $\underline{A}_S=n(n+1)$, it is proved that the base manifold M is isomorphic to $\boldsymbol{R}^n$ and the function $F^i(x, y)$ is written in the form $F^i=\lambda y^i$ for a unique constant $\lambda \in \boldsymbol{R}$. Applying this result to our case, we have

Theorem 7.1. *Let N be a non-linear connection satisfying $Y_j N^i_k = Y_k N^i_j$, and $\underline{A}_1$ be the Lie algebra defined in the above. Then* dim. $\underline{A}_1 \leq n(n+1)$. *Especially, if* dim. $\underline{A}_1 = n(n+1)$, *then the base manifold M is ismorphic to $\boldsymbol{R}^n$, and the given non-linear connection N vanishes identically*: $N=0$. *In this case, $\underline{A}_1$ is the set of all affine vector fields on $\boldsymbol{R}^n$.*

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