# Some specific Examples in Boolean Algebra ( I )

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## 1. Introduction

The statements (sometimes called assertions or prpositions) will be denoted by the letters $p, q, r, \cdots$. When these statements are combined by grammatical connectives "or" "and" and "not", we denote "or" by cup ($\cup$) "and" by cap ($\cap$) and "not" by aprime ($'$). Thus

"$p \cup q$" denotes the statement "$p$ or $q$"

"$p \cap q$" denotes the statement "$p$ and $q$"

"$p'$" denotes the statement "not $p$".

Furthermore, the statement "$p \rightarrow q$" is to mean that the statement "if $p$, then $q$". This may be defined in terms of other connectives, for "$p \rightarrow q$" means that $q$ must hold unless $p$ failes, so that

$$(p \rightarrow q) = (p' \cup q).$$

An equation such as $p \cup p' = I$ means that $p \cup p'$ is logically equivalent to a true statement $I$. Expressions of this sort are known as logical tautologies.

## 2. Some specific Examples in Boolean Algebra

THEOREM 1. *If* $(p \rightarrow q_1) = I$, $(p \rightarrow q_2) = I$, *then* $[p \rightarrow q_1 \cap q_2] = I$.

*Proof.* $(p \rightarrow q_1) \rightarrow [p \rightarrow (q_1 \cap q_2)] \cdots\cdots\cdots\cdots (1)$.

We may prove that the equation (1) is a tautology under $(p \rightarrow q_2) = I$

$$\begin{aligned} & (p \rightarrow q_1) \rightarrow [p \rightarrow (q_1 \cap q_2)] \\ &= [p' \cup q_1]' \cup [p' \cup (q_1 \cap q_2)] \\ &= [p' \cup q_1]' \cup [(p' \cup q_1) \cap (p' \cup q_2)] \\ &= [(p' \cup q_1)' \cup (p' \cup q_1)] \cap [(p' \cup q_1)' \cup (p' \cup q_2)] \\ &= (p' \cup q_1)' \cup (p \cup q_2) \\ &= p' \cup q_2 \\ &= I. \end{aligned}$$

Hence, if $(p \rightarrow q_1) = I$, and $(p \rightarrow q_2) = I$, then $[p \rightarrow q_1 \cap q_2] = I$.

THEOREM 2. *If* $[p \rightarrow q_1 \cap q_2] = I$, *then* $(p \rightarrow q_1) = I$ *and* $(p \rightarrow q_2) = I$.

*Proof.* $[p \rightarrow (q_1 \cap q_2)] \rightarrow (p \rightarrow q_1) \cdots\cdots\cdots\cdots (1)$.

We may prove that the equation (1) is a tautology,

$$\begin{aligned} & [p \rightarrow (q_1 \cap q_2)] \rightarrow (p \rightarrow q_1) \\ &= [p' \cup (q_1 \cap q_2)]' \cup (p' \cup q_1) \\ &= [(p' \cup q_1) \cap (p' \cup q_2)]' \cup (p' \cup q_1) \end{aligned}$$

$=[(p' \cup q_1)' \cup (p' \cup q_2)'] \cup (p' \cup q_1)$

$=I \cup (p' \cup q_2)'$

$=I.$

Similarly, $[p \to (q_1 \cap q_2)] \to (p \to q_2)$ is also a tautology.

*Example.* If $x=a$ always shows numbers of coordinates $[(-3, 0), (4.5, 0)]$ between the intersection of the two curves $y=1/3x^2$ and $y=-x^2+2x+18$, then $-3<a$, $a<4.5$ that is $-3<a<4.5$.

Conversely, if $-3<a<4.5$, then$x=a$ satisfies the inequalities $-3<a$, and $a<4.5$.

THEOREM 3. $[(p_1 \cup p_2) \to q] \to (p_1 \to q) = I$ *and* $[(p_1 \cup p_2) \to q] \to (p_2 \to q) = I$.

Proof. $[(p_1 \cup p_2) \to q] \to (p_1 \to q)$

$=[(p_1 \cup p_2)' \cup q]' \cup (p_1' \cup q)$

$=[(p_1 \cup p_2) \cup (p_1' \cup q)] \cap [q' \cup (p_1' \cup q)]$

$=I.$

Similary $[(p_1 \cup p_2) \to q] \to (p_2 \to q) = I$.

*Example.* If the power (cardinal numbers) of all integers is $\aleph$, then the power of positive integers is $\aleph$ and so is the power of negative integers.

Because let $p_1$ denote all positive integers, $p_2$ all negative integers, and $q$ the power $\aleph$, then

$$[(p_1 \cup p_2) \to q] \to (p_1 \to q) = I$$

and

$$[(p_1 \cup p_2) \to q] \to (p_2 \to q) = I$$

hence we understand the statement immediately.

THEOREM 4. If $(p_1 \to q) = I$, *and* $(p_2 \to q) = I$, *then* $[(p_1 \cup p_2) \to q] = I$.

Proof. $(p_1 \cup p_2) \to q$

$=(p_1 \cup p_2)' \cup q$

$=(p_1' \cap p_2') \cup q$

$=(p_1' \cup q) \cap (p_2' \cup q)$

$=I \cap I$

$=I.$

*Example.* If $E_1$ and $E_2$ are events, then $E_1 \cup E_2$ is an event. Because, let $p_1$ denote $E_1$, $p_2$ denote $E_2$, $q$ denote an event, "$p_1 \to q$" denote that "$E_1$ is an event", "$p_2 \to q$" denote that "$E_2$ is an event", then the above statement is plain from this theorem.

*Lemma. If* $(p_1 \to q) = I$, $(p_2 \to q) = I$, *then* $[(p_1 \cap p_2) \to q] = I$.

Proof. $(p_1 \cap p_2) \to q$
$= (p_1 \cap p_2)' \cup q$
$= (p_1' \cup p_2') \cup q$
$= I.$

*Example.* If $E_1$ and $E_2$ are events, then $E_1 \cap E_2$ is an event.

THEOREM 5. *$(p \to q) \to (q' \to p')$ is a tautology, but $(p \to q) \to (q \to p)$ is not always a tautology.*

Proof. $(p \to q) \to (q' \to p')$
$= (p' \cup q)' \cup [(q')' \cup p']$
$= (p \cap q') \cup (q \cup p')$
$= (p \cup q \cup p') \cap (q' \cup q \cup p')$
$= I,$

but $(p \to q) \to (q \to p)$
$= (p' \cup q)' \cup (q' \cup p)$
$= (p \cup q') \cap (q' \cup p)$
$= q' \cup p \cdots\cdots\cdots\cdots (1).$

Hence, the equation (1) is a tautology when $p = q$, but is not a tautology when $p \neq q$.

*Example* 1. In this theorem, if $p$, $q$, $p'$ and $q'$ denotes $A = B$, $C = D$, $A \neq B$, and $C \neq D$ respectively, then, this theorem shows that "the proposition of a statement is always true." But "$(p \to q) \to (q \to p)$ is not always a tautology" shows that "the inverse of a statement is not always true."

*Example* 2. The obverse of a statement is not always true.

Instead of proving this statement, we may prove that $(p \to q) \to (p' \to q')$ is not always a tautology.

$(p \to q) \to (p' \to q')$
$= (p' \cup q)' \cup (p \cup q')$
$= p \cup q' \cdots\cdots\cdots\cdots (1),$

and the equation (1) is not always a toutology.

THEOREM 6. $(p \cap q \cap r) \to (p \cap q) = I.$

Proof. $(p \cap q \cap r) \to (p \cap q)$
$= (p \cap q \cap r)' \cup (p \cap q)$
$= (p \cap q)' \cup r' \cup (p \cap q)$
$= I.$

*Example* 1. $A$: the real number is enumerable,

$B$: the cardinal number of the real number is the smallest infinite cardinal number.

If $A$ and $B$ denote the above statements respectively, then
$A \to B$: if the real number is enumerable, then the cardinal number of the real number is the smallest infinite cardinal number,
$B'$: the cardinal number of the real number is not the smallest infinite cardinal number,
and $A \to B$ and $B'$ are true. Hence, from $(A \to B) \cap B'$ we have $(A \to B) \cap B' = (A' \cup B) \cap B' = (A' \cup B) \cap (A \cup B') \cap (A' \cup B')$.
Let $A' \cup B = p$, $A' \cup B' = q$, $A \cup B' = r$,
them, from the theorem
$p \cap q = (A' \cup B) \cap (A' \cup B') = A'$.

Hence, we have a statement that "the real number is not enumerable."

*Example* 2. If $A$ and $B$ denote the following statements
$A$: the probability $P(E_1 \cap E_2)$ of the event $E_1 \cap E_2$ is $P(E_1) \cdot P(E_2)$,
$B$: events $E_1$ and $E_2$ are independent,
we have the following statement similarly with example 1,
"the probability $P(E_1 \cap E_2)$ of the event $E_1 \cap E_2$ is not $P(E_1) \cdot P(E_2)$."
That is, we have the case of dependence.

**General References**

(1) Birkhoff, Garrett and Saunders MacLane. "A Survey of Modern Algebra." New York: Macmillan Company 1941. (450 pp.)

(2) Hilbert, D. and W. Ackermann. "Grundzuge der Theoretischen Logik". 1949