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k-mersions and k-morphisms

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Abstract

In this article, we study the existence and classification of k-mersions of manifolds via k-morphisms of vector bundles.

Introduction

A differentiable map $g: M \to N$ between differentiable manifolds without boundary is called a k-mersion if g has rank at least k at each point of M. In particular, a k-mersion is an immersion or a submersion according as $k = \dim M$ or $k = \dim N$. The space of k-mersions of M to N, endowed with C^1-topology, is denoted by $k(M, N)$. The two k-mersions are said to be k-regularly homotopic if they are joined by a path in $k(M,N)$. We denote by $k[M, N]_{[f]}$ the set of k-regular homotopy classes of k-mersions homotopic to a given map $f: M \to N$. In particular, the set $k[M, N]_{[f]}$ is equal to the regular homotopy set $I[M, N]_{[f]}$ of immersions homotopic to f or the regular homotopy set $S[M, N]_{[f]}$ of submersions homotopic to f, according as $k = \dim M$ or $k = \dim N$.

A morphism $g: \xi \to \zeta$ between real vector bundles is called a k-morphism if its restriction to each fiber is of rank at least k. The space of k-morphisms of ξ to ζ with compact open topology is denoted by $k(\xi, \zeta)$. Moreover, for a map $f: X \to Y$ and two real vector bundles ξ over X and ζ over Y, let $k(\xi, \zeta)_f$ and $k(\xi, \zeta)_{[f]}$ be the subspaces of $k(\xi, \zeta)$ consisting of k-morphisms covering f and of k-morphisms covering maps homotopic to f, respectively. Further, let $k[\xi, \zeta]$, $k[\xi, \zeta]_f$ and $k[\xi, \zeta]_{[f]}$ be the homotopy set of $k(\xi, \zeta)$, $k(\xi, \zeta)_f$ and $k(\xi, \zeta)_{[f]}$, respectively.

Under these circumstances, Fait [1, Theorem 1] has proved that if $k < \dim N$ or M is an open manifold then the correspondence that maps k-mersions to their differentials induces a bijection

$$k[M, N]_{[f]} = k[\tau_M, \tau_N]_{[f]} \qquad \text{for any map } f: M \to N,$$

where τ_M is the tangent bundle of M.

In this paper, we study the existence and classification problem for k-meriosns via k-morphisms and we obtain some relations between k-mersions of M to N and

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those of M to the euclidean space. In particular, we get some existence and classification theorems both for immesions and submersions simultaneously (see Theorem 7.6). For our purpose, it is important to study the existence and classification of k-morphisms. We do this by using the way similar to that used by Li and Habegger [3]-[6].

In the forthcoming paper [9], we will study the set $n[\xi, \zeta]_f$ for $n = \dim \zeta$ more exactly and get some results concerning the regular homotopy sets $S[M, P^n]_{[f]}$ of submersions of certain open manifolds to the real projective space.

The remainder of this paper is organized as follows: In § 1, we construct two fiber bundles $\beta_f(\xi, \zeta; k)$ and $\tilde{\beta}(\xi, \zeta; k)$ associated with ξ, ζ, and $f: X \to Y$. In § 2, we see that the space $k(\xi, \zeta)$ and $k(\xi, \zeta)_f$ are homeomorphic, respectively, to the space $\gamma(\tilde{\beta}(\xi, \zeta; k))$ of cross sections of $\tilde{\beta}(\xi, \zeta; k)$ and to the space $\gamma(\beta_f(\xi, \zeta; k))$ of cross sections of $\beta_f(\xi, \zeta; k)$, and the natural inclusion $k(\xi, \zeta)_f \to k(\xi, \zeta)_{[f]}$ leads to a surjection $k[\xi, \zeta]_f \to k[\xi, \zeta]_{[f]}$. In §3, we define an action of the fundamental group $\pi_1(Y^X, f)$ on $k[\xi, \zeta]_f$, and in §6, we show that the above surjection induces a bijection of the quotient set $k[\xi, \zeta]_f/\pi_1(Y^X, f)$ to $[\xi, \zeta]_{[f]}$. In §4, we will give some examples of the trivial actions of the non-trivial groups $\pi_1(Y^X, f)$ on $k[\xi, \zeta]_f$. §5 is devoted to a study of the stabilization map $k[\xi, \zeta]_f \to (k+l)[\xi \oplus \theta^l_X, \zeta \oplus \theta^l_Y]_f$, where θ^l_Z is a trivial l-plane bundle over a space Z. In the last section (§7), we will give some conditions, namely, that for a map $f: M \to N$, the existence of a k-mersion homotopic to f is equivalent to that of a k-mersion of M to the n-dimensional enclidean space R^n ($n = \dim N$) (see Theorem 7.3) and that there exists a bijection between k-regular homotopy sets $k[M, N]_{[f]}$ and $k[M, R^n]$ (see Theorem 7.5).

§1. Bundles $\beta_f(\xi, \zeta; k)$ and $\tilde{\beta}(\xi, \zeta; k)$.

For $k \leq \min\{m, n\}$, let $M^*(n, m; k)$ denote the space consisting of all real $n \times m$-matrices of rank at least k and let $G(m)$ denote either $GL(m, R)$ or $O(m)$. Then $G(m) \times G(n)$ acts on $M^*(n, m; k)$ from the left by the equation $(A, B)C = BCA^{-1}$, where $A \in G(m)$, $B \in G(n)$ and $C \in M^*(n, m; k)$.

In what follows, ξ and ζ will mean, respectively, a real m-plane bundle over a CW-complex X and a real n-plane bundle over Y. The bundle ξ_m and ζ_n mean the principal $G(m)$-and $G(n)$-bundles associated with ξ and ζ, respectively. The space $B(\xi, \zeta; k)$ is defined by

$$B(\xi, \zeta; k) = (\xi_m \times \zeta_n) \times_{G(m) \times G(n)} M^*(n, m; k).$$

It is easily seen that

$$B(\xi, \zeta; k) = \xi_m \times_{G(m)} (\zeta_n \times_{G(n)} M^*(n, m; k)).$$

Let $p: B(\xi, \zeta; k) \to X$ and $q: B(\xi, \zeta; k) \to X \times Y$ be the maps defined by the natural projections. Then we have the following

Lemma 1.1 (cf. Li [3]). (i) $\tilde{\beta}(\xi, \zeta ; k) = (p : B(\xi, \zeta ; k) \to X)$ *is a fiber bundle with fiber* $\zeta_n \times_{G(n)} M^*(n, m ; k)$ *and with structure group* $G(m)$.

(ii) $\beta(\xi, \zeta ; k) = (q : B(\xi, \zeta ; k) \to X \times Y)$ *is a fiber bundle with fiber* $M^*(n, m ; k)$ *and with structure group* $G(m) \times G(n)$.

Remark. The element $[u_x, v_y, C]$ of $B(\xi, \zeta ; k)$ $(x \in X,\ y \in Y)$ can be regarded as a linear map of ξ_x to ζ_y with C as its matrix relative to the bases u_x and v_y, where ξ_x and ζ_y are fibers of ξ at x and of ζ at y, respectively.

Both of the maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ stand for the natural projections. Then we have a commutative diagram

$$\begin{tikzcd} & B(\xi, \zeta ; k) \arrow[dl, "p"'] \arrow[d, "q"] \\ X & X \times Y. \arrow[l, "P_1"] \end{tikzcd}$$

For a map $f : X \to Y$, let $(1_X, f) : X \to X \times Y$ be a map defined by $(1_X, f)(x) = (x, f(x))$ and let

$$\beta_f(\xi, \zeta ; k) = (1_X, f)^* \beta(\xi, \zeta ; k)$$

be the pull-back of $\beta(\xi, \zeta ; k)$ along $(1_X, f)$. Then we have

Lemme 1.2. *The following properties hold*:

(i) (cf. Li and Habegger [6, 3.2]) *for* $f : X' \to X$ *and* $g : Y' \to Y$,

$$(f \times g)^* \beta(\xi, \zeta ; k) = \beta(f^* \xi, g^* \zeta ; k),$$

(ii) *for* $f : X \to X'$ *and* $g : X' \to Y$,

$$\beta_f(\xi, g^* \zeta ; k) = \beta_{gf}(\xi, \zeta ; k),$$

and in particular $\beta_{1_X}(\xi, f^* \zeta ; k) = \beta_f(\xi, \zeta ; k)$ *for* $f : X \to Y$,

(iii) *for* $h : X' \to X$ *and* $f : X \to Y$,

$$h^* \beta_f(\xi, \zeta ; k) = \beta_{fh}(h^* \xi, \zeta ; k).$$

§ 2. k-morphisms and cross sections.

This section is devoted to a study of the relations between k-morphisms of ξ to ζ and cross sections of $\tilde{\beta}(\xi, \zeta ; k)$ and $\beta_f(\xi, \zeta ; k)$. Let $\gamma(\tilde{\beta}(\xi, \zeta ; k))$ be the space of cross sections of the bundle $\tilde{\beta}(\xi, \zeta ; k)$, and let $\gamma(\tilde{\beta}(\xi, \zeta ; k))_f$ and $\gamma(\tilde{\beta}(\xi, \zeta ; k))_{[f]}$ for a map $f : X \to Y$ be its subspaces defined by

$$\gamma(\tilde{\beta}(\xi,\zeta;k))_f=\{s\in\gamma(\tilde{\beta}(\xi,\zeta;k))\mid p_2qs=f\},$$
$$\gamma(\tilde{\beta}(\xi,\zeta;k))_{[f]}=\{s\in\gamma(\tilde{\beta}(\xi,\zeta;k))\mid p_2qs\simeq f\}.$$

Moreover let $\Gamma(\tilde{\beta}(\xi,\zeta;k))$, $\Gamma(\tilde{\beta}(\xi,\zeta;k))_f$ and $\Gamma(\tilde{\beta}(\xi,\zeta;k))_{[f]}$ be the sets of path components of the above respective spaces. Then it is easy to see that the map $\phi:k(\xi,\zeta)\to\gamma(\tilde{\beta}(\xi,\zeta;k))$ defined by $\phi(g)(x)=g\mid\xi_x$, the restriction of g to the fiber ξ_x, is a homeomorphism. This map ϕ induces homeomorphisms $\phi_f:k(\xi,\zeta)_f\to\gamma(\tilde{\beta}(\xi,\zeta;k))_f$ and $\phi_{[f]}:k(\xi,\zeta)_{[f]}\to\gamma(\tilde{\beta}(\xi,\zeta;k))_{[f]}$.

Let $\gamma(\beta_f(\xi,\zeta;k))$ be the space of cross sections of $\beta_f(\xi,\zeta;k)$ and $\Gamma(\beta_f(\xi,\zeta;k))$ the set of its path components. Then we can naturally identify $\gamma(\tilde{\beta}(\xi,\zeta;k))_f$ with $\gamma(\beta_f(\xi,\zeta;k))$. Hence we regard $\Gamma(\tilde{\beta}(\xi,\zeta;k))_f$ and $\Gamma(\beta_f(\xi,\zeta;k))$ as identical. Thus we get

Proposition 2.1. *There are three bijections*

$$\phi_*:k[\xi,\zeta]\to\Gamma(\tilde{\beta}(\xi,\zeta;k)),$$
$$\phi_{f*}:k[\xi,\zeta]_f\to\Gamma(\tilde{\beta}(\xi,\zeta;k))_f=\Gamma(\beta_f(\xi,\zeta;k)),$$
$$\phi_{[f]*}:k[\xi,\zeta]_{[f]}\to\Gamma(\tilde{\beta}(\xi,\zeta;k))_{[f]}.$$

From now no, we frequently identify the sets $k[\xi,\zeta]$, $k[\xi,\zeta]_f$ and $k[\xi,\zeta]_{[f]}$ with $\Gamma(\tilde{\beta}(\xi,\zeta;k))$, $\Gamma(\beta_f(\xi,\zeta;k))$ and $\Gamma(\tilde{\beta}(\xi,\zeta;k))_{[f]}$, respectively, by means of the above bijections, and we study cross sections of $\tilde{\beta}(\xi,\zeta;k)$ and $\beta_f(\xi,\zeta;k)$ instead of k-morphisms of ξ to ζ.

The natural inclusions $i_f:k(\xi,\zeta)_f\to k(\xi,\zeta)_{[f]}$ and $i_f:\gamma(\beta_f(\xi,\zeta;k))\to\gamma(\tilde{\beta}(\xi,\zeta;k))_{[f]}$ induce the maps

$$i_{f*}:k[\xi,\zeta]_f\to k[\xi,\zeta]_{[f]}\quad\text{and}\quad i_{f*}:\Gamma(\beta_f(\xi,\zeta;k))\to\Gamma(\tilde{\beta}(\xi,\zeta;k))_{[f]}.$$

In the same way as in the proof of Li [3, Theorem 3], we have

Proposition 2.2. *The map i_{f*} is surjective.*

From here on, the bundle θ^l_Z will mean the trivial l-plane bundle over a space Z and a map c stands for a constant map of spaces.

Proposition 2.3. *Assume that $f^*\zeta$ is trivial. Then there is a bijection between the sets $k[\xi,\zeta]_f$ and $k[\xi,\theta^n_Z]_c$ for any space Z, and in particular there exists a k-morphism of ξ to ζ covering a map homotopic to f if and only if there is a k-morphism of ξ to θ^n_Z covering a null-homotopic map.*

Proof. The result follows immediately from Lemma 1.2(ii), Propositions 2.1-2.2, and the assumption $f^*\zeta=\theta^n_X$.

§ 3. $\pi_1(Y^X, f)$-action on $k[\xi, \zeta]_f$.

We begin this section by defining a $\pi_1(Y^X, f)$-action on the set $\Gamma(\beta_f(\xi, \zeta ; k))$ for a map $f : X \to Y$. For a cross section $s_0 \in \gamma(\beta_f(\xi, \zeta ; k))$, i.e., a map $s_0 : X \to B(\xi, \zeta ; k)$ such that $qs_0 = (1_X, f)$, and for a homotopy $f_t : X \to Y$ such that $f_0 = f_1 = f$, by the homotopy covering property of bundles there exists a homotopy $s_t : X \to B(\xi, \zeta ; k)$ lifting a homotopy $(1_X, f_t)$. Since $p_2 q s_1 = f$, we regard s_1 as a cross section of $\beta_f(\xi, \zeta ; k)$ and describe $[s_1] \in \Gamma(\beta_f(\xi, \zeta ; k))$ as $[s_0][f_t]$. It is easily seen that the element $[s_1]$ depends only on the homotopy classes $[s_0] \in \Gamma(\beta_f(\xi, \zeta ; k))$ and $[f_t] \in \pi_1(Y^X, f)$, and that the map $\Gamma(\beta_f(\xi, \zeta ; k)) \times \pi_1(Y^X, f) \to \Gamma(\beta_f(\xi, \zeta ; k))$ thus defind is an action on $\Gamma(\beta_f(\xi, \zeta ; k))$ from the right (cf. [4, § 1]).

Therefore we can define a right action of $\pi_1(Y^X, f)$ on $k[\xi, \zeta]_f$ by means of the bijections ϕ_{f*} of Proposition 2.2, as follows: Given a k-morphism $\Psi_0 : \xi \to \zeta$ covering f and a homotopy $f_t : X \to Y$ such that $f_0 = f_1 = f$, there exists a homotopy $\Psi_t : \xi \to \zeta$ of k-morphisms covering f_t. We write $[\Psi_1]$ as $[\Psi_0][f_t]$.

For a map $g : Y \to Z$, let $g' : (Y^X, f) \to (Z^X, gf)$ be a map defined by $g'(h) = gh$ and let $g^{\prime}_{\#} : \pi_1(Y^X, f) \to \pi_1(Z^X, gf)$ be a homomorphism induced from g'. Then the definition of the $\pi_1(Y^X, f)$-action on $k[\xi, \zeta]_f$ and the argument similar to that in [6, § 3] lead to

Proposition 3.1. *If $\tilde{g} : \zeta \to \eta$ is a pull-back of η along $g : Y \to Z$, then the map $g_* : k[\xi, \zeta]_f \to k[\xi, \eta]_{gf}$ defined by $g_*[\Psi] = [\tilde{g}\Psi]$ is a bijection and moreover $g^{\prime}_{\#}$-equivariant, meaning that $g_*(\Psi F) = g_*(\Psi) g^{\prime}_{\#}(F)$ for $\Psi \in k[\xi, \zeta]_f$ and $F \in \pi_1(Y^X, f)$.*

In the same way as in [6, 2.2] we have the following

Corollary 3.2. *If ζ is r-trivial and if* $\dim X < r$, *then the $\pi_1(Y^X, f)$-action on $k[\xi, \zeta]_f$ is trivial.*

Here ζ is said to be r-trivial if its restriction to the r-skeleton is trivial for some CW-decomposition of its base space.

For a map $h : Z \to X$, let $h'' : (Y^X, f) \to (Y^Z, fh)$ be a map defined by $h''(g) = gh$ and $\tilde{h} : h^*\xi \to \xi$ be the pull-back of ξ along h. Both of the maps h'' and $\tilde{h}$ lead to a map $h^* : k[\xi, \zeta]_f \to k[h^*\xi, \zeta]_{fh}$ given by $h^*[\Psi] = [\Psi\tilde{h}]$ and a homomorphism $h^{\prime\prime}_{\#} : \pi_1(Y^X, f) \to \pi_1(Y^Z, fh)$. By simple calculation, we get

Proposition 3.3. *For a map $h : Z \to X$, the map $h^* : k[\xi, \zeta]_f \to k[h^*\xi, \zeta]_{fh}$ is $h^{\prime\prime}_{\#}$-equivariant, meaning thah $h^*(\Psi F) = h^*(\Psi) h^{\prime\prime}_{\#}(F)$ for $\Psi \in k[\xi, \zeta]_f$ and $F \in \pi_1(Y^X, f)$.*

Corollary 3.4. *If $h: Z \to X$ is a homotopy equivalence, then $h^*: k[\xi, \zeta]_f \to k[h^*\xi, \zeta]_{fh}$ is an $h_\#^\#$-equivariant bijection.*

§4. Examples.

In this section, we shall give some examples of the trivial actions of non-trivial groups $\pi_1(Y^X, f)$ on $k[\xi, \zeta]_f$.

Let $P^n(R)$ and λ_n be the real projective n-space and its canonical real line bundle, respectively, and let $f: X \to P^n(R)$ be a base point preserving map. By using the homotopy exact sequence of the fibration $P^n(R)^X \to P^n(R)$ given by the evaluation map, and the Eilenberg classification theorem, we have the following

Lemma 4.1. (Li [4, Lemma 1]). *If* $\dim X < n-1$ $(n > 2)$, *then the evaluation map $P^n(R)^X \to P^n(R)$ induces an isomorphism $\pi_1(P^n(R)^X, f) \cong \pi_1(P^n(R)) = Z_2$.*

We now show the following

Proposition 4.2. *Let λ be a real vector bundle over a CW-complex X and let $f: X \to P^n(R)$ $(n > 2)$ be a map. If* $\dim X < n-1$, *then the $\pi_1(P^n(R)^X, f)$-action is trivial on $k[\lambda, 2r\lambda_n]_f$, where $2r\lambda_n$ is the Whitney sum of $2r$ copies of λ_n.*

Proof. First we consider the case where n is odd and replace n by $2n+1$. Regard R^{2s} as the complex s-space C^s and S^{2s-1} as the unit $(2s-1)$-sphere in C^s. Then $S^1 = \{e^{2\pi it} \mid 0 \leqq t \leqq 1\}$. The S^1-actions both on C^s and S^{2s-1} are defined by

$$e^{2\pi it}(z_1, \cdots, z_s) = (e^{2\pi it}z_1, \cdots, e^{2\pi it}z_s) \quad \text{for } z_j \in C \ (1 \leqq j \leqq s).$$

Moreover Z_2-action on $S^{2s+1} \times C^r$ is given by $e^{h\pi i}(z, x) = (e^{h\pi i}z, e^{h\pi i}x)$ $(h=0, 1)$. It is easy to see that $2r\lambda_{2n+1} = (p: S^{2n+1} \times_{Z_2} C^r \to P^{2n+1}(R))$. The self map Ψ_t on $S^{2n+1} \times_{Z_2} C^r$ and Φ_t on $P^{2n+1}(R)$, defined by

$$\Psi_t[z, x] = [e^{\pi it}z, e^{\pi it}x] \quad \text{and} \quad \Phi_t[z] = [e^{\pi it}z] \quad \text{for } t \in R,$$

are flows on the respective spaces such that

$$\Psi_{t+1} = \Psi_t \quad \text{and} \quad \Phi_{t+1} = \Phi_t \quad \text{for } t \in R$$

and the following diagram is commutative:

$$\begin{array}{ccc} 2r\lambda_{2n+1} & \xrightarrow{\Psi_t} & 2r\lambda_{2n+1} \\ \downarrow p & & \downarrow p \\ P^{2n+1}(R) & \xrightarrow{\Phi_t} & P^{2n+1}(R). \end{array}$$

We note that Ψ_t is linear on each fiber. For a map $f : X \to P^{2n+1}(R)$, the group $\pi_1(P^{2n+1}(R)^X, f)$ is generated by the homotopy class of a homotopy $\Phi_t f$ $(0 \leqq t \leqq 1)$ (see the proof of Li [4, Theorem 1]). Given a k-morphism $g : \lambda \to 2r\lambda_{2n+1}$ covering f, the homotopy $\Psi_t g : \lambda \to 2r\lambda_{2n+1}$ covers the homotopy $\Phi_t f$. Hence $[g][\Phi_t f] = [\Psi_1 g] = [g]$. This completes the proof of the proposition for odd n.

In order to consider the case where n is even, replace n by $2n$ and let $i : p^{2n}(R) \to P^{2n+1}(R)$ be the natural inclusion. Then Proposition 3.1 leads to an $i^{\prime}_{\#}$-equivariant bijection $i_* : k[\lambda, 2r\lambda_{2n}]_f \to k[\lambda, 2r\lambda_{2n+1}]_{if}$ because $i^*\lambda_{2n+1} = \lambda_{2n}$. Here $i^{\prime}_{\#} : \pi_1(P^{2n}(R)^X, f) \to \pi_1(P^{2n+1}(R)^X, if)$ is an isomorphism. For this reason, and because the $\pi_1(P^{2n+1}(R)^X, if)$-action on $k[\lambda, 2r\lambda^{2n+1}]_{if}$ is trivial, the $\pi_1(P^{2n}(R)^X, f)$-action on $k[\lambda, 2r\lambda_{2n}]_f$ is also trivial. The proof is thus complete.

Let η_n be the canonical real 2-plane bundle over a lens space $L^n(p)$ mod p, p being odd prime. Then we have the following result simlar to the above:

Proposition 4.3. *Let λ be a real vector bundle over a CW-complex X, where* dim $X < 2n$, *and let $f : X \to L^n(p)$ be a map. Then the evaluation map induces an isomorphism $\pi_1(L^n(p)^X, f) \cong \pi_1(L^n(p)) = Z_p$ and the $\pi_1(L^n(p)^X, f)$ -action is trivial on $k[\lambda, r\eta_n]_f$.*

Proof is similar to that leading to the above proposition for odd n.

§ 5. Stabilization maps.

The natural inclusions $I_f : k(\xi, \zeta)_f \to (k+l)(\xi \oplus \theta^l_X, \zeta \oplus \theta^l_Y)_f$ and $I_{[f]} : k(\xi, \zeta)_{[f]} \to (k+l)(\xi \oplus \theta^l_X, \zeta \oplus \theta^l_Y)_{[f]}$ induce the maps $I_{f*} : k[\xi, \zeta]_f \to (k+l)[\xi \oplus \theta^l_X, \zeta \oplus \theta^l_Y]_f$ and $I_{[f]*} : k[\xi, \zeta]_{[f]} \to (k+l)[\xi \oplus \theta^l_X, \zeta \oplus \theta^l_Y]_{[f]}$, respectively. In this section we study the map I_{f*}. For our purpose, we construct a map $I'_f : \gamma(\beta_f(\xi, \zeta ; k)) \to \gamma(\beta_f(\xi \oplus \theta^l_X, \zeta \oplus \theta^l_Y ; k+l))$ in such a way that $I'_f\phi_f = \phi_f I_f$ as follows. The two natural inclusions $i_\xi : \xi \to \xi \oplus \theta^l_X$ and $i_\zeta : \zeta \to \zeta \oplus \theta^l_Y$ induce natural inclusions $i_\xi : \xi_m \to (\xi \oplus \theta^l_X)_{m+l}$ and $i_\zeta : \zeta_n \to (\zeta \oplus \theta^l_Y)_{n+l}$, respectively. Let $i : M^*(n, m ; k) \to M^*(n+l, m+l ; k+l)$ be a map defined by $i(C) = \begin{pmatrix} C & O \\ O & E \end{pmatrix}$, E being the unit matrix. Then we have a bundle map

$$I : B(\xi, \zeta ; k) \to B(\xi \oplus \theta^l_X, \zeta \oplus \theta^l_Y ; k+l)$$

covering $1_{X \times Y}$, defind by $I[u, v, C] = [i_\xi(u), i_\zeta(v), i(C)]$. This map I gives rise to a map $I'_f : \gamma(\beta_f(\xi, \zeta ; k)) \to \gamma(\beta_f(\xi \oplus \theta^l_X, \zeta \oplus \theta^l_Y ; k+l))$ for any map $f : X \to Y$. It is easy to verify that $I'_f\phi_f = \phi_f I_f$ and hence we have a commutative diagram

$$\begin{array}{ccc} k[\xi,\zeta]_f & \xrightarrow{\quad I_{f*}\quad} & (k+l)[\xi\oplus\theta_X^l,\ \zeta\oplus\theta_Y^l]_f \\ \downarrow{\scriptstyle\phi_{f*}} & & \downarrow{\scriptstyle\phi_{f*}} \\ \Gamma(\beta_f(\xi,\zeta\,;k)) & \xrightarrow{\quad I'_{f*}\quad} & \Gamma(\beta_f(\xi\oplus\theta_X^l,\ \zeta\oplus\theta_Y^l\,;k+l)). \end{array}$$

Moreover, both maps I_{f*} and I'_{f*} are easily shown to be $\pi_1(Y^X, f)$-equivariant.

Let $c(n, m\,; k)_l$ be the connectivity of the map $i : M^*(n, m\,; k) \to M^*(n+l, m+l\,; k+l)$ above, that is, the map i is *a* $c(n, m\,; k)_l$-equivalence. Then the map I'_f is of the same status and hence we get

Theorem 5.1. *For a map $f : X \to Y$,*

(i) *I_{f*} is surjecive if* $\dim X \leqq c(n, m\,; k)_l$ *and is injective if* $\dim X < c(n, m\,; k)_l$, *and*

(ii) *I_{f*} is $\pi_1(Y^X, f)$-equivariant.*

Remark. (i) The integer $c(n, m\,; k)_l = m-1$ or $n-1$ according as $k = n$ or $k = m$, because if $n > m$ then the natural inclusion of the Stiefel manifold $V_{n,m}$ of orthonormal m-frames in R^n into $M^*(n, m\,; k)$ is a homotopy equivalence and the natural inclusion $V_{n,m} \to V_{n+1,\, m+1}$ is an $(n-1)$-equivalence.

(ii) In general, $M^*(n, m\,; k)$ is $((n+1-k)\ (m+1-k)\ -2)$ -connected (see Phillips [7, §2°]).

In consequence of Theorem 5.1 and Propositions 2.2-2.3, we have

Proposition 5.2. *Assume that $f^*\zeta\oplus\theta_X^l$ is trivial and that* $\dim X \leqq c(n, m\,; k)_l$ *for some integer l. Then there exists a k-morphism of ξ to ζ covering a map homotopic to f if and only if there is a k-morphism of ξ to θ_Z^n covering a null-homotopic map of X to Z for any space Z.*

§6. Classification of k-morphisms.

We have proved that for a map $f : X \to Y$, the map $i_{f*} : k[\xi, \zeta]_f \to k[\xi, \zeta]_{[f]}$ is surjective (see §2) and the group $\pi_1(Y^X, f)$ acts on $k[\xi, \zeta]_f$ (see §3). In this section we first generalize Li's result [4, §1].

Theorem 6.1. *The map $i'_{f*} : k[\xi, \zeta]_f/\pi_1(Y^X, f) \to k[\xi, \zeta]_{[f]}$ defined by $i'_{f*}(g\pi_1(Y^X, f)) = i_{f*}(g)$ $(g \in k[\xi, \zeta]_f)$ is a bijection.*

Proof. For two k-morphisms g_0, $g_1 : \xi \to \zeta$ covering f, the equality $i_{f*}[g_0] = i_{f*}[g_1]$ holds if and only if there exists a homotopy $g_t : \xi \to \zeta$ of k-morphisms covering a homotopy, say f_t, such that $f_0 = f_1 = f$, which implies that $[g_0][f_t] = [g_1]$ for some element $[f_t] \in \pi_1(Y^X, f)$ by the definition of the $\pi_1(Y^X, f)$-action on $k[\xi, \zeta]_f$. This completes the proof.

This theorem, together with Theorem 5.1, leads at once to

Theorem 6.2. *If* $\dim X < c(n, m; k)_l$, *then the map* $I_{[f]*} : k[\xi, \zeta]_{[f]} \to (k+l)[\xi \oplus \theta_X^l, \zeta \oplus \theta_Y^l]_{[f]}$ *is a bijection for any map* $f : X \to Y$.

In consequence of Theorem 6.1 and Corollary 3.4, we have

Proposition 6.3. *If* $h : Z \to X$ *is a homotopy equivalence, then the map* $h^* : k[\xi, \zeta]_{[f]} \to k[h^*\xi, \zeta]_{[fh]}$ *is a bijection for any map* $f : X \to Y$.

Now we consider the condition that the set $k[\xi, \zeta]_{[f]}$ is equivalent to the set $k[\xi, \theta_Z^n]_{[c]}$ for a constant map $c : X \to Z$.

Proposition 6.4. *For a map* $f : X \to Y$, *there is a bijection between the sets* $k[\xi, \zeta]_{[f]}$ *and* $k[\xi, \theta_Z^n]_{[c]}$ *if one of the following conditions is satisfied*:

(1) ζ *is r-trivial and* $\dim X < r$,

(2) $\zeta \oplus \theta_Y^l$ *is r-trivial and* $\dim X < \min\{c(n, m; k)_l, r\}$.

Proof. If the condition (1) is satisfied, then by virtue of Proposition 2.3, Corollary 3.2 and Theorem 6.1, we have our assertion, while if (2) is satisfied our assertion follows from Theorem 6.2 and the case (1) of the proposition.

§7. k-mersions.

Throughout this section we assume that the manifolds M and N both mean the connected smooth manifolds without boundary and are of dimensions m and n, respectively, and that "either $k < n$ or M is open". Under this assumption, Feit [1, Theorem 1] has proved that the differential map $d : k(M, N) \to k(\tau_M, \tau_N)$ is a weak homotopy equivalence and hence we have a bijection

$(7.1)_k$(Feit) $\qquad k[M, N]_{[f]} = k[\tau_M, \tau_N]_{[f]}$ for any $f : M \to N$,

and in particular

$(7.1)_I$(Hirsch) $\qquad I[M, N]_{[f]} = m[\tau_M, \tau_N]_{[f]}$,

$(7.2)_S$(Phillips) $\qquad S[M, N]_{[f]} = n[\tau_M, \tau_N]_{[f]}$ if M is open.

As for the existence and classification problem of k-mersions, we have the following theorems:

Theorem 7.2. (cf. Li [3]). *A map* $f : M \to N$ *is homotopic to a k-mersion* *if and only if the bundle $\beta_f(\tau_M, \tau_{\bar{N}}; k)$ admits a cross section.*

Proof. This follows from (7.1) and Propositions 2.1-2.2.

Theorem 7.3. *A map $f: M \to N$ is homotopic to a k-mersion if and only if there exists a k-mersion of M to R^n, when one of the conditions* (1) *and* (2) *is satisfied;*

(1) *$f^*\tau_N$ is trivial*

(2) *homotopy* dim $M \leqq c(n, m; k)_l$ *and $f^*\tau_N \oplus \theta_M^l$ is trivial for some integer l.*

Proof. We deduce the theorem from (7.1) and Proposition 2.3 under the condition (1), and from (7.1) and Propositions 5.2 and 6.3 under the condition (2).

Corollary 7.4. *Let $f: M \to N$ be a map and assume that one of conditions* (1)′,…,(2)′′ *listed below is satisfied. Then f is homotopic to a k-mersion if and only if there exists a k-mersion of M to R^n.*

(1)′ *f is null-homotopic,*

(1)′′ *homotopy* dim $M \leqq r$ *and τ_N is r-trivial for some integer r,*

(1)′′′ *N is parallelizable,*

(2)′ *homotopy* dim $M \leqq \min\{c(n, m; k)_l, r\}$ *and $\tau_N \oplus \theta_N$ is r-trivial for some integers l and r,*

(2)′′ *homotopy* dim $M \leqq c(n,m; k)_l$ *for some integer l and N is a π-manifold.*

Theorem 7.5. *There is a bijection between the sets $k[M, M]_{[f]}$ for $f: M \to N$ and $k[M, R^n]$, if one of the following conditions holds:*

(1) *homotopy* dim $M < r$ *and τ_N is r-trivial,*

(1)′ *N is parallelizable,*

(2) *homotopy* dim $M < \min\{c(n, m; k)_l, r\}$ *and $\tau_N \oplus \theta_N$ is r-trivial for some integers l and r,*

(2)′ *homotopy* dim $M < c(n, m; k)_l$ *for some integer l and N is a π-manifold.*

Proof. The result follows from (7.1) and Propositions 6.3-6.4.

Considering the case when $k = \min\{m, n\}$ (if $k = n$ then M is understood to be open), we get

Theorem 7.6. *Let $f: M \to N$ be a map.*

(i) *Assume that $f^*\tau_N$ is stably trivial. Then f is homotopic to a submersion or an immersion if and only if there exists such a map of M to R^n according as $n \leqq m$ or $n \geqq m$.*

(ii) *There exists a bijection between the sets* $S[M,N]_{[f]}$ *and* $S[M,R^n]$ *or between* $I[M,N]_{[f]}$ *and* $I[M,R^n]$ *according as* $n \leqq m$ *or* $n \geqq m$, *if one of the following conditions holds*:

(1) *homotopy* dim $M < r$ *and* τ_N *is* r-*trivial*,

(1)′ N *is parallelizable*,

(2) *homotopy* dim $M < \min\{r, m-1\}$ *or* $\min\{r, n-1\}$ *according as* $n \leqq m$ *or* $n \geqq m$, *and* $\tau_N \oplus \theta_N$ *is* r-*trivial*,

(2)′ *homotopy* dim $M < m-1$ *or* $n-1$ *according as* $n \leqq m$ *or* $n \geqq m$, *and* N *is a* π-*manifold.*

Corollary 7.7. *A map* $f: M \to N$ *is homotopic to a submersion or an immersion if and only if there exists such a map of* M *to* R^n *according as* $n \leqq m$ *or* $n \geqq m$, *when one of the following conditions holds*:

(1) f *is null-homotopic*,

(2) *homotopy* dim $M \leqq r$ *and* $\tau_N \oplus \theta_N$ *is* r-*trivial*,

(2)′ N *is a* π-*manifold.*

Remark. In Theorem 7.6 and its Corollary, the part concerning immersions fairly overlaps with the results of Li [3] and [5], while the existence theorem for submesrions is deduced from Phillips′ theorem reworded by Thomas [8].

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