A Sufficient Condition for Cohomological Descent

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Abstract

In this note we give a sufficent condition for a surjective continuous mapping to be of cohomological descent (*de descente cohomologique* cf. Deligne [1]), which leads to a simultenuous proof of two fundamental facts in [2] or [1] (5. 3. 5) :

Fact 1 Any proper surjective mapping is of cohomological descent,

Fact 2 Any surjective continuous mapping with local sections is of cohomological descent.

Notations and Definitions We use notations and definitions in Deligne [1] section 5. We recall them as far as we need.

$\mathfrak{A}^o$ = the opposite category of a category $\mathfrak{A}$.

Hom $(\mathfrak{A}, \mathfrak{B})$ = the category of functors of $\mathfrak{A}$ to $\mathfrak{B}$.

Let n and k be integers ≥ -1.

Δ_n = the totally ordered finite set $[0, n] = \{0, 1, \ldots, n\}$.

$\delta_i : \Delta_n \to \Delta_{n+1}$ = the increasing injection such that $i \notin \delta_i(\Delta_n)$ $(0 \leq i \leq n+1)$.

$s_i : \Delta_{n+1} \to \Delta_n$ = the increasing surjection such that $s_i(i) = s_i(i+1)$ $(0 \leq \leq n)$.

$\epsilon : \Delta_{-1} \to \Delta_n$ = the unique mapping of Δ_{-1} to Δ_n.

(Δ^+) = the category whose objects are Δ_n $(n \geq -1)$, and whose morphisms are increasing mappings between Δ_n.

(Δ) = the full subcategory of (Δ^+) of objects Δ_n $(n \geq 0)$.

For any category $\mathfrak{C}$, a *simplicial object* of $\mathfrak{C}$ is an object of *Hom* $((\Delta)^o, \mathfrak{C})$.

If $X. : (\Delta)^o \to \mathfrak{C}$ is a simplicial object of $\mathfrak{C}$, we put

$$X_n = X.(\Delta_n),$$

$$\delta_i = X.(\delta_i : \Delta_n \to \Delta_{n+1}) : X_{n+1} \to X_n$$

and

$$s_i = X.(s_i : \Delta_{n+1} \to \Delta_n) : X_n \to X_{n+1}.$$

$$X. : \qquad \cdots\cdots X_2 \;\substack{\xrightarrow{\delta_0}\\ \xleftarrow{s_0}\\ \xrightarrow{\delta_1}\\ \xleftarrow{s_1}\\ \xrightarrow{\delta_2}}\; X_1 \;\substack{\xrightarrow{\delta_0}\\ \xleftarrow{s_0}\\ \xrightarrow{\delta_1}}\; X_0$$

Let $S \in \mathrm{Ob}\,\mathfrak{C}$. The *constant simplicial objact* $S.$ is the simplicial object such that $S_n = S$ and $\delta_i = s_i = \mathrm{Id}_S$. A simplicial object of $\mathfrak{C}$, *augmented* to S, is a morphism $a : X. \to S.$ which is also denoted by $a : X. \to S$.

For a morphism $a : X \to S$ in $\mathfrak{C}$, cosq $(X \to S)$ is the simplicial object augmented to S whose components are cartesian products of X in $\mathfrak{C}/S$:

$$\mathrm{cosq}\,(X \to S) = (((X/S)^{\Delta_n})_{n\geq 0} \to S).$$

A *simplicial topological space* is a simplicial object of the category whose objects are topological spaces and whose morphisms are continuous mappings.

A *sheaf* $\mathcal{F}$ on a simplical topological space $X.$ consists of

a) a family $\mathcal{F}^n$ of sheaves on X_n ;

b) for $f : \Delta_n \to \Delta_m$, a $X.(f)$-morphism $\mathcal{F}^{\cdot}(f)$ of $\mathcal{F}^n$ to $\mathcal{F}^m$, such that $\mathcal{F}^{\cdot}(f \circ g) = \mathcal{F}^{\cdot}(f) \circ \mathcal{F}^{\cdot}(g)$.

A morphism u of $\mathcal{F}^{\cdot}$ to $\mathcal{G}^{\cdot}$ is a family of morphisms $u^n : \mathcal{F}^n \to \mathcal{G}^n$ such that for $f \in Hom_{(\Delta)}(\Delta_n, \Delta_m)$, one has $u^m\mathcal{F}^{\cdot}(f) = \mathcal{G}^{\cdot}(f)\, u^n$.

Let $a : X. \to S$ be a simplicial topological space augmented to S. If $\mathcal{F}$ is a sheaf on S, $a^*\mathcal{F} = (a_n{}^*\mathcal{F})_{n\geq 0}$ is a sheaf on $X.$. The functor a^* has a right adjoint

$$a_* : \mathcal{F}^{\cdot} \to \mathrm{Ker}\,(a_{0*}\mathcal{F}^0 \underset{\delta_1^*}{\overset{\delta_0^*}{\rightrightarrows}} a_{1*}\mathcal{F}^1).$$

Definition 1. *Let* $a : X. \to S$ *be an augmented simplicial topological space. One says that* a *is of* cohomological descent *if for any abelian sheaf* $\mathcal{F}$ *on* S, *one has*

$$\mathcal{F} \xrightarrow{\sim} \mathrm{Ker}\,(a_{0*}a_0{}^*\mathcal{F} \rightrightarrows a_{1*}a_1{}^*\mathcal{F})$$

and

$$R^i a_* a^* \mathcal{F} = 0 \quad \text{for } i > 0.$$

Definition 2. *A continuous mapping* $a : X \to S$ *is of cohomological descent if the augmentation morphism of* $\mathrm{cosq}\,(X \to S)$:

$$((X/S)^{\Delta_n})_{n\geq 0} \to S$$

is of cohomological descent.

Lemma 1. *For a surjective continuous mapping* $X \to S$, *we denote the augmentation morphism of* $\mathrm{cosq}\,(X \to S)$ *by* $a : ((X/S)^{\Delta_n})_{n\geq 0} \to S$. *Then for any abelian sheaf* $\mathcal{F}$ *on* S, *we have*

$$R^i a_* a^* \mathcal{F} = 0 \quad \text{for } i > 0.$$

Proof. We construct an injective resolution $\mathcal{T}^{\cdot\cdot}$ of the sheaf $a^*\mathcal{F}$ on the simplicial topological space $((X/S)^{\Delta_n})_{n\geq 0}$ as follows :

Let $\mathcal{T}^{0\cdot}$ be an injective resolution of $a_0{}^*\mathcal{F}$. Since $\delta_0{}^*a_0{}^*\mathcal{F} = \delta_1{}^*a_0{}^*\mathcal{F} = a_1{}^*\mathcal{F}$, we define $\mathcal{M}^0$ by the short exact sequence

$$0 \to a_1{}^*\mathcal{F} \to \delta_0{}^*\mathcal{T}^{00} \oplus \delta_1{}^*\mathcal{T}^{00} \to \mathcal{M}^0 \to 0,$$

where $a_1{}^*\mathcal{F} \to \delta_0{}^*\mathcal{T}^{00} \oplus \delta^*\mathcal{T}^{00}$ is the monomorphism defined by $x \mapsto (x', -x'')$, where $x \mapsto x'$(resp. $x \mapsto x''$) is the natural injection $\delta_0{}^*a_0{}^*\mathcal{F} \to \delta_0{}^*\mathcal{T}^{00}$ (resp. $\delta_1{}^*a_0{}^*\mathcal{F} \to \delta_1{}^*\mathcal{T}^{00}$). For $(u, v) \in \delta_0{}^*\mathcal{T}^{00} \oplus \delta_1{}^*\mathcal{T}^{00}$, we denote the image of (u, v) in $\mathcal{M}^0$ by $[u, v]$. Then for any $x \in a_1{}^*\mathcal{F}$, $[x', 0] - [0, x''] = [x', -x''] = 0$. Thus we have natural monomorphisms $\delta_i{}^*\mathcal{T}^{00} \to \mathcal{M}^0$ $i = 0, 1$ and $a_1{}^*\mathcal{F} \to \mathcal{M}^0$ defined by $x \mapsto [x', 0] = [0, x'']$. And if we embedd $\mathcal{M}^0$ into some injective $\mathcal{T}^{10}$, we have monomorphisms $p_i^0 : \delta_i{}^*\mathcal{T}^{00} \to \mathcal{T}^{10}$ $i = 0, 1$ and $a_1{}^*\mathcal{F} \to \mathcal{T}^{10}$ which make a commutative diagram :

$$\begin{array}{ccccccc}
0 & \to & \delta_0{}^*a_0{}^*\mathcal{F} & \to & \delta_0{}^*\mathcal{T}^{00} & \to \\
 & & \| & & \downarrow p_0{}^0 & \\
0 & \to & a_1{}^*\mathcal{F} & \to & \mathcal{T}^{10} & \\
 & & \| & & \uparrow p_1{}^0 & \\
0 & \to & \delta_1{}^*a_0{}^*\mathcal{F} & \to & \delta_1{}^*\mathcal{T}^{00} & \to
\end{array}$$

If we have a commutative diagram

$$(*_i)\qquad \begin{array}{ccccccccccc} 0 & \to & \delta_0{}^*a_0{}^*\mathcal{F} & \to & \delta_0{}^*\mathcal{T}^{00} & \to & \cdots & \xrightarrow{d'^{i-1}} & \delta_0{}^*\mathcal{T}^{0i} & \to & \\ & & \| & & \downarrow p_0{}^0 & & & & \downarrow p_0{}^i & & \\ 0 & \to & a_1{}^*\mathcal{F} & \to & \mathcal{T}^{10} & \to & \cdots & \xrightarrow{d^{i-1}} & \mathcal{T}^{1i} & & \\ & & \| & & \uparrow p_1{}^0 & & & & \uparrow p_1{}^i & & \\ 0 & \to & \delta_1{}^*a_0{}^*\mathcal{F} & \to & \delta_1{}^*\mathcal{T}^{00} & \to & \cdots & \xrightarrow{d''^{i-1}} & \delta_1{}^*\mathcal{T}^{0i} & \to & , \end{array}$$

where each row is exact and $\mathcal{T}^{1k}$ ($k=0, 1, \ldots, i$) are injective, then we can define a commutative diagram :

$$\begin{array}{ccc} \delta_0{}^*\mathcal{T}^{0i} & \xrightarrow{d'^i} & \delta_0{}^*\mathcal{T}^{0i+1} \\ \downarrow p_0{}^i & & \downarrow p_0{}^{i+1} \\ \mathcal{T}^{1i} & \xrightarrow{d^i} & \mathcal{T}^{1i+1} \\ \uparrow p_1{}^i & & \uparrow p_1{}^{i+1} \\ \delta_1{}^*\mathcal{T}^{0i} & \xrightarrow{d''^i} & \delta_1{}^*\mathcal{T}^{0i+1}, \end{array}$$

where $\mathcal{T}^{1i+1}$ is injective, as follows : Let $\mathcal{N}^i$ be the subsheaf of $\delta_0{}^*\mathcal{T}^{0i+1} \oplus \mathcal{T}^{1i} \oplus \delta_1{}^*\mathcal{T}^{0i+1}$ generated by

$$\{(d'^i(x), -p_0{}^i(x), 0) \mid x \in \delta_0{}^*\mathcal{T}^{0i}\} \cup \{(0, -p_1{}^i(x), d''^i(x)) \mid x \in \delta_1{}^*\mathcal{T}^{0i}\}.$$

We embedd $(\delta_0{}^*\mathcal{T}^{0i+1} \oplus \mathcal{T}^{1i} \oplus \delta_1{}^*\mathcal{T}^{0i+1})/\mathcal{N}^i$ into some injective $\mathcal{T}^{1i+1}$. Then we have natural homomorphisms $d^i : \mathcal{T}^{1i} \to \mathcal{T}^{1i+1}$, $p_0{}^{i+1} : \delta_0{}^*\mathcal{T}^{0i+1} \to \mathcal{T}^{1i+1}$, and $p_1{}^{i+1} : \delta_1{}^*\mathcal{T}^{0i+1} \to \mathcal{T}^{1i+1}$ which make the above diagram commutative and $\mathrm{Im}\ d^{i-1} = \mathrm{Ker}\ d^i$. Thus for any $i \geq 0$ we have the commutative diagram $(*_i)$. Thus we have defined injectuve resolutions $\mathcal{T}^{j\cdot}$ of sheaves $a_j{}^*\mathcal{F}$ for $j=0, 1$, and we can also define them for $j \geq 2$ such that $\mathcal{T}^{\cdot\cdot}$ gives an injective resolution of $a^*\mathcal{F}$. Now applying $a_1{}^* = a_{0*}\delta_{0*} = a_{0*}\delta_{1*}$ to the diagrm $(*_i)$, we have a commutative diagram :

$$(**_i)\qquad \begin{array}{ccccccccccc} 0 & \to & a_{0*}\delta_{0*}\delta_0{}^*a_0{}^*\mathcal{F} & \to & a_{0*}\delta_{0*}\delta_0{}^*\mathcal{T}^{00} & \to & \cdots & \xrightarrow{d'^{i-1}} & a_{0*}\delta_{0*}\delta_0{}^*\mathcal{T}^{0i} & \to \\ & & \| & & \downarrow p_0{}^0 & & & & \downarrow p_0{}^i & \\ 0 & \to & a_{1*}a_1{}^*\mathcal{F} & \to & a_{1*}\mathcal{T}^{10} & \to & \cdots & \xrightarrow{d^{i-1}} & a_{1*}\mathcal{T}^{1i} & \\ & & \| & & \uparrow p_1{}^0 & & & & \uparrow p_1{}^i & \\ 0 & \to & a_{0*}\delta_{1*}\delta_1{}^*a_0{}^*\mathcal{F} & \to & a_{0*}\delta_{1*}\delta_1{}^*\mathcal{T}^{00} & \to & \cdots & \xrightarrow{d''^{i-1}} & a_{0*}\delta_{1*}\delta_1{}^*\mathcal{T}^{0i} & \to \end{array}$$

The two homomorphisms

$$a_{0*}\mathcal{T}^{0\cdot} \rightrightarrows a_{1*}\mathcal{T}^{1\cdot}$$

corresponding to the two homomorphisms

$$a_{0*}a_0{}^*\mathcal{F} \rightrightarrows a_{1*}a_1{}^*\mathcal{F}$$

are given by $p_0{}^\cdot$ and $p_1{}^\cdot$, where we denote the composed homomorphisms of

$$a_{0*}\mathcal{T}^{0\cdot} \to a_{0*}\delta_{i*}\delta_i{}^*\mathcal{T}^{0\cdot}$$

and $p_i{}^\cdot$ also by $p_i{}^\cdot$ ($i=0, 1$). Define the complex $\mathcal{K}^\cdot$ of abelian sheaves on S by

$$\mathcal{K}^\cdot = \mathrm{Ker}\,(a_{0*}\mathcal{T}^{0\cdot} \rightrightarrows a_{1*}\mathcal{T}^{1\cdot}).$$

Then we have

$$R^i a_* a^*\mathcal{F} = H^i(\mathcal{K}^\cdot).$$

Let $f \in (\mathcal{K}^i)_s$, $s \in S$, $i > 0$. Then for some open neighbourhood U of s, there is a section in $\Gamma(\mathcal{T}^{0i}, a_0{}^{-1}(U))$ representing f which is also denoted by f. By the assumption the images of $\delta_0{}^*f$ and $\delta_1{}^*f$ in $\mathcal{T}^{1i}$ are the same, so for any point $t \in a_1{}^{-1}(s) \subset X_1 = X_0 \times_S X_0$ there exist $x_{i-1}{}' \in (\delta_0{}^*\mathcal{T}^{0i-1})_t$

and $x_{i-1}'' \in (\delta_1{}^*\mathcal{T}^{0i-1})_t$ such that

$$(\delta_0{}^*f)_t = d'^{i-1}(x_{i-1}'),\ (\delta_1{}^*f)_t = d''^{i-1}(x_{i-1}''),\ p_0{}^{i-1}(x_{i-1}') = p_1{}^{i-1}(x_{i-1}'').$$

Since $p_0{}^{i-1}(x_{i-1}') = p_1{}^{i-1}(x_{i-1}'')$ means that the images of x_{i-1}' and x_{i-1}'' in $\mathcal{T}^{1i-1}$ are the same, there exist $x_{i-2}' \in (\delta_0{}^*\mathcal{T}^{0i-2})$ and $x_{i-2}'' \in (\delta_1{}^*\mathcal{T}^{0i-2})$ such that

$$x_{i-1}' = d'^{i-2}(x_{i-2}'),\ x_{i-1}'' = d''^{i-2}(x_{i-2}''),\ p_0{}^{i-2}(x_{i-2}') = p_1{}^{i-2}(x_{i-2}''),$$

where $\delta_0{}^*\mathcal{T}^{0,-1} = \delta_1{}^*\mathcal{T}^{0,-1} = \mathcal{T}^{1,-1} = \delta_0{}^*a_0{}^*\mathcal{F} = \delta_1{}^*a_0{}^*\mathcal{F} = a_1{}^*\mathcal{F}$ and $p_0{}^{-1} = p_1{}^{-1} = id_{a_1{}^*\mathcal{F}}$.

Hence we have

$$(\delta_0{}^*f)_t = d'^{i-1}(x_{i-1}') = d'^{i-1}d'^{i-2}(x_{i-2}') = 0,\ (\delta_1{}^*f)_t = d''^{i-1}(x_{i-1}'') = d''^{i-1}d''^{i-2}(x_{i-2}'') = 0.$$

Since δ_i is surjective we have $f=0$ and therefore $\mathcal{K}^i=0$ for $i>0$. q. e. d.

It follows from the above proof that

$$a_*a^*\mathcal{F} = \mathrm{Ker}\,(a_{0*}a_0{}^*\mathcal{F} \rightrightarrows a_{1*}a_1{}^*\mathcal{F}) = H^0(\mathcal{K}^\cdot) = \mathcal{K}^0.$$

And for any point $s \in S$,

$$(\mathcal{K}^0)_s = \varinjlim_{U \ni s} \{ f \in \Gamma\,(a_0{}^*\mathcal{F},\ a_0{}^{-1}(U)) \mid (\delta_0{}^*f)_t = (\delta_1{}^*f)_t \text{ for any } t \in a_1{}^{-1}(s) \}.$$

If we write $t \in a_1{}^{-1}(s) \subset X_0 \times_S X_0 = X_1$ as $t=(t_0, t_1)$, $t_i \in a_0{}^{-1}(s) \subset X_0$, then we have

$$(\delta_0{}^*f)_{(t_0, t_1)} = (f)_{t_0},\ (\delta_1{}^*f)_{(t_0, t_1)} = (f)_{t_1}.$$

Therfore

$$(\mathcal{K}^0)_s = \varinjlim_{U \ni s} \{ f \in \Gamma\,(a_0{}^*\mathcal{F},\ a_0{}^{-1}(U) \mid (f)_t \in (a_0{}^*\mathcal{F})_t = (\mathcal{F})_s \text{ is constant for all } t \in a_0{}^{-1}(s) \}.$$

Obviously, the image of natural injection $\mathcal{F} \to a_{0*}a_0{}^*\mathcal{F}$ is contained in $\mathcal{K}^0$.

Proposition. *A surjective continuos mapping $\varphi : X \to S$ is of cohomological descent if the following condition (♯) is satisfied :*

(♯) *For any point $s \in S$ and any open neighbourhood V of $\varphi^{-1}(s)$ in X, the image $\varphi(V)$ of V contains a neighbourhood of s.*

Proof. We denote the augmentation mapping of cosq $(\varphi : X \to S)$ by $a = a.: ((X/S)^{\Delta^n})_{n\geq 0} \to S$. Especially $a_0 : X_0 \to S$ is $\varphi : X \to S$. Assume that (♯) is satisfied. By the above lemma it suffices to prove that for every abelian sheaf $\mathcal{F}$ on S, $\mathcal{F}=\mathcal{K}^0$, where $\mathcal{K}^0$ is defined as in Lemma 1 for $a_0=\varphi$. Let $s \in S$. For any open neighbourhood U of s in S, take

$$f \in \Gamma\,(\mathcal{K}^0,\ U) \subset \Gamma\,(a_{0*}a_0{}^*\mathcal{F},\ U) = \Gamma\,(\varphi_*\varphi^*\mathcal{F},\ U) = \Gamma\,(\varphi^*\mathcal{F},\ \varphi^{-1}(U)).$$

For every point $u \in \varphi^{-1}(U)$, f defines an element $(f)_u \in (\varphi^*\mathcal{F})_u = (\mathcal{F})_{\varphi(u)}$, and since f is a local section of $\mathcal{K}^0$, $(f)_u$ depends only on $\varphi(u)$. For every $t \in \varphi^{-1}(s)$, take an open neighbourhood V_t of t in $\varphi^{-1}(U) \subset X$ such that a section $f'_t \in \Gamma\,(\varphi^*\mathcal{F},\ V_t)$ represents $(f)_t$ and $(f)_u = (f'_t)_u$ for any $u \in V_t$, and that there exist an open set $U'_t \supset \varphi(V_t)$ and a local section $f''_t \in \Gamma\,(\mathcal{F}, U'_t)$ which repersents f'_t, i. e. $(f'_t)_u = (f''_t)_{\varphi(u)}$ for any $u \in V_t$. Let $V = \cup_t V_t$, then by the assumption (♯), $\varphi(V)$ contains an open neighbourhood U_0 of s in S, and since $U_0 \subset \varphi(V) = \varphi(\cup_t V_t) \subset \cup_t U'_t$, we have an open covering $\{U_{0t}\}$ of U_0 with $U_{0t} = U_0 \cap U'_t$. Then for any point $x \in U_0$, there exist $t \in \varphi^{-1}(s)$ and $u \in V_t$ such that $\varphi(u) = x$ and $(f''_t)_x = (f''_t)_{\varphi(u)} = (f'_t)_u = (f)_u$. And for any other pair of u' and $V_{t'}$ such that $x=\varphi(u')$ and $V_{t'} \ni u'$, we have $(f''_{t'})_x = (f)_{u'}$, and since $\varphi(u) = \varphi(u') = x$, we have $(f)_u = (f)_{u'}$ and therefore $(f''_t)_x = (f''_{t'})_x$. Thus the collection of local sections $\{f''_t | U_{0t}\}_t$ forms a section in $\Gamma\,(\mathcal{F}, U_0)$ which we denote by f''. Then for any point $x \in U_0$ and for any point $u \in \varphi^{-1}(x)$, we have

$$(f)_u = (f'')_x\ ,$$

which means that

$$f \mid U_0 = f'' \in \Gamma(\mathcal{F}, U_0)$$

as an element in $\Gamma(\varphi_*\varphi^*\mathcal{F}, U_0)$. Therefore we have $(\mathcal{K}^0)_s \subset (\mathcal{F})_s$ and this completes the proof of the assertion that $\mathcal{F} = \mathcal{K}^0$. q. e. d.

Lemma 2. *Any proper surjective mapping satisfies the condition* (#) *in Proposition.*

Proof. Let $\varphi : X \to S$ be a proper surjective mapping. For any point $s \in S$, take an open neighbourhood V of $\varphi^{-1}(s)$ in X. Since φ is proper, $\varphi(X-V)$ is closed in S, and therefore $U = S - \varphi(X-V)$ is open. But since φ is surjective and $V \supset \varphi^{-1}(s)$, it is obvious that $\varphi(V) \supset U \ni s$. q. e. d.

Lemma 3. *Any surjective continuous mapping with local sections satisfies the condition* (#) *in Proposition.*

Proof. Let $\varphi : X \to S$ be a surjective continuous mapping with local sections. For any point $s \in S$, take an open neighbourhood U of s and a local section $\alpha : U \to X$. For any open neighbourhood V of $\varphi^{-1}(s)$, since $V \ni \alpha(s)$ and α is continuous, there exists an open neighbourhood U_0 of s in S such that $\alpha(U_0) \subset V$. Hence $\varphi(V) \supset \varphi(\alpha(U_0)) = U_0$. q. e. d.

Combining lemma 2 and 3 with Proposition, we obtain Fact 1 and Fact 2.

Remark. The condition (#) in Proposition is weaker than the following condition (#'):
(#') *For any point* $s \in S$, *the family of open sets* $\{\varphi^{-1}(U) \mid U$ is a open neighbourhood of $s\}$ *is a fundamental system of neighbourhoods of* $\varphi^{-1}(s)$ *in* X.

If φ is a proper surjective mapping, (#') is satisfied. Here we give some examples of surjective continuous mappings which are not proper.

Example 1. The exponential mapping

$$\mathbf{R} \to S^1 \simeq \mathbf{R} / \mathbf{Z} \ (x \mapsto e^{2\pi i x})$$

is not proper but has local sections, hence is of cohomological descent.

Example 2. $\mathrm{Id}_X : X_{Ann} \to X_{Zar}$ does not satisfy the condition (#), where X_{Zar} is an algebraic variety over the complex number field C with Zariski topology and X_{Ann} is the corresponding analytic variety with usual complex topology.

Example 3. For a topological space X, $\mathrm{Id}_X : X_{dis} \to X$ does not satisfy the condition (#) in general, where X_{dis} is defined by giving the discrete topology to the underlying set of X.
It can be easily seen that example 3 is not of cohomological descent.

References

[1] Deligne, P., *Théorie de Hodge, III,* Publ. Math. IHES 44, 5-77 (1974).

[2] Saint-Donat, B., *Techniques de descente cohomologique,* (*SGA 4 Exposé* V^{bis}), Lecture Notes in Math. 270, Springer-Verlag (1972).