LINEAR ESTIMATION USING COVARIANCE INFORMATION IN THE PRESENCE OF UNCERTAIN OBSERVATIONS

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Abstract In the conventional estimation problem, we assume that the observed value always contains a signal to be estimated and observation noise. However, in communication systems, there arises the case where the observed value consists of noise alone during observing the signal.

This paper, in the estimation problem with the uncertain observation, designs algorithms for fixed-point smoothing and filtering estimates by use of a probability that the signal exists, a crosscovariance function of the signal with the observed value, an autocovariance function of the signal plus colored noise besides the observed value when the signal is observed with additional white Gaussian and colored noises. The signal might be correlated with colored noise. In the current approach, it is advantageous over the existing one in that the present estimator necessitates the covariance information without requiring complete knowledge of state-space model of the signal.

A numerical simulation example is shown to demonstrate the feasibilty of the current estimation technique.

1. Introduction

In the Kalman filter [1], it is a usual assumption that the observed value always contains the signal and observation noise. However, in practical estimation problems in communication systems, we encounter the case of "false alarm" where the observed value consists of noise alone with nonzero probability in spite of positive decision on the existence of the signal. This case occurs, for example, in tracking the target trajectory in space, and we must decide the existence of a target by means of the decision rule based on a likelihood ratio test etc..

Previously, linear least-squares estimator is developed regarding the estimation problem with the uncertain observation [2]. In [2], recursive estimation algorithm is designed by assuming complete knowledge of a state-space model for the signal in linear discrete-time systems.

By the way, there is a study on the recursive Wiener filter for the estimation problem in linear stochastic systems, given a specified covariance function of the observed value, when observation noise is white Gaussian [3]. This approach differs from that based on the Kalman filter on the kind of information used. The recursive Wiener filter uses the information of the system matrix in the state-space model of the signal, the observation matrix, the crosscovariance function of the signal with the observed value and the observed value in linear continuous

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systems. Whereas the Kalman filter requires complete knowledge of the state-space model of the signal. As an example, the state-space model is realized from the information of the Markov parameter of the system [4]. Also, still in the recursive Wiener filter, there remains a task to estimate the system matrix before implementing the computation of the estimate. To avoid the realization step for the state-space model, the estimators, by use of the covariance information, have been devised [5].

In this paper, on the estimation problem with the uncertain observation, new fixed-point smoother and filter are designed in linear continuous stochastic systems. The estimator is efficient for recursive estimation from the uncertain observed value. Let the observation equation be given by $y(t) = z(t) + v(t)$, $z(t) = x_u(t) + v_c(t)$, $x_u(t) = U(t)x(t)$. The signal $x(t)$ might be correlated with colored observation noise $v_c(t)$. We assume that the probabilty on the existence of the signal $x(t)$ at time t is $p(t)$ as introduced in [2]. Besides the probability, the proposed estimator assumes the knowledge on the information of the observed value $y(t)$, the crosscovariance function $K_{uy}(t, s)$ of $x_u(t)$ with $y(s)$, the autocovariance function $K_z(t, s)$ of $z(t)$ and the variance of white Gaussian observation noise $v(t)$. Since $K_{uy}(t, s) = p(t)K_{xy}(t, s)$, we see that the information $K_{uy}(t, s)$ is decomposed to $p(t)$ and $K_{xy}(t, s)$. We assume that the covariance functions $K_{uy}(t, s)$ and $K_z(t, s)$ are expressed in the semi-degenerate kernel form. The semi-degenerate kernel is appropriate for expressing the covariance functions of stationary stochastic signal etc. by finite sum of nonrandom functions of the variables t and s [5]. Therefore, the current algorithms can be applied to the estimation of the stochastic signal in general.

The advantage of the present approach over the existing one [2] is that it is suitable for on-line estimation of the signal by using the covariance information, and does not take in the step to realize the state-space model.

2. Estimation problem with uncertain observations

Let an observation equation be described by

$$y(t) = z(t) + v(t), \quad z(t) = x_u(t) + v_c(t), \quad x_u(t) = U(t)x(t), \tag{1}$$

where $y(t)$ is an $n \times 1$ observed value vector, $x(t)$ is an $n \times 1$ zero-mean signal vector, $v_c(t)$ is a zero-mean colored observation noise, and $v(t)$ is white Gaussian observation noise satisfying

$$E[v(t)] = 0, \tag{2}$$

$$E[v(t)v^T(s)] = R(t)\delta(t - s), \quad 0 \leqq t, \ s < \infty. \tag{3}$$

We assume that the signal $x(t)$ and colored noise $v_c(t)$ are uncorrelated with white Gaussian noise $v(s)$ as

$$E[x(t)v^T(s)] = 0, \quad E[v_c(t)v^T(s)] = 0, \quad 0 \leq t, \ s < \infty, \tag{4}$$

where the signal might be correlated with coloured noise $v_c(s)$. Let $p(t)$ be the probabilty that the observation at time t contains the signal $x(t)$. Then, $U(t)$ is a scalar quantity taking on values of 0 or 1 with

$$p(t) = Pr\{U(t) = 1\}, \tag{5}$$

$$1 - p(t) = Pr\{U(t) = 0\}. \tag{6}$$

Hence,

$$E[U(t)] = p(t) \tag{7}$$

and

$$E[U(t)U(s)] = p(t)p(s), \quad t \neq s, \tag{8}$$

$$E[U^2(t)] = p(t) \tag{9}$$

[2]. Then, the probability for $U(t) = 1$ is $p(t)$ and for $U(t) = 0$ is $1 - p(t)$, although the ideal value of $U(t)$ might be 1 when the signal $x(t)$ exists, and $U(t) = 0$ for the case where the observed value contains noises only. The uncertain observation is caused by $U(t)$ in Eq. (1), and $x_u(t) (= U(t)x(t))$ equals $x(t)$ with the probability $p(t)$ and 0 with probability $1 - p(t)$. Also, the probability of false alarm becomes $1 - p(t)$, since the "false-alarm probability" means the probability that the signal $x(t)$ is not present in spite of a priori assumption that the signal exists. Thus, we are lead to the estimation problem, where we estimate the uncertain process $x_u(t)$ with the uncertain observed value.

We consider the fixed-point smoothing and filtering problems. Let the fixed-point smoothing estimate $\hat{x}_u(t, T)$ of $x_u(t)$ be given by

$$\hat{x}_u(t, T) = \int_0^T h(t, s', T)y(s')ds' \tag{10}$$

as a linear integral transform of the observation set $\{y(s'), 0 \leq s' \leq T\}$, where t is the fixed-point and $h(t, s, T)$ is an optimal impulse response function.

Minimizing a cost function

$$J = E[\|x_u(t) - \hat{x}_u(t, T)\|^2], \tag{11}$$

we obtain the Wiener-Hopf integral equation

$$E[x_u(t)y^T(s)] = \int_0^T h(t, s', T)E[y(s')y^T(s)]ds' \tag{12}$$

by an orthogonal projection lemma [5]

$$x_u(t) - \int_0^T h(t, s', T)y(s')ds' \perp y(s), \quad 0 \leq s, \ t \leq T. \tag{13}$$

Here, " $\perp$ " denotes the notation of the orthogonality. Substituting Eq. (1) into Eq. (12), and using Eq. (3), we have

$$h(t, s, T)R(s) = K_{uy}(t, s) - \int_0^T h(t, s', T)K_z(s', s)ds', \tag{14}$$

where $K_{uy}(t, s)$ denotes the crosscovariance function of $x_u(t)$ with $y(s)$, and $K_z(t, s)$ the autocovariance function of $z(t)$. Considering the stochastic property of $U(t)$ above, we might express the

crosscovariance function $K_{uy}(t, s)$ in the semi-degenerate kernel form [5] by

$$
\begin{aligned}
K_{uy}(t, s) &= E[x_u(t)y^T(s)] \\
&= E[U(t)x(t)y^T(s)] \\
&= p(t)K_{xy}(t, s) \\
&= \left[\begin{array}{ll} C(t)H^T(s), & 0 \leqq s \leqq t, \\ M(t)N^T(s), & 0 \leqq t \leqq s, \end{array} \right.
\end{aligned} \tag{15}
$$

where $C(t)$ and $H(s)$ are $n \times n'$ bounded matrices, and $M(t)$ and $N(s)$ are $n \times m'$ bounded matrices. The autocovariance function $K_z(t, s)$ is also expressed in the semi-degenerate kernel form by

$$
\begin{aligned}
K_z(t, s) &= E[z(t)z^T(s)] \\
&= \left[\begin{array}{ll} G(t)L^T(s), & 0 \leqq s \leqq t, \\ L(t)G^T(s), & 0 \leqq t \leqq s, \end{array} \right.
\end{aligned} \tag{16}
$$

where $G(t)$ and $L(s)$ are $n \times l$ bounded matrices.

The specific estimation problem pursued in this paper necessitates the information of the probability $p(t)$, the observed value $y(t)$, the crosscovariance function $K_{uy}(t, s)$, the autocovariance function $K_z(t, s)$ and the variance $R(t)$ of white Gaussian observation noise. This certifies that the approach taken in this paper is distinct from that in [2] for the estimation problem with the uncertain observation. In [2], complete description of the state-space model is assumed as for the necessary information regarding the estimation problem with the uncertain observation. Since the kind of covariance information might be useful before realizing the state-space model, we recommend, in the confronted estimation problem, rather the approach using the covariance information than that by use of the state-space model.

In section 3, on the basis of the preliminary statement above, we design the recursive algorithms for the linear least-squares fixed-point smoothing and filtering estimates of $x_u(t)$ by use of the covariance information.

3. Recursive fixed-point smoothing algorithm using covariance information

In Theorem 1, we present the recursive least-squares algorithm for the fixed-point smoothing estimate in linear continuous stochastic systems.

Theorem 1. Let the probability for $U(t) = 1$ be $p(t)$ in the observation equation (1) for the signal observed with additional white Gaussian plus coloured noise. Here, the signal might be correlated with colored noise. Let the crosscovariance function $K_{xy}(t, s)$ of $x(t)$ with $y(s)$ and the autocovariance function $K_z(t, s)$ of $z(t)$ be expressed in the semi-degenerate kernel form. Also, let the observed value and the variance of white Gaussian observation noise be given. Then, the recursive algorithm for linear least-squares fixed-point smoothing estimate consists of Eqs. (17)–(26) in continuous stochastic systems.

Fixed-point smoothing estimate:

$$\partial \hat{x}_u(t, T)/\partial T = h(t, T, T)(y(T) - G(T)e(T)) \tag{17}$$

Filtering estimate:

$$\hat{x}_u(T, T) = C(T)V(T) \tag{18}$$

$$dV(T)/dT = \Phi(T, T)(y(T) - G(T)e(T)), \quad V(0) = 0 \tag{19}$$

$$de(T)/dT = J(T, T)(y(T) - G(T)e(T)), \quad e(0) = 0 \tag{20}$$

$$\Phi(T, T) = (H^T(T) - W(T)G^T(T))R^{-1}(T) \tag{21}$$

$$J(T, T) = (L^T(T) - r(T)G^T(T))R^{-1}(T) \tag{22}$$

$$dr(T)/dT = J(T, T)(L(T) - G(T)r(T)), \quad r(0) = 0 \tag{23}$$

$$h(t, T, T) = (M(t)N^T(T) - S(t, T)G^T(T))R^{-1}(T) \tag{24}$$

$$\partial S(t, T)/\partial T = h(t, T, T)(L(T) - G(T)r(T)), \quad S(t, t) = C(t)W(t) \tag{25}$$

$$dW(T)/dT = \Phi(T, T)(L(T) - G(T)r(T)), \quad W(0) = 0 \tag{26}$$

(Proof)

Let us differentiate Eq. (14) with respect to T.

$$\partial h(t, s, T)/\partial TR(s) = -h(t, T, T)K_z(T, s) - \int_0^T \partial h(t, s', T)/\partial TK_z(s', s)\,ds' \tag{27}$$

If we introduce an auxiliary function $J(T, s)$ which satisfies

$$J(T, s)R(s) = L^T(s) - \int_0^T J(T, s')K_z(s', s)\,ds', \tag{28}$$

we have a partial differential equation for $h(t, s, T)$

$$\partial h(t, s, T)/\partial T = -h(t, T, T)G(T)J(T, s). \tag{29}$$

Similarly, if we differentiate Eq. (28) with respect to T, we have

$$\partial J(T, s)/\partial TR(s) = -J(T, T)K_z(T, s) - \int_0^T \partial J(T, s')/\partial TK_z(s', s)\,ds'. \tag{30}$$

From Eqs. (16), (28) and (30), we obtain a partial differential equation for $J(T, s)$

$$\partial J(T, s)/\partial T = -J(T, T)G(T)J(T, s). \tag{31}$$

Now, from Eq. (28), the function $J(T, T)$ in Eq. (31) satisfies

$$J(T, T)R(T) = L^T(T) - \int_0^T J(T, s')K_z(s', T)\,ds'. \tag{32}$$

Substituting the expression $K_z(s', T) = L(s')G^T(T)$ for $0 \leq s' \leq T$ from Eq. (16) into Eq. (32), we have

$$J(T, T)R(T) = L^T(T) - \int_0^T J(T, s')L(s')\,ds'\,G^T(T). \tag{33}$$

If we introduce a function $r(T)$ defined by

$$r(T) = \int_0^T J(T, s')L(s')\,ds',\tag{34}$$

we obtain Eq. (22) for $J(T, T)$.

If we differentiate Eq. (34) with respect to T and substitute Eq. (31) into the resultant equation, we have

$$dr(T)/dT = J(T, T)L(T) - J(T, T)G(T)\int_0^T J(T, s')L(s')\,ds'.\tag{35}$$

From Eq. (34), We can rewrite Eq. (35) as Eq. (23), where the initial condition on the differential equation (23) at $T = 0$ is $r(0) = 0$ from Eq. (34).

From Eq. (14), the function $h(t, T, T)$, which appeared in Eq. (29), satisfies

$$h(t, T, T)R(T) = K_{uy}(t, T) - \int_0^T h(t, s', T)K_z(s', T)\,ds'.\tag{36}$$

If we use the expressions $K_{uy}(t, T) = M(t)N^T(T)$ for $0 \leqq t \leqq T$ and $K_z(s', T) = L(s')G^T(T)$ for $0 \leqq s' \leqq T$ from Eqs. (15)–(16) in Eq. (36), we have

$$h(t, T, T)R(T) = M(t)N^T(T) - \int_0^T h(t, s', T)L(s')\,ds'\,G^T(T).\tag{37}$$

If we introduce a function $S(t, T)$ defined by

$$S(t, T) = \int_0^T h(t, s', T)L(s')\,ds',\tag{38}$$

we obtain Eq. (24) for $h(t, T, T)$.

If we differentiate Eq. (38) with respect to T and substitute Eq. (29) into the resultant equation, we have

$$\partial S(t, T)/\partial T = h(t, T, T)L(T) - h(t, T, T)G(T)\int_0^T J(T, s')L(s')\,ds'.\tag{39}$$

From Eq. (34), we can rewrite Eq. (39) as Eq. (25).

If we put $t = T$ in Eq. (14), we have

$$h(T, s, T)R(s) = K_{uy}(T, s) - \int_0^T h(T, s', T)K_z(s', s)\,ds'.\tag{40}$$

If we substitute $K_{uy}(T, s) = C(T)H^T(s)$ from Eq. (15) into Eq. (40), we have

$$h(T, s, T)R(s) = C(T)H^T(s) - \int_0^T h(T, s', T)K_z(s', s)\,ds'.\tag{41}$$

Let us introduce an auxiliary function $\Phi(T, s)$ which satisfies

$$\Phi(T, s)R(s) = H^T(s) - \int_0^T \Phi(T, s')K_z(s', s)\,ds'.\tag{42}$$

From Eqs. (41) and (42), we obtain

$$h(T, s, T) = C(T)\Phi(T, s). \tag{43}$$

The initial condition on the partial differential equation (25) at $T = t$ is $S(t, t)$. From Eq. (38), $S(t, t)$ is formulated as

$$S(t, t) = \int_0^t h(t, s', t)L(s')\,ds'. \tag{44}$$

From Eq. (43), we can rewrite Eq. (44) as

$$S(t, t) = C(t)\int_0^t \Phi(t, s')L(s')\,ds'. \tag{45}$$

If we introduce a function $W(T)$ defined by

$$W(T) = \int_0^T \Phi(T, s')L(s')\,ds', \tag{46}$$

we obtain the initial condition as $S(t, t) = C(t)W(t)$.

If we differentiate Eq. (42) with respect to T, we have

$$\partial\Phi(T, s)/\partial T R(s) = -\Phi(T, T)K_z(T, s) - \int_0^T \partial\Phi(T, s')/\partial T K_z(s', s)\,ds'. \tag{47}$$

If we substitute $K_z(T, s) = G(T)L^T(s)$ for $0 \leqq s \leqq T$ from Eq. (16) into Eq. (47) and compare the resultant equation with Eq. (28), we obtain a partial differential equation for $\Phi(T, s)$

$$\partial\Phi(T, s)/\partial T = -\Phi(T, T)G(T)J(T, s). \tag{48}$$

From Eq. (42), the function $\Phi(T, T)$ in Eq. (48) satisfies

$$\Phi(T, T)R(T) = H^T(T) - \int_0^T \Phi(T, s')K_z(s', T)\,ds'. \tag{49}$$

Since $K_z(s', T) = L(s')G^T(T)$ for $0 \leqq s' \leqq T$, we can rewrite Eq. (49) as

$$\Phi(T, T)R(T) = H^T(T) - \int_0^T \Phi(T, s')L(s')\,ds'\,G^T(T). \tag{50}$$

Also, by use of Eq. (46), Eq. (50) becomes Eq. (21).

If we differentiate Eq. (46) with respect to T, we have

$$dW(T)/dT = \Phi(T, T)L(T) + \int_0^T \partial\Phi(T, s')/\partial T L(s')\,ds'. \tag{51}$$

If we substitute Eq. (48) into Eq. (51) and use Eq. (34), we obtain Eq. (26). The initial condition on the differential equation for $W(T)$ at $T = 0$ is $W(0) = 0$ from Eq. (46).

If we differentiate Eq. (10) with respect to T, we have

$$\partial\hat{x}_u(t, T)/\partial T = h(t, T, T)y(T) + \int_0^T \partial h(t, s', T)/\partial T y(s')\,ds'. \tag{52}$$

If we substitute Eq. (29) into Eq. (52) and introduce a function

$$e(T) = \int_0^T J(T, s')y(s')\,ds',$$ (53)

we obtain the partial differential equation (17) for the fixed-point smoothing estimate $\hat{x}_u(t, T)$.

If we differentiate Eq. (53) with respect to T, we have

$$de(T)/dT = J(T, T)y(T) + \int_0^T \partial J(T, s')/\partial T y(s')\,ds'.$$ (54)

If we substitute Eq. (31) into Eq. (54) and use Eq. (53), we obtain Eq. (20). The initial condition on the differential equation (20) at $T = 0$ is $e(0) = 0$ from Eq. (53).

The filtering estimate $\hat{x}_u(T, T)$ of $x_u(T)(= U(T)x(T))$ is formlated as

$$\hat{x}_u(T, T) = \int_0^T h(T, s', T)y(s')\,ds'$$ (55)

by putting $t = T$ in Eq. (10). If we substitute Eq. (43) into Eq. (55), and introduce a function $V(T)$ defined by

$$V(T) = \int_0^T \Phi(T, s')y(s')\,ds',$$ (56)

we obtain Eq. (18).

If we differentiate Eq. (56) with respect to T, we have

$$dV(T)/dT = \Phi(T, T)y(T) + \int_0^T \partial \Phi(T, s')/\partial T y(s')\,ds'.$$ (57)

If we substitute Eq. (48) into Eq. (57) and use Eq. (53), we obtain Eq. (19). The initial condition on the differential equation for $V(T)$ at $T = 0$ is $V(0) = 0$ from Eq. (56) (Q.E.D.).

In [Theorem 1] the recursive least-squares algorithms for the fixed-point smoothing and filtering estimates are proposed in linear continuous stochastic systems when the covariance information is given. In section 4, we show a simple numerical simulation example which calculates the fixed-point smoothing and filtering estimates by the algorithm of [Theorem 1].

4. A numerical simulation example

Let the observation equation be given by Eq. (1) for a scalar signal when observation noise is white Gaussian plus colored. Let the signal $x(t)$ be generated by

$$dx(t)/dt = -5x(t) + u(t), \quad E[u(t)u(s)] = 100\delta(t - s), \quad E[x^2(0)] = 10,$$ (58)

where the autocovariance function $K_x(t, s)$ of $x(t)$ is expressed by $K_x(t, s) = 10e^{-5|t-s|}$ [6]. Also, let the process of colored noise $v_c(t)$ be generated by

$$dv_c(t)/dt = w(t), \quad E[w(t)w(s)] = 10\delta(t - s), \quad E[v_c^2(0)] = 0,$$ (59)

where the autocovariance function $K_c(t, s)$ of $v_c(t)$ is given by $K_c(t, s) = 10\min(t, s)$ [6]. The crosscovariance function $K_{uy}(t, s)$ of $x_u(t)$ with $y(s)$ is expressed by Eq. (15). Since $x(t)$ is

uncorrelated with $v_c(s)$, we obtain $C(t) = 10p(t)e^{-5t}(= N(t))$ and $H(s) = e^{5s}(= M(s))$ from $K_x(t, s) = 10e^{-5|t-s|}$. Here, we put the value of $p(t)$ as $p(t) = 0.9$. Also, the autocovariance function $K_z(t, s)$ of $z(t)(= x_u(t) + v_c(t), x_u(t) = U(t)x(t))$ is given by Eq. (16). The functions $G(t)$ and $L(s)$ become $G(t) = [p(t)10e^{-5t} \ 10]$ and $L(s) = [p(s)e^{5s} \ s]$. If we substitute the functions $C(T)$, $H(T)$, $N(T)$, $M(t)$, $G(T)$ and $L(T)$ into [**THEOREM 1**], we can calculate the filtering estimate $\hat{x}_u(T, T)$ and the fixed-point smoothing estimate $\hat{x}_u(t, T)$. Graphs (a) and (b) in **Fig. 1** illustrate the colored noise processes generated by Eq. (59), starting with initial conditions $v_c(0) = -0.1$ and $v_c(0) = -0.3$ respectively. **Fig. 2** illustrates the process $x_u(t)$ (graph (a)) and its filtering estimate $\hat{x}_u(t, t)$ vs. t. Graphs (b), (c) and (d) show $\hat{x}_u(t, t)$ for white Gaussian observation noises $N(0, 0.1^2)$,
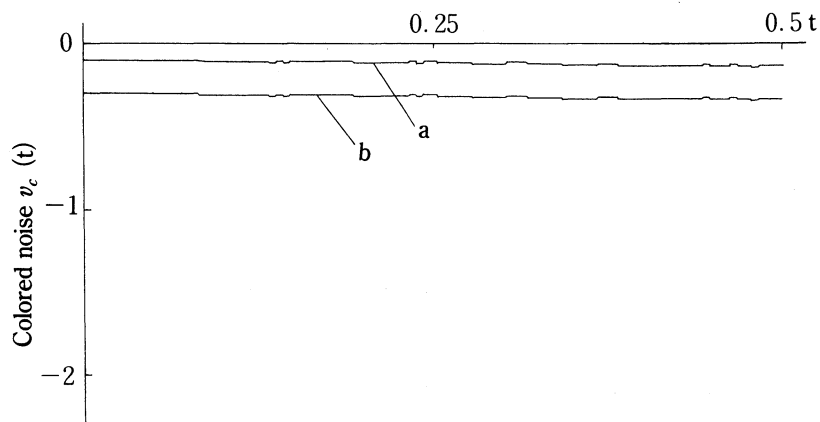

Fig. 1 The colored noise process v_c (t) generated by Eq. (59) vs. t.
(a) The colored noise process for the initial condition v_c (0) = −0.1.
(b) The colored noise process for the initial condition v_c (0) = −0.3.
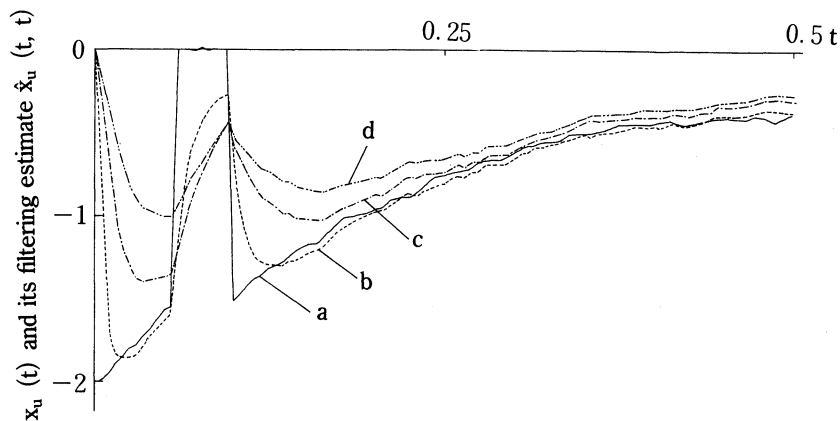

Fig. 2 The process x_u (t) and its filtering estimate $\hat{x}_u$ (t, t) vs. t.
(a) The signal process x_u (t).
(b) The filtering estimate $\hat{x}_u$ (t, t) for white Gaussian observation noise N (0, 0.1²).
(c) The filtering estimate $\hat{x}_u$ (t, t) for white Gaussian observation noise N (0, 0.3²).
(d) The filtering estimate $\hat{x}_u$ (t, t) for white Gaussian observation noise N (0, 0.5²).

$N(0, 0.3^2)$ and $N(0, 0.5^2)$ respectively. **Fig. 3** illustrates the fixed-point smoothing estimate $\hat{x}_u(0.13, T)$ vs. T. Graphs (a), (b), (c), (d), (e) and (f) show $\hat{x}_u(0.13, T)$ vs. T for white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.2^2)$, $N(0, 0.3^2)$, $N(0, 0.4^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$ respectively. Here, the value of $x_u(t)$ at the fixed-point $t = 0.13$ is $x_u(0.13) = -1.2772(= x(0.13))$. **Table 1** shows
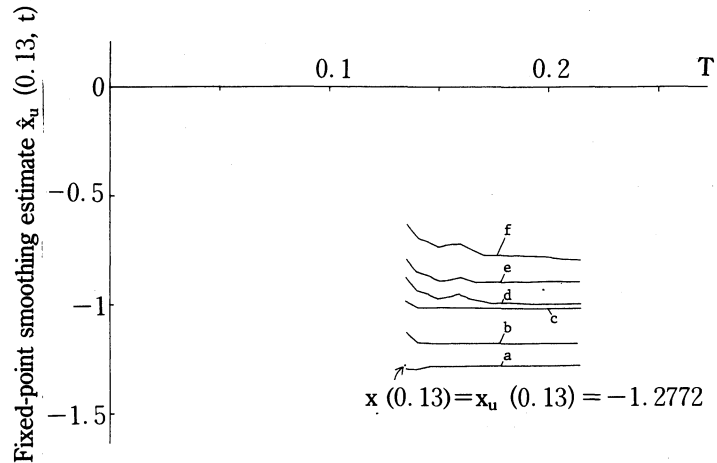

Fig. 3 　 The fixed-point smoothing estimate $\hat{x}_u$ *(0.13, T)* vs. *T*.

 (a) The fixed-point smoothing estimate $\hat{x}_u$ *(0.13, T)* for white Gaussian observation noise N (0, 0.1²).

 (b) The fixed-point smoothing estimate $\hat{x}_u$ *(0.13, T)* for white Gaussian observation noise N (0, 0.2²).

 (c) The fixed-point smoothing estimate $\hat{x}_u$ *(0.13, T)* for white Gaussian observation noise N (0, 0.3²).

 (d) The fixed-point smoothing estimate $\hat{x}_u$ *(0.13, T)* for white Gaussian observation noise N (0, 0.4²).

 (e) The fixed-point smoothing estimate $\hat{x}_u$ *(0.13, T)* for white Gaussian observation noise N (0, 0.5²).

 (f) The fixed-point smoothing estimate $\hat{x}_u$ *(0.13, T)* for white Gaussian observation noise N (0, 0.7²).

Table 1 　 The mean-square values of the filtering error and the fixed-point smoothing error for white Gaussian observation noises N (0, 0.1²), N (0, 0.2²), N (0, 0.3²), N (0, 0.4²), N (0, 0.5²) and N (0, 0.7²).

White Gaussian noise sequence	$v_c(0) = -0.1$		$v_c(0) = -0.3$	
	M.S.V. of the filtering error	M.S.V. of the fixed-point smoothing error	M.S.V. of the filtering error	M.S.V. of the fixed-point smoothing error
N (0, 0.1²)	0.19286	0.019094	0.36359	0.047612
N (0, 0.2²)	0.38346	0.036371	0.46377	0.052578
N (0, 0.3²)	0.56022	0.054127	0.56055	0.056694
N (0, 0.4²)	0.72864	0.074201	0.66492	0.064096
N (0, 0.5²)	0.89623	0.097973	0.78314	0.076780
N (0, 0.7²)	1.2325	0.15610	1.0563	0.11811