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Notes on enumerating embeddings of certain n-manifolds in Euclidean $(2n-2)$- and $(2n-3)$-spaces

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Abstract

Denote by $[M \subset R^m]$ the set of isotopy classes of embeddings of an n-manifold M in Euclidean m-space. In this note, we shall study the set $[M \subset R^{2n-k}]$ for $k=2$ and 3 in the case when M is an n-manifold satisfying the condition $\tilde{H}_i(M; Z) \otimes Z_2 = 0$ for $i<k$, and generalize some results in [15] and [19].

§1. Introduction

Throughout this nots, an n-manifold and an embedding mean respectively a closed connected differentiable manifold of dimension n and a differentiable embedding. Let $[M \subset R^m]$ denote the set of isotopy classes of embeddings of a manifold M in Euclidean m-space R^m. The set $[M \subset R^{2n-k}]$ has so far been studied (see [17] - [19] and [15]), when M is an n-manifold and $k=1$, when M is a homologically $(k-1)$-connected n-manifold $(k \geqq 2)$, and when M is a lens space $L^{(n-1)/2}(p)$ mod p and $1 \leqq k \leqq 5$. These results make us interested in $[M \subset R^{2n-2}]$ or $[M \subset R^{2n-k}]$ for an n-manifold M satisfying the condition

$$(*) \qquad \tilde{H}_i(M; Z) \otimes Z_2 = 0 \quad \text{for} \quad i<k.$$

In this note we shall study the set $[M \subset R^{2n-k}]$ for an n-manifold M satisfying the above condition (*) for $k=2$ and 3, and prove the following theorems:

Theorem A. *Assume that M is an n-manifold $(n \geqq 8)$ satisfying the condition $H_1(M; Z) \otimes Z_2 = 0$. Then, when it is not empty, the set $[M \subset R^{2n-2}]$ is given as follows:*

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$$[M\subset R^{2n-2}]=H^{n-1}(M;Z)\otimes H^{n-2}(M;Z)\times(1+t^*)(H^{n-1}(M;Z)*H^{n-1}(M;Z))$$

$$\times\begin{cases} H^{n-3}(M;Z_2) & \text{if } n\equiv 0\,(4),\ w_2(M)\neq 0, \\ & \text{or if } n\equiv 2\,(4),\ w_2(M)=0, \\ H^{n-3}(M;Z_2)\times Z_2 & \text{if } n\equiv 2\,(4),\ w_2(M)\neq 0, \\ H^{n-3}(M;Z_2)\times H^{n-2}(M;Z_2)/Sq^2H^{n-4}(M;Z_2) & \\ & \text{if } n\equiv 0\,(4),\ w_2(M)=0, \\ H^{n-3}(M;Z)\times H^{n-2}(M;Z_2) & \text{if } n\equiv 1\,(4),\ w_3(M)\neq 0, \\ H^{n-3}(M;Z)\times H^{n-2}(M;Z_2)\times Z_2 & \text{if } n\equiv 3\,(4),\ w_2(M)\neq 0 \\ & \text{or if } n\equiv 1\,(4),\ w_2(M)\neq 0,\ w_3(M)=0, \\ H^{n-3}(M;Z)\times H^{n-2}(M;Z_2)/(Sq^2\rho_2H^{n-4}(M;Z)+Sq^1H^{n-3}(M;Z_2)) & \\ \quad\times H^{n-2}(M;Z_2) & \text{if } n\equiv 1\,(4),\ w_2(M)=0, \\ H^{n-3}(M;Z)\times H^{n-2}(M;Z_2)/Sq^2\rho_2H^{n-4}(M;Z)\times H^{n-2}(M;Z_2) & \\ & \text{if } n\equiv 3\,(4),\ w_2(M)=0. \end{cases}$$

From now on $t: M\times M\to M\times M$ is the map defined by $t(x,y)=(y,x)$ and the symbol $*$ denotes the torsion product.

Corollary. *If M is an odd torsion n-manifold, i.e. if M is an n-manifold such that $\tilde{H}_i(M;Z)\otimes Z_2=0$ for $i<n$ (cf. [9]), then*

$$[M\subset R^{2n-2}]=H^{n-1}(M;Z)\otimes H^{n-2}(M;Z)\times(1+t^*)(H^{n-1}(M;Z)*H^{n-1}(M;Z))$$

$$\times\begin{cases} 0 & n\equiv 0\,(2), \\ H^{n-3}(M;Z) & n\equiv 1\,(2). \end{cases}$$

Theorem B. *Assume that M is an n-manifold ($n\geqq 10$) satisfying the condition $\tilde{H}_i(M;Z)\otimes Z_2=0$ for $i<3$ and that the first Pontrjagin class* mod 3 *$P_1(M)$ of M vanishes if n is even. Then, when it is not empty, the set $[M\subset R^{2n-3}]$ is given by*

$$[M\subset R^{2n-3}]=H^{n-1}(M;Z)\otimes H^{n-3}(M;Z)\times H^{n-1}(M;Z)*H^{n-2}(M;Z)$$

$$\times(1-t^*)(H^{n-2}(M;Z)\otimes H^{n-2}(M;Z))$$

$$\times\begin{cases} H^{n-4}(M;Z_2)\times H^{n-3}(M;Z_2)/Sq^2H^{n-5}(M;Z_2) & \text{if } n\equiv 1\,(4), \\ H^{n-4}(M;Z_2) & \text{if } n\equiv 3\,(4), \\ H^{n-4}(M;Z)\times H^{n-3}(M;Z_2)\times H^{n-1}(M;Z_3) & \\ \quad\times\begin{cases} H^{n-3}(M;Z_2)/(Sq^2\rho_2H^{n-5}(M;Z)+Sq^1H^{n-4}(M;Z_2)) & \\ & \text{if } n\equiv 2\,(4), \\ H^{n-3}(M;Z_2)/Sq^2\rho_2H^{n-5}(M;Z) & \text{if } n\equiv 0\,(4). \end{cases} \end{cases}$$

These are the generalization both of the Main Theorem for $k=2$ and 3 in [19] and Theorem $A(2)-(3)$ in [15].

This note is, in a sense, a sequel to the papers [15] and [17]–[19]. Hence the defintitions and notations used here are exactly the same as those explained in [18]-[19].

The remainder of this note is organized as follows : In § 2, we give some definitions and notations, and restate Haefliger's theorem [3, Théorème 1′] by using the homotopy set of liftings and the reduced symmetric product M^* of M. In § 3, we state the cohomology of M^*, postponing the proof of the integral case till § 5. The proofs of Theorems A and B are given in § 4.

§ 2. Preliminaries

We study the set $[M\subset R^{2n-k}]$ along the lines of Haefliger [3]–[4]. The cyclic group Z_2 of order 2 acts on the product X^2 of X via the map t above. The diagonal ΔX in X^2 is the fixed point set of this action. The quotient spaces

$$X^*=(X^2-\Delta X)/Z_2 \quad \text{and} \quad \Lambda^2 X=X^2/Z_2$$

are defined. The former is called the reduced symmetric product of M. Here the projection $p: X^2-\Delta X\rightarrow X^*$ is a double covering, whose classifying map we denote by

$$\xi: X^*\longrightarrow P^\infty.$$

Haefliger's theorem [3, Théorème 1′] can be restated as follows (cf. [18, Theorem 1.1]) :

Theorem 2.1 (Haefliger). *If $2m>3(n+1)$, then for an n-manifold M, there is a bijection*

$$[M\subset R^m]=[M^*, P^{m-1}; \xi].$$

Here the right hand side of this equality is the homotopy set of liftings of $\xi: M^*\rightarrow P^\infty$ to $(S^\infty\times_{Z_2}S^{m-1})\simeq P^{m-1}$.

To compute $[M\subset R^{2n-k}]=[M^*, P^{2n-k-1}; \xi]$, we may use Proposition 4 in [1] or Proposition on p.414 of [14] if $k=2$, and Proposition 1.1 in [15] if $k=3$.

we give some notations which will be used later.

$Z_r\langle a\rangle$ denotes the cyclic group of order r generated by a ($r\leqq\infty$, $Z_\infty=Z$).

The non-trivial elements $u\in H^1(P^\infty; Z_2)$ and $v\in H^1(X^*; Z_2)$ denote the first Stiefel-Whitney classes of the double coverings $S^\infty\rightarrow P^\infty$ and $X^2-\Delta X\rightarrow X^*$, respectively.

The relation $\xi^* u = v$ holds.

For $x \in H^1(M; Z_2)$, $Z_r[x]$ $(r \leqq \infty)$ denotes the sheaf of coefficients over X, locally isomorphic to Z_r, twisted by x, and $\underline{Z}_r$ denotes either Z_r or $Z_r[x]$. Let

$$\tilde{\rho}_r : H^i(X; Z_s[x]) \longrightarrow H^i(X; Z_r[x]) \qquad (s \equiv 0\ (r) \text{ or } s = \infty)$$

and

$$\tilde{\beta}_r : H^{i-1}(X; Z_r[x]) \longrightarrow H^i(X : Z[x]) \qquad (r < \infty)$$

denote the reduction mod r and Bockstein operator, respectively, twisted by x. Then $\tilde{\rho}_r$ and $\tilde{\beta}_r$ for $x=0$ are the ordinary ρ_r and β_r, respectively. $\bar{\rho}_r$ and $\bar{\beta}_r$ denote either $\tilde{\rho}_r$ and $\tilde{\beta}_r$ or the ordinary ρ_r and β_r, respectively. By [2] and [11], we have

$$(2.2) \qquad \bar{\rho}_2 \bar{\beta}_2 = \begin{cases} Sq^1 & \text{if } \underline{Z} = Z, \\ Sq^1 + x & \text{if } \underline{Z} = Z[x]. \end{cases}$$

For an orientable n-manifold M, there is a short exact sequence

$$(2.3) \qquad 0 \to H^i(M; Z_r) \xrightarrow{\phi_1} H^{i+n}(M^2; Z_r) \xrightarrow{\tilde{i}^*} H^{i+n}(M^2 - \Delta M; Z_r) \to 0 \ (r \leqq \infty),$$

where $\tilde{i} : M^2 - \Delta M \to M^2$ is the natural inclusion,

$$(2.4) \qquad \phi_1(x) = U(1 \otimes x) \qquad \text{for } x \in H^i(M; Z_r),$$

and $U \in H^n(M^2; Z)$ is called the Thom class or the diagonal cohomology class of M [8, §11]. Further there are the following relations (cf. [12, p.305] and [8, Theorem 11.11]):

$$(2.5) \qquad t^* \phi_1(x) = (-1)^n \phi_1(x) \qquad \text{for } x \in H^i(M; Z_r)$$

and

$$(2.6) \qquad U \equiv \pm(1 \otimes M + (-1)^n M \otimes 1)$$

$$\bmod \sum_{1 \leqq j \leqq n-1} H^j(M; Z) \otimes H^{n-j}(M; Z) + \sum_{1 \leqq j \leqq n-2} H^j(M; Z) * H^{n+1-j}(M; Z)$$

where

$$H^n(M; Z_r) = Z_r \langle M \rangle \ (r \leqq \infty).$$

§ 3. The cohomology of M^*

Throughout this section we assume that M is an n-manifold satisfying the condition

$$(*) \qquad \tilde{H}_i(M\,;Z)\otimes Z_2=0 \qquad \text{for } i<k \quad (k\geqq 2).$$

It is equivalent to the condition $H^{n-i}(M\,;Z_2)=0$ for $0<i<k$.

Under this condition we should like to determine the cohomology of M^*. The notations used here are the same as those explained in [18]–[19] (most of them are the same as those in [13, § 2]). Let

$$\sigma=1+t^*: H^*(M^2\,;Z_2)\longrightarrow H^*(M^2\,;Z_2).$$

Then Lemma 3.1 of and (4.1)–(4.2) of [19] are valid if the condition $\tilde{H}_i(M\,;Z)=0$ for $i<k$ in [19] is replaced by $(*)$.

Lemma 3.1 (cf. [19]). *Assume that M is an n-manifold satisfying the condition $\tilde{H}_i(M\,;Z)\otimes Z_2=0$ for $i<k$. Then*

(i) $H^i(M^*\,;Z_2)=0$ *for* $i>2n-k$,

(ii) $H^{2n-k}(M^*\,;Z_2)=\{\rho\sigma(M\otimes x)\mid x\in H^{n-k}(M\,;Z_2)\}$ $(\cong H^{n-k}(M\,;Z_2))$,

(iii) $H^{2n-k-1}(M^*\,;Z_2)=\{\rho(u^{k-1}\otimes x^2)\mid x\in H^{n-k}(M\,;Z_2)\}$ $(\cong H^{n-k}(M\,;Z_2))$
$+\{\rho(u^{k+1}\otimes x^2)\mid x\in H^{n-k-1}(M\,;Z_2)\}$ $(\cong H^{n-k-1}(M\,;Z_2))$,

(iv) $H^{2n-k-2}(M^*\,;Z_2)=\{\rho(u^{k}\otimes x^2)\mid x\in H^{n-k-1}(M\,;Z_2)\}$ $(\cong H^{n-k-1}(M\,;Z_2))$
$+\{\rho(u^{k-2}\otimes x^2)\mid x\in H^{n-k}(M\,;Z_2)\}$ $(\cong H^{n-k}(M\,;Z_2))$
$+\{\rho(u^{k+2}\otimes x^2)\mid x\in H^{n-k-2}(M\,;Z_2)\}$ $(\cong H^{n-k-2}(M\,;Z_2))$
$+[\{\rho\sigma(x\otimes y)\mid x,\ y\in H^{n-2}(M\,;Z_2),\ x\neq y\}]$,

where the term in the square brackets [ ] *is present only when* $k=2$.

(v) *there are equalities*

$$\rho(u^k\otimes x^2)=\rho(U(1\otimes x))=\rho\sigma(M\otimes x)\in H^{2n-k}(M^*\,;Z_2) \quad \textit{for } x\in H^{n-k}(M\,;Z_2)$$

and an isomorphism

$$\chi: H^{n-k}(M\,;Z_2)\xrightarrow{\cong} H^{2n-k}(M\,;Z_2) \quad (\chi(x)=\rho\sigma(M\otimes x)).$$

Further we have the following theorem, postponing its proof till § 5 :

Theorem 3.2. *Let $k \geqq 2$ and assume that M is an n-manifold satisfying the condition $\tilde{H}_i(M; Z) \otimes Z_2 = 0$ for $i < k$. Then*

(i) *for $i=2$, $k \geqq 3$ or $i=1$,*

$$H^{2n-k-i}(M^*; \underline{Z})$$

$$= p^{*-1}\bar{i}^*(1+(-1)^k t^*)\left(\begin{array}{c}\sum_{1\leqq j\leqq[(k+i)/2]} H^{n-j}(M; Z)\otimes H^{n-k-i+j}(M; Z)\\ +\\ \sum_{1\leqq j\leqq[(k+i-1)/2]} H^{n-j}(M; Z)*H^{n-k-i+1+j}(M; Z)\end{array}\right)$$

$$+\begin{cases}0 & \text{if } n\text{-}k \text{ is even},\\ p^{*-1}\bar{i}^*(1+(-1)^k t^*)(H^n(M; Z)\otimes H^{n-k-i}(M; Z)) & \text{if } n\text{-}k \text{ is odd},\end{cases}$$

$$+\begin{cases}\{\bar{\beta}_2\rho(u^k\otimes x^2) \mid x\in H^{n-k-1}(M; Z_2)\} & \text{if } i=1,\ n\text{-}k \text{ is even},\\ \{\bar{\beta}_2\rho(u^{k-2}\otimes x^2) \mid x\in H^{n-k}(M; Z_2)\} & \text{if } i=1,\ n\text{-}k \text{ is odd},\\ \{\bar{\beta}_2\rho(u^{k-3}\otimes x^2) \mid x\in H^{n-k}(M; Z_2)\} & \\ \quad +\{\bar{\beta}_2\rho(u^{k+1}\otimes x^2) \mid x\in H^{n-k-2}(M; Z_2)\} & \text{if } i=2,\ n\text{-}k \text{ is even},\\ \{\bar{\beta}_2\rho(u^{k-1}\otimes x^2) \mid x\in H^{n-k-1}(M; Z_2)\} & \text{if } i=2,\ n\text{-}k \text{ is odd};\end{cases}$$

(ii) $\bar{\rho}_2 H^{2n-k-2}(M^*; \underline{Z})$

$$= \{\rho(u^{k-2}\otimes x^2) \mid x\in H^{n-k}(M; Z_2)\} + \{\rho(u^{k+2}\otimes x^2) \mid x\in H^{n-k-2}(M; Z_2)\}$$
$$+ [\{\rho\sigma(x\otimes y) \mid x,\ y\in H^{n-2}(M; Z_2),\ x\neq y\}] \qquad \text{if } n\text{-}k \text{ is even},$$
$$= \{\rho(u^k\otimes x^2) \mid x\in H^{n-k-1}(M; Z_2)\} + \{\rho\sigma(M\otimes\rho_2 x) \mid x\in H^{n-k-2}(M; Z)\}$$
$$+ [\{\rho\sigma(x\otimes y) \mid x,\ y\in H^{n-2}(M; Z_2),\ x\neq y\}] \qquad \text{if } n\text{-}k \text{ is odd};$$

where $\underline{Z}=Z$ or $Z[v]$ according as k is even or odd, and the terms in the square brackets appear only when $k=2$.

Let q be an odd prime. If we consider the cohomology spectral sequence (cf. [7]) for a fibration $M^2-\Delta M \to S^\infty \times_{Z_2}(M^2-\Delta M) \to P^\infty$, which is homotopically equivalent to $M^2-\Delta M \xrightarrow{p} M^* \xrightarrow{\xi} P^\infty$, then it follows that the map p induces an isomorphism

$$p^*: H^i(M^*; \underline{Z}_q) \cong H^i(M^2-\Delta M; Z_q)^{(-1)^k t^*} (=\{x\in H^i(M^2-\Delta M; Z_q) \mid (-1)^k t^* x = x\})$$

where $\underline{Z}_q = Z_q$ or $Z_q[v]$ according as k is even or odd. By the Künneth formula, we get

$$H^{2n-i}(M^2 ; Z_q)=(1+(-1)^k t^*)(\sum_{0\leqq j\leqq[i/2]} H^{n-j}(M ; Z_q)\otimes H^{n-i+j}(M ; Z_q))$$
$$+(1-(-1)^k t^*)(\sum_{0\leqq j\leqq[i/2]} H^{n-j}(M ; Z_q)\otimes H^{n-i+j}(M ; Z_q)).$$

Here $(1+(-1)^k t^*)$−image is $(-1)^k t^*$−invariant, and by (2.5), $\phi_1 H^{n-i}(M ; Z_q) \subset H^{2n-i}(M^2 ; Z_q)^{(-1)^n t^*}$. This, together with (2.4) and (2.6), leads to a commutative diagram of isomorphisms

$$\begin{array}{ccc} (H^{2n-i}(M^2 ; Z_q)/\phi_1 H^{n-i}(M ; Z_q))^{(-1)^k t^*} & \xrightarrow[\cong]{\tilde{i}^*} & H^{2n-i}(M^2-\Delta M ; Z_q)^{(-1)^k t^*} \\ \nwarrow\cong & & \tilde{i}^* \nearrow\cong \end{array}$$
$$(1+(-1)^k t^*)(\sum_{1\leqq j\leqq[i/2]} H^{n-j}(M ; Z_q)\otimes H^{n-i+j}(M; Z_q))$$
$$+\begin{cases} 0 & \text{if } n\text{-}k \text{ is even,} \\ (1+(-1)^k t^*)(H^n(M ; Z_q)\otimes H^{n-i}(M ; Z_q)) & \text{if } n\text{-}k \text{ is odd,} \end{cases}$$

and hence the following lemma holds:

Lemma 3.3. *For any odd prime q,*

$$H^{2n-i}(M^* ; \underline{Z}_q)=$$
$$p^{*-1}\tilde{i}^*(1+(-1)^k t^*)(\sum_{1\leqq j\leqq[i/2]} H^{n-j}(M ; Z_q)\otimes H^{n-i+j}(M ; Z_q))$$
$$+\begin{cases} 0 & \textit{if } n\text{-}k \textit{ is even,} \\ p^{*-1}\tilde{i}^*(1+(-1)^k t^*)(H^n(M ; Z_q)\otimes H^{n-i}(M ; Z_q)) & \textit{if } n\text{-}k \textit{ is odd,} \end{cases}$$

where $\underline{Z}_q = Z_q$ or $Z_q[v]$ according as k is even or odd.

Corollary 3.4. *There is an isomorphism*

$$\tilde{i}^{*-1}p^* : H^{2n-1}(M^* ; \underline{Z}_3)\cong 0 \qquad \textit{if } n\text{-}k \textit{ is even,}$$
$$\cong\{(1+(-1)^k t^*)(M\otimes x) \mid x\in H^{n-1}(M ; Z_3)\} \qquad \textit{if } n\text{-}k \textit{ is odd.}$$

§ 4. Proofs of Theorems A and B

We prove only Theorem B and not Theorem A because the proof of the latter is similar to, and moreover rather simpler than, that of the former.

Assume that M is an n-manifold ($n\geqq 10$) satisfying the condition

$$(*) \qquad \tilde{H}_i(M ; Z)\otimes Z_2= 0 \qquad \text{for } i<3,$$

and that it is embedded in Euclidean $(2n-3)$-space. By Theorem 2.1, Lemma 3.1 and [15, Proposition 1.1], we have a filtration

$$[M\subset R^{2n-3}]=[M^*,\ P^{2n-4}\ ;\ \xi]=F_0\supset F_1\supset F_2\supset F_3\supset 0$$

such that

$$F_0/F_1=H^{2n-4}(M^*\ ;\ Z[v]),\qquad F_2/F_3=0\ ,$$
$$F_1/F_2=\operatorname{Coker}\theta,\qquad F_3=0\ ,$$

where

$$\theta : H^{2n-5}(M^*\ ;\ Z[v])\longrightarrow H^{2n-3}(M^*\ ;\ Z_2)\times H^{2n-1}(M^*\ ;\ Z_3[v]),$$
$$\theta=((Sq^2+\binom{2n-3}{2}v^2)\tilde{\rho}_2,\ \mathcal{P}^1\tilde{\rho}_3).$$

Hence we have

$$(4.1)\qquad [M\subset R^{2n-3}]=H^{2n-4}(M^*\ ;\ Z[v])\times\operatorname{Coker}\theta$$

We first consider Coker θ. If n is odd, then by Corollary 3.4, $H^{2n-1}(M^*\ ;\ Z_3[v])=0$ and hence $\operatorname{Coker}\theta=H^{2n-3}(M^*\ ;\ Z_2)/(Sq^2+\binom{2n-3}{2}v^2)\tilde{\rho}_2H^{2n-5}(M^*\ ;\ Z[v])$, which is obtained in exactly the same way as in [19, (4.6)], i.e.

$$(4.2)\qquad \operatorname{Coker}\theta\cong\begin{cases}0 & n\equiv 3\ (4),\\ H^{n-3}(M\ ;\ Z_2)/Sq^2H^{n-5}(M\ ;\ Z_2) & n\equiv 1\ (4).\end{cases}$$

If n is even, there is a commutative diagram

$$(4.3)\qquad \begin{array}{ccc} H^{2n-5}(M^*\ ;\ Z[v]) & \xrightarrow{\ \mathcal{P}^1\tilde{\rho}_3\ } & H^{2n-1}(M^*\ ;\ Z_3[v])\ (\cong H^{n-1}(M\ ;\ Z_3)) \\ \downarrow{\scriptstyle \tilde{i}^{*-1}p^*} & & \cong\ \downarrow{\scriptstyle \tilde{i}^{*-1}p^*} \\ (1-t^*)\begin{pmatrix}\sum_{0\leq j\leq 2} H^{n-j}(M\ ;\ Z)\otimes H^{n-5+j}(M\ ;\ Z) \\ + \\ \sum_{1\leq j\leq 2} H^{n-j}(M\ ;\ Z)*H^{n-4+j}(M\ ;\ Z)\end{pmatrix} & \xrightarrow{\ \mathcal{P}^1\rho_3\ } & (1-t^*)(H^n(M\ ;\ Z_3)\otimes H^{n-1}(M\ ;\ Z_3)), \end{array}$$

where by Corollary 3.4, $\tilde{i}^{*-1}p^*$ in the right hand side is an isomorphism, and by (5.8)–(5.9) below, the one in the left is an epimorphism. To study $\mathcal{P}^1\tilde{\rho}_3$, recall Yo's operation Q^1 [20, p.1481 and p.1485],

$$Q^1 : H^i(M\ ;\ Z_3)\longrightarrow H^{i+4}(M\ ;\ Z_3)$$

such that

$$Q^1=0\ \text{ for }\ i\geq n-5\quad\text{ and }\quad Q^1x+\mathcal{P}^1x=W_3^1(M)x,$$

where $W_3^1(M)$ is the first Wu class mod 3, which is equal to $P_1(M)$, the first Pontrjagin class mod 3 of M [8, p.229]. Hence

$$\mathscr{P}^1 x=0 \quad \text{for } x\in H^i(M\,;Z_3),\ i>n-5,$$

because of the assumption $P_1(M)=0$. Using this relation, the diagram (4.3) and the relations (6.1) below, we get $\mathscr{P}^1\tilde{\rho}_3 H^{2n-5}(M^*\,;Z[v])=0$, and so

(4.4) $\operatorname{Coker}\theta\cong H^{2n-3}(M^*\,;Z_2)/Sq^2\tilde{\rho}_2 H^{2n-5}(M^*\,;Z[v])\times H^{n-1}(M\,;Z_3) \qquad n\equiv 0\,(2)$,

The group $Sq^2\tilde{\rho}_2 H^{2n-5}(M^*\,;Z[v])$ for even n is obtained in exactly the same way as in [19, (4.10)] and is given as follows:

(4.5) $$Sq^2\tilde{\rho}_2 H^{2n-5}(M^*\,;Z[v])\cong\begin{cases} Sq^2\rho_2 H^{n-5}(M\,;Z) & n\equiv 0\,(4),\\ Sq^2\rho_2 H^{n-5}(M\,;Z)+Sq^1 H^{n-4}(M\,;Z_2) & n\equiv 2\,(4).\end{cases}$$

Thus $\operatorname{Coker}\theta$ is determined by (4.2) and (4.4)−(4.5).

On the other hand, the group $H^{2n-4}(M\,;Z[v])$ is given by Theorem 3.2. Therefore by (4.1), the set $[M\subset R^{2n-3}]$ is determined and so Theorem B is established.

§ 5. Proof of Theorem 3.2

Let $k\geq 2$ and assume that M is an n-manifold satisfying the condition

(*) $$\tilde{H}_i(M\,;Z)\otimes Z_2=0 \qquad \text{for } i<k.$$

Case I: n-k is even. If we consider the cohomology spectral sequence [7] for $M^2-\Delta M\xrightarrow{p}M^*\xrightarrow{\varepsilon}P^\infty$, then it follows that the rank and the odd torsion subgroup of $H^{2n-k-i}(M^*\,;\underline{Z})$ are equal to and isomorphic to those of $H^{2n-k-i}(M^2-\Delta M\,;Z)^{(-1)^k t^*}$ by p^*. Since by (2.5), $\phi_1 H^{n-k-i}(M\,;Z)\subset H^{2n-k-i}(M^2\,;Z)^{(-1)^k t^*}$ because n-k is even, there is an isomorphism

$$\tilde{i}^*: H^{2n-k-i}(M^2\,;Z)^{(-1)^k t^*}/\phi_1 H^{n-k-i}(M\,;Z)\cong H^{2n-k-i}(M^2-\Delta M\,;Z)^{(-1)^k t^*}.$$

Here by (2.4) and (2.6), ϕ_1 is a split monomorphism. Hence the left hand side of this equality is isomorphic by $\tilde{i}^*$ to the subgroup of $H^{2n-k-i}(M^2\,;Z)$,

$$(1+(-1)^k t^*)\begin{pmatrix}\sum_{1\leq j\leq[(k+i)/2]} H^{n-j}(M\,;Z)\otimes H^{n-k-i+j}(M\,;Z)\\ +\\ \sum_{1\leq j\leq[(k+i-1)/2]} H^{n-j}(M\,;Z)*H^{n-k-i+1+j}(M\,;Z)\end{pmatrix}$$

which is an odd torsion group for $i=2$ ($k\geqq 3$) and $i=1$, under the condition (*). This shows that for $i=2$ ($k\geqq 3$) and $i=1$, $p^*: H^{2n-k-i}(M^*; \underline{Z})\to H^{2n-k-i}(M^2-\Delta M: Z)^{(-1)^k t^*}$ is an epimorphism, whose kernel is a 2-primary component. This component is easily calculated in the same way as in [19, § 5], i.e. by using (2.2), Lemma 3.1, the Bockstein exact sequence and [19, Lemma 3.2], and is given as follows:

$$H^{2n-k-1}(M^*; \underline{Z})\equiv\{\bar\beta_2\rho(u^k\otimes x^2)\mid x\in H^{n-k-1}(M; Z_2)\} \text{ mod odd torsion},$$
$$H^{2n-k-2}(M^*; \underline{Z})\equiv\{\bar\beta_2\rho(u^{k+1}\otimes x^2)\mid x\in H^{n-k-2}(M; Z_2)\}$$
$$+\{\bar\beta_2\rho(u^{k-3}\otimes x^2)\mid x\in H^{n-k}(M; Z_2)\} \text{ mod odd torsion } (k\geqq 3),$$
$$\bar\rho_2 H^{2n-k-2}(M^*; \underline{Z})=\{\rho(1\otimes x^2)\mid x\in H^{n-2}(M; Z_2)\}$$
$$+\{\rho(u^4\otimes x^2)\mid x\in H^{n-4}(M; Z_2)\}$$
$$+\{\rho\sigma(x\otimes y)\mid x,\ y\in H^{n-2}(M; Z_2),\ x\neq y\}\quad (k=2).$$

Here there is a relation

$$\bar\rho_2\bar\beta_2\rho(u^j\otimes x^2)=\rho(u^{j+1}\otimes x^2)\ \text{ if } (j,\ \dim x)=(k+1,\ n-k-2),\ (k-3,\ n-k).$$

The argument above establishes Theorem 3.2(i) and (ii) for *n-k* even.

Case II : *n-k* is odd. We make an argument similar to that used in [19, § 5]. For the natural embedding $j: PM\to M^*$, write j^*v as v in $H^1(PM; Z_2)$ and consider the exact sequence in [19, (5.3)],

$$(5.1)\qquad \cdots\to H^{i-1}(PM; \underline{Z})\xrightarrow{\delta} H^i(\Lambda^2 M,\ \Delta M; \underline{Z})\xrightarrow{i^*} H^i(M^*; \underline{Z})\xrightarrow{j^*} H^i(PM; \underline{Z})\to\cdots.$$

The cohomology of PM has been given by Rigdon [10, § 9] (cf. [19, Lemma 5.4]).

Lemma 5.2 (Rigdon). *Assume that M is an n-manifold satisfying the condition (*) above and that n-k is odd. Then*

(i) $H^{2n-k-1}(PM; \underline{Z})=\{\beta_2(v^{n-2}x+v^{n-k-2}Sq^k x)\mid x\in H^{n-k}(M; Z_2)\}$
$+Z_2\langle\beta_2(v^{n-k-2}M)\rangle$ *if k is even,*
$=\{\tilde\beta_2(v^{n-2}x)\mid x\in H^{n-k}(M; Z_2)\}$ *if k is odd;*

(ii) $H^{2n-k-2}(PM; \underline{Z})=\{\beta_2(v^{n-2}x)\mid x\in H^{n-k-1}(M; Z_2)\}$, *if k is even,*
$=\{\tilde\beta_2(v^{n-2}x+v^{n-k-3}Sq^{k+1}x)\mid x\in H^{n-k-1}(M; Z_2)\}$
$+Z_2\langle\tilde\beta_2(v^{n-k-3}M)\rangle$ *if k is odd.*

The cohomology of $(\Lambda^2 M, \Delta M)$ has been investigated by Larmore [5]. There are elements

$$\Lambda x\in H^r(\Lambda^2 M,\ \Delta M; Z_p[v])\ \text{ for } x\in H^r(M; Z_p)\ (p\leqq\infty),$$
$$\Delta(x,\ y)\in H^{r+s}(\Lambda^2 M,\ \Delta M; Z_p[v])\ \text{ for } x\in H^r(M; Z_p),\ y\in H^s(M; Z_p)\ (p\leqq\infty),$$

satisfying the conditions

(5.3) $$\pi^* \Lambda x = x\otimes 1 - 1\otimes x, \qquad \pi^*\Delta(x, y) = x\otimes y - (-1)^{rs} y\otimes x,$$

where $\pi : (M^2, \Delta M)\to(\Lambda^2 M, \Delta M)$ is the natural projection. Let

(5.4) $$\underline{\Lambda}(x, y) = \begin{cases} \Lambda x \Lambda y & \text{if } \underline{Z}_p = Z_p, \\ \Delta(x, y) & \text{if } \underline{Z}_p = Z_p[v], \end{cases}$$

and assume that the integral cohomology groups of M are of the form

$$H^n(M; Z) = Z\langle M\rangle \quad (\rho_r M = M),$$

$$H^m(M; Z) = \sum_{1\leq i\leq \gamma(m)} Z_{r(m,i)}\langle x_{m,i}\rangle \quad \text{(direct sum)} \quad \text{for } m<n,$$

$$x_{m,i} = \beta_{r(m,i)} y_{m,i} \quad (y_{m,i}\in H^{m-1}(M; Z_{r(m,i)}) \quad \text{for } \alpha(m)<i\leq\gamma(m),$$

where the order $r(m, i)$ is infinite for $1\leq i\leq\alpha(m)$ and a power of a prime for $\alpha(m)<i\leq\gamma(m)$, and if $\alpha(m)<i<j\leq\gamma(m)$, then either $(r(m, i), r(m, j))=1$ or $r(m, i) \mid r(m, j)$ holds. Then using these notations we have

Lemma 5.5 (Larmore*)). *Assume that M is an n-manifold satisfying the condition (*) above and that $n-k$ is odd. Then*

(i) *$H^{2n-k}(\Lambda^2 M, \Delta M; \underline{Z})$ has $Z_2\langle\beta_2(v^{n-k-1}\Lambda M)\rangle$ as a direct summand if k is even,*

(ii) *if $i=2$ $(k\geq 3)$ or $i=1$, then*

$$H^{2n-k-i}(\Lambda^2 M, \Delta M; \underline{Z}) = \varepsilon Z_2 + \sum_{1\leq j\leq\alpha(n-k-i)} Z\langle\underline{\Lambda}(M, x_{n-k-i,j})\rangle$$
$$+ \sum_{1\leq j\leq k+i} \sum_{(\lambda,\mu)\in A_j\cup A_j'} Z_r\langle\beta_r\underline{\Lambda}(y_{n-j,\lambda}, \rho_r x_{n-k-i+j,\mu})\rangle$$
$$+ \sum_{1\leq j< k+i} \sum_{(\lambda,\mu)\in B_j\cup B_j'} Z_r\langle\beta_r\underline{\Lambda}(y_{n-j,\lambda}, \rho_r y_{n-k-i+1+j,\mu})\rangle$$

where $r=r(n-j, \lambda)$, $x_{n,\mu}=M$ and

$$\varepsilon Z_2 = \begin{cases} Z_2\langle\beta_2(v^{n-k-i-1}\Lambda M)\rangle & \textit{if } k\equiv 0\,(2),\ i=1 \textit{ or } k\equiv 1\,(2),\ i=2, \\ 0 & \textit{otherwise}, \end{cases}$$

$$A_j = \left\{ (\lambda, \mu) \;\middle|\; \begin{matrix} \alpha(n-j)<\lambda\leq\gamma(n-j),\ 1\leq\mu\leq\gamma(n-k-i+j) \\ r(n-j, \lambda)<r(n-k-i+j, \mu) \end{matrix} \right\},$$

*) The author has proved this lemma using results on pp. 908–915 in [5]. He thinks that the expresions "*r is a power of* 2 *or*" in I(iv) and II(v) of [5, Theorem 20] should be omitted.

$$A'_j = \begin{cases} \{(\lambda,\ \mu) \mid r(n-j,\ \lambda)=r(n-k-i+j,\ \mu)\} & 1\leqq j<(k+i)/2, \\ \left\{(\lambda,\ \mu) \,\middle|\, \begin{array}{l} r(n-j,\ \lambda)=r(n-j,\ \mu),\ \textit{and}\ \lambda<\mu\ \textit{or} \\ \lambda\leqq\mu\ \textit{according as}\ j\equiv 0\,(4)\ \textit{or}\ 2\,(4) \end{array}\right\} & j=(k+i)/2, \\ \phi & (k+i)/2<j, \end{cases}$$

$$B_j = \left\{ (\lambda,\ \mu) \,\middle|\, \begin{array}{l} \alpha(n-k-i+j+1)<\mu\leqq\gamma(n-k-i+j+1), \\ \alpha(n-j)<\lambda\leqq\gamma(n-j),\ r(n-j,\ \lambda)<r(n-k-i+j+1,\ \mu) \end{array}\right\},$$

$$B'_j = \begin{cases} \{(\lambda,\ \mu) \mid r(n-j,\ \lambda)=r(n-k-i+j+1,\ \mu)\} & 1\leqq j<(k+i-1)/2, \\ \left\{(\lambda,\ \mu) \,\middle|\, \begin{array}{l} r(n-j,\ \lambda)=r(n-j,\ \mu),\ \textit{and}\ \lambda\leqq\mu\ \textit{or} \\ \lambda<\mu\ \textit{according as}\ j\equiv 1\,(4)\ \textit{or}\ 3\,(4) \end{array}\right\} & j=(k+i-1)/2, \\ \phi & (k+i-1)/2<j \end{cases}$$

Using this lemma, the two relations

$$\delta(v^i x)=v^{i+1}\Delta x \qquad \text{for } x\in H^*(M\,;Z_2),$$

$$j^*\rho(u^r\otimes x^2)=\sum_{1\leqq t\leqq s} v^{r+s-t}Sq^t x \qquad \text{for } x\in H^s(M\,;Z_2),$$

contained in [16, Lemma 1.5] and [13, §2], the exact sequence (5.2) and the relations (5.4)−(5.5) above, and (6.1)−(6.2) below, we have the following two lemmas:

Lemma 5.6. *Let $k\geqq 2$ and assume that M is an n-manifold satisfying the condition $(*)$ above and n-k is odd. Then for $i=2\,(k\geqq 3)$ and $i=1$,*

$$H^{2n-k-i}(M^*\,;\underline{Z})=\{\bar\beta_2\rho(u^{k+i-3}\otimes x^2) \mid x\in H^{n-k-i+1}(M\,;Z_2)\} + i^*H^{2n-k-i}(\Delta^2 M,\ \Delta M\,;\underline{Z})$$

and $i^*(\varepsilon Z_2)=0$.

Lemma 5.7. *Let $x\in H^*(M\,;Z_r)$ and $y\in H^*(M\,;Z_s)$ be of order r and s, respectively, with $\dim x+\dim y>n$. Then the following three relations hold:*

(i) $\pi^*\underline{\Delta}(x,\ y)=(-1)^{k+1}(1+(-1)^k t^*)(x\otimes y)$ *if* $r=s\leqq\infty$,

(ii) $\pi^*\bar\beta_r\underline{\Delta}(x,\ \rho_r y)=(-1)^{k+1}(1+(-1)^k t^*)(\beta_r x\otimes y)$ *if* $r<s=\infty$,

(iii) $\pi^*\bar\beta_r\underline{\Delta}(x,\ \rho_r y)=(-1)^{k+1+\dim x}(1+(-1)^k t^*)(\beta_r x*\beta_s y)$ *if* $r\,|\,s,\ s<\infty$.

The above relations lead to an isomorphism

$$i'^*\pi^* : H^{2n-k-i}(\Lambda^2M,\ \Delta M\,;Z)/\varepsilon Z_2 \xrightarrow{\cong} H^{2n-k-i}(M^2\,;Z)^{(-1)^k t^*}$$

$$\| $$

(5.8)
$$(1+(-1)^k t^*)\begin{pmatrix} \sum_{0\leq j\leq[(k+i)/2]} H^{n-j}(M\,;Z)\otimes H^{n-k-i+j}(M\,;Z) \\ + \\ \sum_{1\leq j\leq[(k+i-1)/2]} H^{n-j}(M\,;Z)*H^{n-k-i+1+j}(M\,;Z)\end{pmatrix}$$

where $i' : M^2\subset(M^2,\ \Delta M)$ is the natural inclusion ; while by the relations (2.3)–(2.6), the map $\tilde{i}$ induces an isomorphism

$$\tilde{i}^* : H^{2n-k-i}(M^2\,;Z)^{(-1)^k t^*} \xrightarrow{\cong} H^{2n-k-i}(M^2-\Delta M\,;Z)^{(-1)^k t^*}.$$

Therefore we have a commutative diagram of isomorphisms

(5.9)
$$\begin{array}{ccc} H^{2n-k-i}(\Lambda^2M,\ \Delta M\,;\underline{Z})/\varepsilon Z_2 & \xrightarrow[\cong]{i^*} & \mathrm{Im}\, i^*\,(\subset H^{2n-k-i}(M^*\,;\underline{Z})) \\ \cong\Big\downarrow i'^*\pi^* & & \cong\Big\downarrow p^* \\ H^{2n-k-i}(M^2\,;Z)^{(-1)^k t^*} & \xrightarrow[\cong]{\tilde{i}^*} & H^{2n-k-i}(M^2-\Delta M\,;Z)^{(-1)^k t^*}. \end{array}$$

Theorem 3.2(i) for n-k odd follows from Lemma 5.6 and (5.8)–(5.9).

The proof of (ii) for n-k odd is given in the same way as in the case where M is a homologically (k-1)-conneted n-manifold [19].

§ 6. Remarks on the torsion product

By the Künneth formula, there is a split short exact sequence

$$0\to\sum_{p+q=i} H^p(X\,;Z)\otimes H^q(Y\,;Z)\to H^i(X\times Y\,;Z)\to\sum_{p+q=i+1} H^p(X\,;Z)*H^q(Y\,;Z)\to 0\,.$$

Both for $x\in H^p(X\,;Z)$ and $y\in H^q(Y\,;Z)$ of finite order, we should like to express their torsion product $x*y\in H^{p+q-1}(X\times Y\,;Z)$ by using the tensor product and the Bockstein operator.

Let r and s be the order of x and y, respectively, such that $r|s$, and let $x'\in H^{p-1}(X\,;Z_r)$ and $y'\in H^{q-1}(Y\,;Z_s)$ be the elements such that

$$\beta_r x'=x \qquad \text{and} \qquad \beta_s y'=y.$$

Under these circumstances, we shall show the relations

(6.1) $$x*y=(-1)^{p-1}\beta_r(x'\otimes\rho_r y'),\qquad y*x=(-1)^{q-1}\beta_r(\rho_r y'\otimes x'),$$

(6.2) $$t^*(x*y)=(-1)^{pq-1}y*x.$$

Since (6.2) immediately follows from (6.1), we now prove (6.1). Let $S^*(X;G)$ and $S^*(Y;G)$ $(G=Z,\ Z_r$ or $Z_s)$ be the cochain groups of X and Y with coefficients in G, and let $\partial : S^i(\ ;G)\to S^{i+1}(\ ;G)$ be the coboundary operator and let $[c]$ mean the cohomology class of a cocycle c. For x' and y' above, there are two cochains $\bar{x}'\in S^{p-1}(X;Z)$ and $\bar{y}'\in S^{q-1}(Y;Z)$ such that

$$[\rho_r\bar{x}']=x' \qquad \text{and} \qquad [\rho_s\bar{y}']=y'.$$

Put

$$\bar{x}=(1/r)\partial\bar{x}' \qquad \text{and} \qquad \bar{y}=(1/s)\partial\bar{y}'.$$

Then

$$[\bar{x}]=\beta_r x'=x, \qquad [\bar{y}]=\beta_s y'=y,$$

and

$$\partial((s/r)\bar{x}')=s\bar{x} \qquad \partial\bar{y}'=s\bar{y}.$$

By the definition of torsion product and its property, respectively, on p.150 and p.170 of [6], we get

$$x*y=(-1)^{p-1}[(1/s)\partial((s/r)\bar{x}'\otimes\bar{y}')]$$
$$=(-1)^{p-1}[\bar{x}\otimes\bar{y}'+(-1)^{p-1}(s/r)\bar{x}'\otimes\bar{y}];$$

while

$$\beta_r(x'\otimes\rho_r y')=\beta_r([\rho_r\bar{x}']\otimes[\rho_r\bar{y}'])=\beta_r[\rho_r(\bar{x}'\otimes\bar{y}')]$$
$$=[(1/r)(r\bar{x}\otimes\bar{y}'+(-1)^{p-1}\bar{x}'\otimes s\bar{y})]$$
$$=[\bar{x}\otimes\bar{y}'+(-1)^{p-1}(s/r)\bar{x}'\otimes\bar{y}].$$

This show the first relation of (6.1). The second is obtained in a similar way.

References

[1] D.R. Bausum, Embeddings and immersions of manifolds in Euclidean space, Trans Amer. Math. Soc., 213 (1975), 263-303.

[2] R. Greenblatt, The twisted Bockstein coboundary, Proc. Cambridge Phillos. Soc..61 (1965), 295-297.

[3] A. Haefliger, Plongements différentiables dans le domaine stable, Comment. Math. Helv., 37 (1962), 155-176.

[4] A. Haefliger, Plongements de variétés dans le domaine stable, Séminaire Bourbaki 15 (1962/63), n°. 245.

[5] L.L. Larmore, The cohomology of $(A^2X,\ \Delta X)$, Canad. J. Math., 25 (1973), 908-921.

[6] S.S. MacLane, Homology, Grund. Math. Wis., 114. Springer-verlag, 1963.

[7] J.F. McClendon, Obstruction theory in fiber spaces, Math. Z., 120 (1971), 1-17.

[8] J.W. Milnor and J.D. Stasheff, Characteristic Classes, Ann. of Math. Studies, 76, Princeton Univ. Press., 1974.

[9] E. Rees, Embedding odd torsion manifolds, Bull. London Math. Soc., 3 (1971), 356-362.

[10] R.D. Rigdon, Immersions and embeddings of manifolds in Euclidean space, Thesis, Univ. of California at Berkelay, 1970.

[11] H. Samelson, A note on the Bockstein operator, Proc. Amer. Math. Soc., 15 (1964), 450-453.

[12] E.H. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.

[13] E. Thomas, Embedding manifolds in Euclidean space, Osaka J. Math., 13 (1976), 163-186.

[14] T. Yasui, Note on the enumeration of embeddings of real projective spaces, Hiroshima Math. J., 3 (1973), 409-418.

[15] T. Yasui, The enumeration of embeddings of lens spaces and projective spaces, Hiroshima Math. J., 8 (1976), 235-253.

[16] T. Yasui, On the map defined by regarding embeddings as immersions, Hiroshima Math. J., 13 (1983), 457-476.

[17] T. Yasui, Notes on enumerating embeddings of unorientable n-manifolds in Euclidean $(2n-1)$-space for odd n, Bull. of Yamagata Univ., Nat. Sci., 11 (1984), 39-49.

[18] T. Yasui, Enumerating embeddings of n-manifolds in Euclidean $(2n-1)$-space, J. Math. Soc. Japan, 36 (1984), 555-576.

[19] T. Yasui, Eunmerating embeddings of homologically $(k-1)$-connected n-manifolds in Euclidean $(2n-k)$-space, J. Math. Soc. Japan, 38 (1986), 113-126.

[20] G. Yo, Cohomology operations and duality in a manifold, Sci. Sinica, 12 (1963), 1469-1487.