Periodic Solution of Autonomous Nonlinear Equation of Motion with One-degree-of-freedom

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Summary

We discuss the difference between the cosine Fourier series and the sine Fourier series which are applied to obtain the approximate solution of autonomons nonlinear equation of motion with one-degree-of-freedom in conservative field. And, it is shown that the approximate solution obtained by applying the cosine Fourier series converges to the exact periodic solution of the equation, but that obtained by applying the sine Fourier series does not converges to the exact solution generally. Then we examine the condition where the approximate solution obtained by the sine Fourier series converges to the exact solution. Applying the cosine Fourier series, we numerically analyse system where the equation of motion is derived by considering the finite deformation theroy in elasticity.

1. Introduction

There are many papers which deal with the nonlinear free vibration of the conservative system with one-degree-of-freedom. The free vibration of the equation in which nonlinear spring term is expressed as a odd function such as

$$\ddot{x}+\omega_0^2x+bx^3=0,$$
$$\ddot{x}+\sin x=0 \tag{1}$$

is solved analytically by using elliptic functions. Meanwhile, the free vibration of the equation in which nonlinear spring term is not expressed by odd functions is not analytically solved. But the orbit of the free vibration on the phase plane is examined by some authors.[1)2)] And it is expected that the periodic solution obtained by applying the Galerkin's approximation where sine and cosine functions are completed, converges to the exact solution of the equation.[3)]

In this paper,[4)5)] we discuss the difference the cosine Fourier series and the sine Fourier series which are applied to obtain the approximate solution of the conservative system with one-degree-of-freedom. And it is shown that the approximate solution obtained by applying the cosine Fourier series converges to the exact solution of the equation, but that obtained by applying the sine Fourier series doesn't generally converge to the exact solution of the equation. Then the condition in which the approximate solution obtained by the sine Fourier series converges to the exact solution are examined.

2. The phase plane

The equation of motion with one-degree-of-freedom in conservation field is given by

$$\ddot{x}+K(x)=0 \tag{2}$$

Then, Eq. (2) is transformed into the following equation

$$\dot{x}=\pm\sqrt{2E_0^2-2V(x)} \tag{3-a}$$

where $V(x)=\int_0^x K(\xi)d\xi$:potential energy,

$E_0^2=\dot{x}_0^2/2+V(x_0)$:kinematic energy, (3–b)

$(x_0,\ \dot{x}_0)$:initial values.

The orbits of Eq. (3–a) depend on the function $K(x)$. But we place the focus on the periodic solution of Eq. (2), so it is assumed that there is a closed orbit in Eq. (3–a). Then the solution corresponding to the closed orbit is examined here.

3. The relations between a symmetry of the orbit and a symmetry of solution

3–1 A general discussion in conservative field

If there is a closed orbit in Eq. (3), the orbit is symmetric with respect to the x-axis. An example of the orbit is shown in Fig. 1. Now, we examine a feature of the periodic solution.

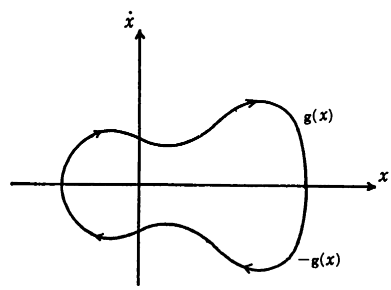

Fig.1 An Orbit

Let's suppose the initial value on the intersection of the closed orbit and the x-axis. Since the orbit is symmetric with respect to the x-axis, the orbit is expressed by the following function

$$\begin{cases}\dot{x}=g(x) & (t\leqq 0),\\ \dot{x}=-g(x) & (t\geqq 0).\end{cases} \tag{4}$$

Here, the following equation is applicable

$$dx=y\cdot dx \tag{5}$$

where $y=\dot{x}$, and substituting Eq. (4) into Eq. (5) and calculating the definite integral from initial values $(t,\ x(t))=(0,\ x(0))$ to $(t,\ x(t))$ or $(-t,\ x(-t))$, the following equations are obtained

$$x(t)=x(0)+\int_0^t\{-g(x)\}d\tau, \tag{6–a}$$

$$x(-t)=x(0)+\int_0^{-t}g(x)d\tau. \tag{6–b}$$

From Eq. (6), we can establish the expression

$$x(t)=x(-t)=x(0)-\int_0^t g(x)d\tau. \tag{7}$$

The above equation makes it clear that the solution $x(t)$ is an even function with respect to t. And the solution $x(t)$ is not only an even function with respect to t but a continuous periodic function because the solution is corresponding to a closed orbit of Eq. (3). If a function is an even function and a periodic function, it can be expressed by the cosine Fourier series. And the approximate solution obtained by applying the cosion Fourier series to the function converges to the exact periodic solution uniformly. Then let's examine the feature of the cosine Fourier series on the phase plane

$$x(t) = C_0 + \sum_{k=1}^{m} C_k \cos k\omega t. \tag{8}$$

Differentiating $x(t)$ in Eq. (8) with respect to t, the following equation is obtained

$$\dot{x}(t) = -\sum_{k=1}^{m} k\omega\, C_k \sin k\omega t. \tag{9}$$

From Eqs. (8) and (9), we have

$$(x(-t),\ \dot{x}(-t)) = (x(t), -\dot{x}(t)). \tag{10}$$

The above equation shows that the orbit of the periodic function expressed by Eq. (8) is symmetric with respect to the x-axis on the phase plane ($x-\dot{x}$ plane).

Then, if there exist a periodic solution in Eq. (2), the approximate solution which uniformly converges to the exact periodic solution is obtained by applying the consine Fourier series expressed in Eq. (8).

3–2 A special discussion in conservative field

In this section, a case where the closed orbit of Eq. (3) is symmetric with respect to $x=C_0$, as shown in Fig. 2, is considered.

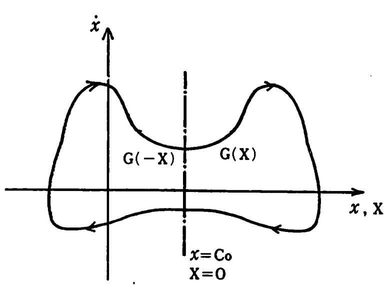

Fig. 2 An Orbit

Then, Eq. (3) is transformed by

$$x = C_0 + X \tag{11}$$

where C_0=constant.

Substituting Eq. (11) into Eq. (3), we have

$$\dot{X}(C_0+X) = \pm\sqrt{2E_0^2 - 2V(C_0+X)}. \tag{12}$$

As we are considering the case where the closed orbit is symmetric with respect to $x=C_0$

$$\dot{X}(C_0+X) = \dot{X}(C_0-X) \tag{13-a}$$

or

$$\dot{x}(X) = \dot{x}(-X). \tag{13-b}$$

From Eqs. (12) and (13), we have

$$V(C_0+X) = V(C_0-X). \tag{14}$$

From Eqs. (14), and (3–b), the following equation is obtained

$$K(C_0+X) = -K(C_0-X) \tag{15}$$

If we examine the case expressed by Eq. (13), it is shown that the system has the potential energy function $V(C_0+X)$ which is an even function with respect to X, and has the spring function $K(C_0+X)$ which is an odd function with respect to X.

Then, let's investigate the feature of the periodic solution of the system.

Set the initial value on the intersection of the closed orbit and $X=0$ ($x=C_0$). Since the closed orbit is symmetric with respect to $X=0$, it is expressed by

$$\dot{X}=G(X) \qquad (t\geqq 0) \tag{16-c}$$

$$\dot{X}=G(-X) \qquad (t\leqq 0) \tag{16-b}$$

where $G(X)=G(-X)$, $\dot{X}=\dot{x}$.

Substituting Eq. (16) into Eq. (15) and calculating the definite integral from initial values $(t, X(t))=(0, 0)$ to $(t, X(t))$ or $(-t, X(-t))$, we have

$$X(t)=X(0)+\int_0^t G(X)d\tau, \tag{17-a}$$

$$X(-t)=X(0)-\int_0^t G(X)d\tau. \tag{17-b}$$

Considering $X(0)=0$ in Eq. (17), the following equation is obtained

$$X(t)=-X(-t)=\int_0^t G(X)d\tau. \tag{18}$$

The above equation shows that the solution $X(t)$ is an odd function with respect to t. And the solution $X(t)$ is not only an odd function but a continuous periodic function because it is corresponding to a closed orbit.

If a function is an odd function and a periodic function, it can be expressed by the sine Fourier series. And the approximate solution obtained by applying the sine Fourier series converges to the exact periodic solution uniformly. Then, let's study the feature of the sine Fouries series on the phase plane,

$$X(t)=\sum_{k=1}^{m} S_k \sin k\omega t \tag{19-a}$$

or

$$x(t)=C_0+\sum_{k=1}^{m} S_k \sin k\omega t. \tag{19-b}$$

Differentiating $X(t)$ with respect to t, we have

$$\dot{X}(t)=\sum_{k=1}^{m} k\omega\, S_k \cos k\omega t. \tag{20}$$

From Eqs. (19) and (20), a point on the phase plane is expressed by

$$(X(-t),\ \dot{X}(-t))=(-X(t),\ \dot{X}(t)) \tag{21}$$

The above equation shows that the orbit of the periodic function expressed by Eq. (19) is symmetric with respect to $X=0$ ($x=C_0$) on the phase plane.

Then, if there is a periodic solution in Eq. (2) of which orbit on the phase plane is symmetric with respect to $x=C_0$, the approximate solution which uniformly converges to the exact periodic solution is obtained by applying the sine Fourier series expressed in Eq. (19).

4. The problems in elasticity considering the finite deformation theory

Considering the finite deformation theory in elasticity, we have the following equation of motion with one-degree-of-freedom in conservative field

$$\ddot{x}+\omega_0^2x+ax^2+bx^3=0 \tag{22}$$

The above equation is included in Eq. (2), and the discussion described in 3-1 is applied to it. Then, let's seek the condition where Eq. (22) is regarded as the system treated in 3-2. The condition examined in 3-2 is originally expressed by Eq. (13), and it is equivalent to Eq. (14) or Eq. (15).

Set $K(x)=\omega_0^2x+ax^2+bx^3$, the condition in Eqs. (13), (14) and (15) is given by

$$\begin{cases} a+3bC_0=0, \\ C_0(bC_0^2+aC_0+\omega_0^2)=0. \end{cases} \tag{23}$$

From Eq. (23), we have

$$a=0, \quad (C_0=0), \tag{24}$$

$$a^2=\frac{9}{2}b\omega_0^2\left(C_0=-\frac{a}{3b}\right). \tag{25}$$

If in the system where Eq. (24) or Eq. (25) is satisfied and there is a closed orbit which intersects the line $x=C_0$, the sine Fouries series in Eq. (19-b) may express the approximate periodic solution of the system which uniformly converges to the exact periodic solution.

The equation of motion corresponding to Eq. (24) is the Duffing equation, and the periodic solution of the equation may expressed by both cosine and sine Fourier series.

5. Numerical Analysis

Here, sinusoidal shallow arch models are adopted as the system governed by Eq. (22), because the coefficients of Eq. (22) for the arch are explicitly calculated in papers.[6)7)]
Then, we have

$$\ddot{x}+\omega_0^2x+ax^2+bx^3=0 \tag{26}$$

where $\omega_0^2=1+H^2/2$, $a=-3H/4$, $b=1/4$, H:the nondimentionalized rise of the arch.

We analyze the case which corresponds to the nondimentionalized rise $H=3$. The orbits on the phase plane is shown in Fig.3. In the system Eq. (24) or Eq. (25) is not satisfied and we

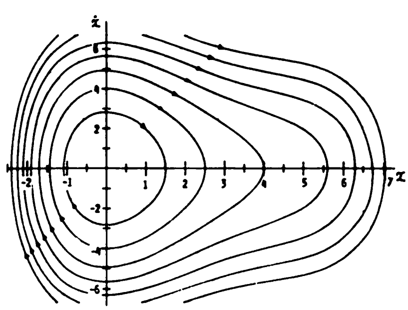

Fig.3 Orbits

have to apply the consine Fourier series in order to obtain the periodic solution which is uniformly converges to the exact periodic solution of the system.

Then let assume the following cosine Fourier series

$$x(t)=C_0+\sum_{k=1}^{5}C_k\cos k\omega t. \tag{27}$$

The backbone curve which is obtained by assuming Eq. (27) is shown in Fig.4. The response shape and the value of each coefficient in Eq. (27) at the points A, B, and C in Fig.4 are depicted in Figs. 5,6 and 7, respectively.

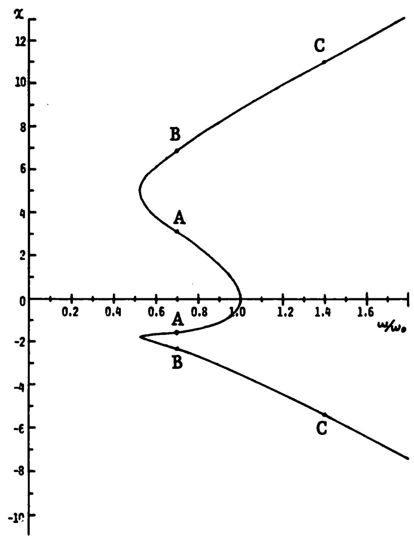

Fig.4 Backbone Curve

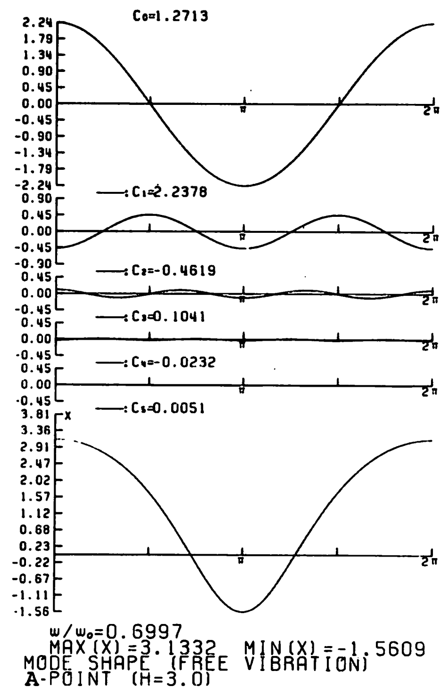

Fig.5 Response Shape

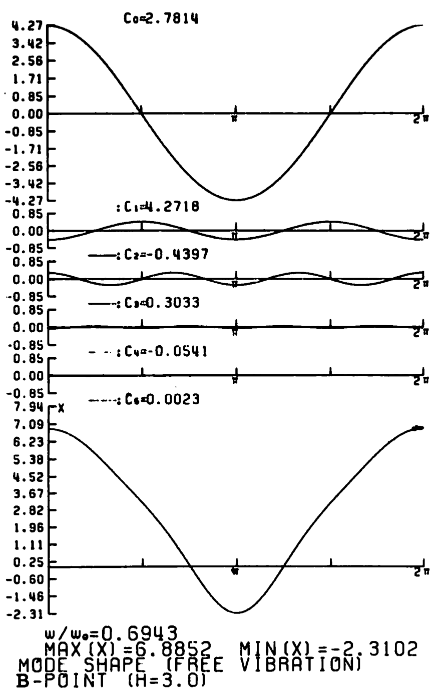

Fig.6 Response Shape

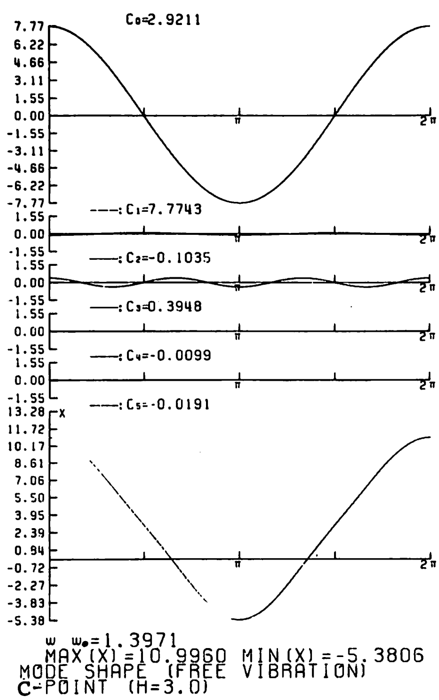

Fig.7 Response Shape

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