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Immersion groups of complex projective spaces

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Introduction

Let M be a closed connected smooth n-dimensional manifold and let R^m be the m-dimensional Euclidean space. Let $[M \subseteq R^m]$ be the set of regular homotopy classes of immersions of M into R^m. If $2m > 3n+1$ and there exists an immersion of M into R^m, then the set $[M \subseteq R^m]$ has the structure of an abelian group (see § 1 or [13]). In this article, we determine the group structure of $[CP^n \subseteq R^{4n-i}]$ for $2 \leqq i \leqq 5$, when there exists an immersion of CP^n, the complex projective space of complex dimension n, into R^{4n-i}.

Theorem. (1) *If n is odd and $n \geqq 5$, then*

$[CP^n \subseteq R^{4n-2}] = Z + Z_2 + Z_2$.

(2) *If n is odd and $n \geqq 5$, then*

$[CP^n \subseteq R^{4n-3}] = Z_2 \qquad n \equiv 1(4)$,

$= Z_2 + Z_2 \quad n \equiv 3(4)$.

(3) *Let $n \geqq 6$ and assume that there is an immersion $CP^n \subseteq R^{4n-4}$. Then*

$[CP^n \subseteq R^{4n-4}] = Z + Z_2 \quad n \equiv 3(4)$,

$= Z \qquad n \not\equiv 3(4)$.

(4) *Let $n \geqq 6$ and assume that there is an immersion $CP^n \subseteq R^{4n-5}$. Then*

$[CP^n \subseteq R^{4n-5}] = Z_2 + Z_2 \quad n \equiv 3(4)$,

$= 0 \qquad n \not\equiv 3(4)$.

For completeness' sake, we mention that

$[CP^n \subseteq R^{4n-2}] = Z$ for $n > 4$, $n \neq 2^r$, $n \equiv 0(2)$,

$[CP^n \subseteq R^{4n-3}] = 0$ or ϕ for $n > 4$, $n \equiv 0(2)$.

These results are given by [9], [12] and [16]*).

In § 1, we give the set $[CP^n \subseteq R^{4n-i}]$ the structure of an abelian group. The proofs of (1) and (2) of the Theorem are given in § § 2 – 4. The proofs of (3)

*) In [16, Corollary 3], the results on $[CP^n \subseteq R^m]$ for $m = 4n-4$, $4n-5$ and $n \equiv 1(2)$ are stated, but they are incorrect except the case when $m = 4n-5$ and $n \equiv 3(4)$. In Y.Nomura's private letter, he tells the author that Y.Nomura has made some mistakes in that article.

and (4) are obtained more easily than those of (1) and (2), and will be omitted.

§ 1. Group structures on $[M \subseteq R^{n+m}]$

Let $p: E \to B$ be a fibration, where the fibre is $(r-1)$-connected for some r. If X is a CW-complex of dimension n and if $f: X \to B$ has a lifting $g: X \to E$, denote the set of rel-f-homotopy classes of liftings of f to E by $[X, E; f]$. If $n \leqq 2r-1$, then the set $[X, E; f]$ naturally has the structure of an abelian group with identity element $[g]$, according to J.C.Becker [3, Theorem 7.23]. Let $E^2 \to E$ denote the pull-back of $p: E \to B$ by p. Then the diagonal map $d: E \to E^2$ is a cross section of $E^2 \to E$ and $dg: X \to E^2$ is a lifting of g to E^2. The group $[X, E^2; g]$ with identity $[dg]$ is isomorphic to $[X, E; f]$ by [8, Theoren 3.1].

Now let M be a closed connected smooth n-dimensional manifold and $p: BO(m) \to BO$ be the fibre bundle with fibre $V_m = O/O(m)$. Assume that $2m-2 \geqq n$ and that there is an immersion $M \subseteq R^{n+m}$, i. e. $[M \subseteq R^{n+m}] \neq \phi$. Let $\nu: M \to BO$ classify the stable normal bundle of M. Then the set of regular homotopy classes of immersions of M into R^{n+m} is in one-to-one correspondence with $[M, BO(m); \nu]$ and so with $[M, BO(m)^2; f]$, if $f: M \to BO(m)$ classifies the normal bundle of any immersion $M \subseteq R^{n+m}$. Since $2m-2 \geqq n, [M, BO(m)^2; f]$ has the structure of an abelian group. The set $[M \subseteq R^{n+m}]$, therefore, has the structure of an abelian group via this one-to -one correspondnce. L.L.Larmore and E.Thomas called this group the immersion group and denoted it by $Im_{n+m}(M)$ in [13]. If M is an orientable manifold or a spin manifold, we can replace $p: BO(m) \to BO$ by $p: BSO(m) \to BSO$ or by $p: BSpin(m) \to BSpin$, respectively.

§ 2. The group $[CP^n \subseteq R^{4n-2}]$ for $n \equiv 1(2)$

Let n be odd and $n \geqq 5$. Then CP^n is a spin manifold and there is an immersion $CP^n \subseteq R^{4n-2}$ by [7, Theorem 2]. As is stated in § 1, $[CP^n \subseteq R^{4n-2}] = [CP^n, BSpin(2n-2)^2; f]$ where $f: CP^n \to BSpin(2n-2)$ classifies the normal bundle of any immersion $CP^n \subseteq R^{4n-2}$. In the table of [18], C.A. Robinson gives k-invariants of the Postnikov tower of $BO(r) \to BO$. By a slight modification of them, we have the Postnikov tower of $BSpin(2n-2)^2 \to BSpin(2n-2)$ and its k-invariants as follows:

$$
\begin{array}{ccccc}
 & & BSpin(2n-2)^2 \xrightarrow{\ h\ } B \times P_\varphi & & \\
 & & \downarrow & & \\
B \times K(Z_2, 2n-1) \times K(Z_2, 2n) & \xrightarrow{1 \times i} & B \times P_\beta & \xrightarrow{1 \times \varphi} & B \times K(Z_2, 2n+1) \\
 & & \downarrow & & \\
 & & B \times K(Z, 2n-2) \times K(Z_2, 2n-1) & \xrightarrow{1 \times \beta} & B \times K(Z_2, 2n) \times K(Z_2, 2n+1) \\
 & & \downarrow & & \\
 & & B = BSpin(2n-2), & &
\end{array}
$$

$\beta = (Sq^2 \rho_2 \iota_{2n-2} \times 1, 1 \times Sq^2 \iota_{2n-1}),$

$\varphi i = Sq^2 \iota_{2n-1} \times 1,$

where $P_\beta \to K(Z, 2n-2)\times K(Z_2, 2n-1)$ is the principal fibration classified by β and $\varphi \in H^{2n+1}(P_\beta : Z_2) = Z_2$, and h is a $(2n+1)$-equivalence.

By [14, Theorem 3], there is a decreasing filtration

$$[CP^n,\ BSpin(2n-2)^2\,;f] = F_0 \supset F_1 \supset F_2 \supset 0$$

such that

$$F_0/F_1 = \operatorname{Ker}\Phi^{2n-2}, \qquad F_1/F_2 = \operatorname{Ker}\Gamma^{2n-2}/\operatorname{Im}\Theta^{2n-3}, \qquad F_2 = \operatorname{Coker}\Phi^{2n-3}.$$

Here $\Phi^i : \operatorname{Ker}\Theta^i \to \operatorname{Coker}\Gamma^i$ is the twisted secondary operation due to the relation $\Gamma^{i+1}\Theta^i = 0$, and

$$\Theta^i : H^i(CP^n\,; Z) \times H^{i+1}(CP^n\,; Z_2) \to H^{i+2}(CP^n\,; Z_2) \times H^{i+3}(CP^n\,; Z_2),$$
$$\Theta^i(a,\ b) = (Sq^2\rho_2 a,\ Sq^2 b),$$
$$\Gamma^i : H^{i+1}(CP^n\,; Z_2) \times H^{i+2}(CP^n\,; Z_2) \to H^{i+3}(CP^n\,; Z_2),$$
$$\Gamma^i(a,\ b) = Sq^2 a.$$

It is well-known that

$$H^*(CP^n\,; Z) = Z[z]/(z^{n+1}), \quad H^*(CP^n\,; Z_2) = Z_2[z]/(z^{n+1}) \ (z=\rho_2 z),$$

where $\deg z = 2$. A simple calculation, using the above results, shows that

$$F_0/F_1 = \operatorname{Ker}\Phi^{2n-2} = \operatorname{Ker}\Theta^{2n-2} = H^{2n-2}(CP^n\,; Z) \times 0,$$
$$F_1/F_2 = \operatorname{Ker}\Gamma^{2n-2}/\operatorname{Im}\Theta^{2n-3} = 0 \times H^{2n}(CP^n\,; Z_2),$$
$$F_2 = \operatorname{Coker}\Phi^{2n-3},$$
$$\Phi^{2n-3} : 0 \times H^{2n-2}(CP^n\,; Z_2) \to H^{2n}(CP^n\,; Z_2).$$

By considering the Postnikov tower of $BSpin(2n-2)^2 \to BSpin(2n-2)$, the secondary operation Φ^{2n-3} is an ordinary double secondary operation due to the two relations

$$Sq^2(Sq^2\rho_2) = 0, \qquad Sq^2 0 + 0 Sq^2 = 0.$$

Moreover this operation satisfies the Peterson-Stein type formula (cf. [2, Theorem 6.4]). Let $g : CP^n \to K(Z, 2n-2)$ correspond to $z^{n-1} \in H^{2n-2}(CP^n\,; Z)$ and let $\rho_2 : K(Z, 2n-2) \to K(Z_2, 2n-2)$ correspond to $\rho_2\iota_{2n-2} \in H^{2n-2}(K(Z, 2n-2)\,; Z_2)$. Consider the diagram

$$\begin{array}{ccccc} & & K(Z_2, 2n-2)\times K(Z_2, 2n-1) \xrightarrow{i} P_\beta \xrightarrow{\varphi} K(Z_2, 2n) & & \\ & & \downarrow & & \\ CP^n \xrightarrow{g} K(Z, 2n-2) & \xrightarrow{(*,\ \rho_2)} & K(Z, 2n-3) \times K(Z_2, 2n-2) & \xrightarrow{\beta} & K(Z_2, 2n-1) \times K(Z_2, 2n). \end{array}$$

Then the second formula of Peterson-Stein implies that

$$\begin{aligned}\Phi^{2n-3}(0,\ z^{n-1}) &= \Phi^{2n-3}(g^*0,\ g^*\rho_2\iota_{2n-2}) \\ &= (\varphi i)_g(Sq^2\rho_2 0,\ Sq^2\rho_2\iota_{2n-2}) \bmod Q,\end{aligned}$$

where

$$\begin{aligned} Q &= (\varphi i)_\# [CP^n,\ K(Z_2, 2n-2)\times K(Z_2, 2n-1)] + g^\# [K(Z, 2n-2),\ K(Z_2, 2n)], \\ &= Sq^2 H^{2n-2}(CP^n\,; Z_2) + g^* H^{2n}(K(Z, 2n-2)\,; Z_2) \\ &= 0. \end{aligned}$$

Since $(\varphi i)(a,\ b) = Sq^2 a$, we can easily verify that $(\varphi i)_g(0,\ Sq^2\rho_2\iota_{2n-2}) = 0$ and so $\Phi^{2n-3}(0,\ z^{n-1}) = 0$. This implies $F_2 = H^{2n}(CP^n\,; Z_2)$. The group extension of $0 \to F_2 \to F_1 \to F_1/F_2 \to 0$ is trivial by [12, Corollary 3.7]. Therefore $F_1 = Z_2 + Z_2$. Further the group extension of $0 \to F_1 \to F_0 \to F_0/F_1 (= Z) \to 0$ is trivial. This shows $F_0 = [CP^n \subseteq R^{4n-2}] = Z + Z_2 + Z_2$ for $n \equiv 1(2)$.

§ 3. The group $[CP^n \subseteq R^{4n-3}]$ for $n \equiv 3(4)$

Assume that $n \geqq 5$ and $n \equiv 3(4)$. Then $[CP^n \subseteq R^{4n-3}]$ is not an empty set by [7]. Since $w_2(CP^n) = 0$ and $w_4(CP^n) = 0$, we consider the principal fibration $q : P_{w_4} \to BSpin$ classified by $w_4 \in H^4(BSpin ; Z_2)$. Let $B \to P_{w_4}$ be the pull-back of $BSpin(2n-3) \to BSpin$ by q. Then the classifying map of the stable normal bundle of any immersion $CP^n \subseteq R^{4n-3}$ has a lifting $f : CP^n \to B$ and $[CP^n \subseteq R^{4n-3}] = [CP^n, B^2 ; f]$, where $B^2 \to B$ is the pull-back of $p : B \to P_{w_4}$ by p. The Postnikov tower of $B^2 \to B$ is given by modifying that of $BO(2n-3) \to BO$ constructed in [18], as follows:

$$
\begin{array}{ccccc}
B^2 & \xrightarrow{\quad h \quad} & B \times P_\varphi & & \\
 & & \downarrow & & \\
B \times K(Z_2, 2n-1) \times K(Z_2, 2n) & \xrightarrow{1 \times i} & B \times P_\beta & \xrightarrow{1 \times \varphi} & B \times K(Z_2,\ 2n+1) \\
 & & \downarrow & & \\
 & & B \times K(Z_2,\ 2n-3) & \xrightarrow{1 \times \beta} & B \times K(Z_2,\ 2n) \times K(Z_2,\ 2n+1) \\
 & & \downarrow & & \\
 & & B & &
\end{array}
$$

where

$$\beta = (Sq^2 Sq^1 \iota_{2n-3},\ Sq^4 \iota_{2n-3}),\quad \varphi i = Sq^2 \iota_{2n-1} \times 1$$

and h is a $(2n+1)$-equivalence.

Therefore $[CP^n, B^2 ; f] = [CP^n, B \times P_\varphi ; f]$ and there is a decreasing filtration

$$[CP^n,\ B \times P_\varphi ; f] = F_0 \supset F_1 \supset F_2 \supset 0,$$

such that

$$F_0/F_1 = \mathrm{Ker}\, \Phi^{2n-3},\quad F_1/F_2 = \mathrm{Ker}\, \Gamma^{2n-3}/\mathrm{Im}\, \Theta^{2n-4},\quad F_2 = \mathrm{Coker}\, \Phi^{2n-4},$$

where $\Phi^i : \mathrm{Ker}\, \Theta^i \to \mathrm{Coker}\, \Gamma^i$ is the twisted secondary operation due to the relation $\Gamma^{i+1} \Theta^i = 0$ and

$$\Theta^i : H^i(CP^n ; Z_2) \to H^{i+3}(CP^n ; Z_2) \times H^{i+4}(CP^n ; Z_2),$$

$$\Theta^i(a) = (Sq^2 Sq^1 a,\ Sq^4 a),$$

$$\Gamma^i : H^{i+2}(CP^n ; Z_2) \times H^{i+3}(CP^n ; Z^2) \to H^{i+4}(CP^n ; Z_2),$$

$$\Gamma^i(a,\ b) = Sq^2 a.$$

We briefly have

$$F_0/F_1 = 0,\qquad F_1/F_2 = 0 \times H^{2n}(CP^n ; Z_2) = Z_2,$$

$$F_2 = \mathrm{Coker}\, \Phi^{2n-4} : H^{2n-4}(CP^n ; Z_2) \to H^{2n}(CP^n ; Z_2).$$

By considering the Postnikov tower of $B^2 \to B$, the secondary operation Φ^{2n-4} is an ordinary one due to the relation $Sq^2(Sq^2 Sq^1) + 0 Sq^4 = 0$. Let $\rho_2 : K(Z, 2n-4) \to K(Z_2, 2n-4)$ correspond to $\rho_2 \iota_{2n-4} \in H^{2n-4}(K(Z, 2n-4) ; Z_2)$ and let $g : CP^n \to K(Z, 2n-4)$ correspond to $z^{n-2} \in H^{2n-4}(CP^n ; Z)$. By the second formula of Peterson-Stein [1, Theorem 5.2], we have

$$\Phi^{2n-4}(z^{n-2}) = \Phi^{2n-4}(g^* \rho_2 \iota_{2n-4})$$
$$= (\varphi i)_g (Sq^2 Sq^1 \rho_2 \iota_{2n-4},\ Sq^4 \rho_2 \iota_{2n-4}) \bmod Q,$$

where

$$Q = Sq^2 H^{2n-2}(CP^n\,;\,Z_2) + 0H^{2n-1}(CP^n\,;\,Z_2) + g^* H^{2n}(K(Z,\ 2n-4)\,;\,Z_2).$$

Since $(\varphi i)(a,\ b) = Sq^2 a$, we have $(\varphi i)_g(0,\ Sq^4 \rho_2 \iota_{2n-4}) = 0$. Moreover $Q = 0$ follows from the facts that $H^{2n}(K(Z,\ 2n-4)\,;\,Z_2) = Z_2$ generated by $Sq^4 \rho_2 \iota_{2n-4}$, $g^* Sq^4 \rho_2 \iota_{2n-4} = Sq^4 z^{n-2} = 0$ for $n \equiv 3(4)$ and $Sq^2 z^{n-1} = 0$ for $n \equiv 1(2)$. This shows that $\Phi^{2n-4} = 0$ and so

$$F_2 = H^{2n}(CP^n\,;\,Z_2) = Z_2.$$

The group extension of $0 \to F_2 \to F_1 \to F_1/F_2 \to 0$ is trivial by [12, Corollary 3.7]. The argument made above implies $[CP^n \subseteq R^{4n-3}] = Z_2 + Z_2$ for $n \equiv 3(4)$.

Remark. As for the case $n \equiv 1(4)$, we can obtain $F_0/F_1 = 0$, $F_1/F_2 = 0$ and $F_2 = \text{Cokre}\ \Phi^{2n-4} = H^{2n}(CP^n\,;\,Z_2)$ or 0 by the same method as in the case $n \equiv 3(4)$. Hence it follows that $[CP^n \subseteq R^{4n-3}] = Z_2$ or 0 for $n \equiv 1(4)$. In the next section, we will show that $[CP^n \subseteq R^{4n-3}] = Z_2$ for $n \equiv 1(4)$ by a different way.

§ 4. The group $[CP^n \subseteq R^{4n-3}]$ for $n \equiv 1(4)$

For $n \geqq 5$, $n \equiv 1(4)$, there exists an immersion $CP^n \subseteq R^{4n-3}$. To show that $[CP^n \subseteq R^{4n-3}] = Z_2$ for $n \equiv 1(4)$, we consider another method statad below.

4.1. Another group structure on $[M \subseteq R^{n+m}]$. S.Feder stated in [4] the theorem concerning immersion due to Haefliger-Hirsch [6] as follows:

Assume that there is an immersion of an n-dimensional manifold M in R^{n+m}. If $2m-2 \geqq n$, then the set $[M \subseteq R^{n+m}]$ is in one-to-one correspondence with the set of Z_2-equivariant homotopy classes of Z_2-equivariant maps of $S(M)$ to S^{n+m-1}, where $S(M)$ denotes the tangent sphere bundle of M.

Let $P(M)$ be the real projective tangent bundle of M and let η be the canonical real line bundle over $P(M)$. Then the Z_2-equivariant homotopy set of Z_2-equivariant maps of $S(M)$ to S^{n+m-1} is in one-to-one correspondence with the set of vertical homotopy classes of cross sections of the sphere bundle $(S(M) \times S^{n+m-1})/Z_2 \to P(M)$ associated with $(n+m)\eta$ over $P(M)$. This bundle is induced from $(S^\infty \times S^{n+m-1})/Z_2 \to RP^\infty$ (homotopically equivalent to the natural inclusion $RP^{n+m-1} \subset RP^\infty$) by $\eta : P(M) \to RP^\infty$, the classifying map of η. Therefore $[M \subseteq R^{n+m}]$ is in one-to-one correspondence with $[P(M),\ RP^{n+m-1}\,;\,\eta]$, which has the structure of an abelian group by [3]. Thus the set $[M \subseteq R^{n+m}]$ has the structure of an abelian group via this one-to-one correspondence.

R.Rigdon stated in his dissertation [17, § 8] that this group and the one defined in § 1 are coincident with each other. However we do not use his result in this article because the group of order 2 is uniquely determined and is Z_2.

4.2. **Rreliminaries.** By [19, Proposition 1.1], there is a filtration

$$[CP^n \subseteq R^{4n-3}] = [P(CP^n), RP^{4n-4}; \eta] = F_0 \supset F_1 \supset F_2 \supset F_3 \supset 0,$$

such that

$$F_0/F_1 = \mathrm{Ker}\,\chi^{4n-3}, \qquad F_1/F_2 = \mathrm{Ker}\,\Psi^{4n-3},$$
$$F_2/F_3 = \mathrm{Coker}\,\Phi^{4n-4}, \qquad F_3 = \mathrm{Coker}\,\chi^{4n-4}.$$

Here $\chi^i : \mathrm{Ker}\,\Phi^i \to \mathrm{Coker}\,\Psi^i$ is a twisted tertiary operation, $\Phi^i : \mathrm{Ker}\,\Theta^i \to \mathrm{Ker}\,\Delta^i/\mathrm{Im}\,\Gamma^{i-1}$ and $\Psi^i : \mathrm{Ker}\,\Gamma^i/\mathrm{Im}\,\Theta^{i-1} \to \mathrm{Coker}\,\Delta^i$ are the twisted secondary operations due to the relations $\Gamma^{i+1}\Theta^i = 0$, $\Delta^{i+1}\Gamma^i = 0$, and

$$\Theta^i : H^{i-1}(X; \underline{Z}) \to H^{i+1}(X; Z_2) \times H^{i+3}(X; Z_2) \times H^{i+3}(X; \underline{Z}_3),$$
$$\Theta^i(a) = (Sq^2\tilde{\rho}_2 a,\ Sq^4\tilde{\rho}_2 a,\ \mathscr{P}_3^1\tilde{\rho}_3 a),$$
$$\Gamma^i : H^i(X; Z_2) \times H^{i+2}(X; Z_2) \times H^{i+2}(X; \underline{Z}_3) \to H^{i+2}(X; Z_2) \times H^{i+3}(X; Z_2),$$
$$\Gamma^i(a, b, c) = ((Sq^2+vSq^1+v^2)a,\ (Sq^2Sq^1+v^2Sq^1)a + (Sq^1+v)b),$$
$$\Delta^i : H^{i+1}(X; Z_2) \times H^{i+2}(X; Z_2) \to H^{i+3}(X; Z_2),$$
$$\Delta^i(a, b) = (Sq^2+v^2)a + (Sq^1+v)b,$$

where $X = P(CP^n)$, $\underline{Z}$ and $\underline{Z}_3$ are the non-trivial local systems on $P(CP^n)$ induced by $v \in H^1(P(CP^n); Z_2)$(the first Stiefel-Whitney class of the double covering $S(CP^n) \to P(CP^n)$), $\tilde{\rho}_p : H^i(X; \underline{Z}) \to H^i(X; \underline{Z}_p)$ is the reduction mod p, and $\mathscr{P}_3^1$ is the reduced power operation mod 3 in local system [5].

As is well-known, $1, v, \cdots, v^{2n-1}$ form a base of the $H^*(CP^n; Z_2)$-module $H^*(P(CP^n); Z_2)$ and the ring structure is given by the relation

$$v^{2n} = \sum_{i=1}^{2n} w_i(CP^n)v^{2n-i},$$

while the twisted integral cohomology group $H^i(P(CP^n); \underline{Z})$ is the direct sum of some copies of Z_2 for $i \geq 2n$ by [17, Proposition 9.2]. Let $\tilde{\beta}_2 : H^{i-1}(X; Z_2) \to H^i(X; \underline{Z})$ be the Bockstein operator, where $\underline{Z}$ is the local system due to $v \in H^1(X; Z_2)$. Then there is a relation

$$\tilde{\rho}_2\tilde{\beta}_2(x) = (Sq^1+v)x \quad \text{for } x \in H^*(X; Z_2).$$

By the above results and the Bockstein exact sequence of $P(CP^n)$, we have the following results:

$$H^{4n-4}(P(CP^n); \underline{Z}) = 0,$$
$$H^{4n-5}(P(CP^n); \underline{Z}) = Z_2 + Z_2 + Z_2$$

generated by $\{\tilde{\beta}_2(v^{2n-2}z^{n-2}),\ \tilde{\beta}_2(v^{2n-4}z^{n-1}),\ \tilde{\beta}_2(v^{2n-6}z^n)\}$,

$\tilde{\rho}_2 : H^{4n-5}(P(CP^n); \underline{Z}) \to H^{4n-5}(P(CP^n); Z_2)$ is an isomorphism,

$H^i(P(CP^n); \underline{Z}_3) = 0$ for $i \geq 2n$.

4.3. **Calculation of $[CP^n \subseteq R^{4n-3}]$ for $n \equiv 3(4)$.** A simple calculation yields

$$F_0/F_1 = 0, \qquad F_1/F_2 = 0, \qquad F_3 = 0$$

and

$$F_2/F_3 = \mathrm{Coker}\,\Phi^{4n-4} : \mathrm{Ker}\,\Theta^{4n-4} \to \mathrm{Ker}\,\Delta^{4n-3}/\mathrm{Im}\,\Gamma^{4n-4},$$

where

$\operatorname{Ker}\Theta^{4n-4} = Z_2$ generated by $\tilde{\beta}_2(v^{2n-6}z^n+v^{2n-4}z^{n-1}+v^{2n-2}z^{n-2})$,

$\operatorname{Ker}\Delta^{4n-3}/\operatorname{Im}\Gamma^{4n-4} = H^{4n-2}(P(CP^n);Z_2)\times 0.$

For the rest of this section, we devote ourselves to showing that $\Phi^{4n-4}=0$ on $P(CP^n)$.

Let $(CP^n)^* = (CP^n\times CP^n - \Delta CP^n)/Z_2$ be the reduced symmetric product of CP^n (ΔCP^n is the diagonal of CP^n) and let $j: P(CP^n)\to(CP^n)^*$ be the embedding induced from the Z_2-equivariant map $\bar{j}: S(CP^n)\to CP^n\times CP^n-\Delta CP^n$ defined by $\bar{j}(u) = (\exp(u), \exp(-u))$. Then, if v stands for the first Stiefel-Whitney class of the double covering $CP^n\times CP^n-\Delta CP^n \to (CP^n)^*$, $j^*(v)= v \in H^1(P(CP^n);Z_2)$. Hence we study Φ^{4n-4} on $(CP^n)^*$. Using the results of [4, Theorem 4.3] and [19, (4.8−10)], we have

$\operatorname{Ker}\Phi^{4n-4} = H^{4n-5}((CP^n)^*;\underline{Z})$,

$= Z_2$ genertaed by $\tilde{\rho}_2^{-1}(vx^{2^{r+1}-3}y^s)$ $(n = 2^r+s,\ 0<s<2^r)$,

$\operatorname{Ker}\Delta^{4n-3}/\operatorname{Im}\Gamma^{4n-4} = 0,$

and so we have

$\Phi^{4n-4} = 0$ on $(CP^n)^*$.

To prove $\Phi^{4n-4} = 0$ on $P(CP^n)$, it is sufficient to show that $j^*(\tilde{\rho}_2^{-1}(vx^{2^{r+1}-3}y^s)) = \tilde{\beta}_2(v^{2n-6}z^n+v^{2n-4}z^{n-1}+v^{2n-2}z^{n-2})$, because the secondary operation Φ^{4n-4} is natural for maps. Since $\tilde{\rho}_2$ is an isomorphism on $H^{4n-5}(P(CP^n);\underline{Z})$, it is sufficient to show that

$$(*)\qquad j^*(vx^{2^{r+1}-3}y^s) = v^{2n-5}z^n+v^{2n-3}z^{n-1}+v^{2n-1}z^{n-2}.$$

Let $A^2M\,(=M\times M/Z_2)$ be the 2−fold symmetric product of M, the set of unordered pairs of elements of M, and let ΔM denote the set of unordered pairs $\{x, x\}$. Then $A^2CP^n-\Delta CP^n = (CP^n)^*$. As the $Z_2[v]$−algebra, the cohomology $H^*(A^2CP^n, \Delta CP^n;Z_2)$ is completely described in [10, Theorem 11] and the action of the Steenrod algebra on $H^*(A^2CP^n, \Delta CP^n;Z_2)$ is given by [10, Lemma 10], Moreover, the following results are known in [20, Lemmas 1.4−5];

$$H^{4n-5}((CP^n)^*;Z_2)\xrightarrow{j^*}H^{4n-5}(P(CP^n);Z_2)\xrightarrow{\delta}H^{4n-4}(A^2CP^n, \Delta CP^n;Z_2)$$

is exact (cf. [11, §5]), where

$\delta(v^iz^j) = v^{i+1}Az^j.$

We now return to the proof of (∗). By this result, we have

$\delta(v^{2n-5}z^n) = v^{2n-4}Az^n,\qquad \delta(v^{2n-3}z^{n-1}) = v^{2n-2}Az^{n-1},$

$\delta(v^{2n-1}z^{n-2}) = v^{2n}Az^{n-2}.$

Further we have

$v^{2n}Az^{n-2} = v^4(v^{2n-4}Az^{n-2}) = v^{2n-2}Az^{n-1}+v^{2n-4}Az^n$

by [10, Theorem 11, (ii) and (iv)]. Therefore $\operatorname{Im} j^* = \operatorname{Ker}\delta = Z_2$ generated by $v^{2n-5}z^n+v^{2n-3}z^{n-1}+v^{2n-1}z^{n-2}$. Hence (∗) follows from the above result and the fact that $H^{4n-5}((CP^n)^*;Z_2) = Z_2$ generated by $vx^{2^{r+1}-3}y^s$ [19, (4, 9)]. This implies that $\Phi^{4n-4}=0$ and so $F_2/F_3 = Z_2$. This completes the proof of $[CP^n\subseteq R^{4n-3}] = Z_2$

for $n \equiv 3(4)$.

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