

# On the Propagation of Error in Numerical Integrations

著者	NAKASHIMA Masaharu
journal or publication title	鹿児島大学理学部紀要. 数学・物理学・化学
volume	6
page range	1-5
別言語のタイトル	積分近似式の誤差伝播について
URL	<a href="http://hdl.handle.net/10232/00000492">http://hdl.handle.net/10232/00000492</a>

## On the Propagation of Error in Numerical Integrations

By

Masaharu NAKASHIMA

(Received September 30, 1973)

We consider the first order differential equation

$$(1) \quad \begin{cases} y' = f(x, y), \\ y(x_0) = y_0. \end{cases}$$

And in [5] we have tried to approximate the equation by some difference equation, which is of open type, and have considered the propagation of error of the formula. Here we consider the closed type approximation formula, i.e., Adams-Bashforth type, and investigate the propagation of error of the formula. We denote the value of the solution of the differential equation (1) at the point  $x_n$  by  $y(x_n)$  and the  $i$ -th approximation to  $y(x_n)$  by  $y_n^{(i)}$ . We shall now try to approximate the equation (1) by the difference equation

$$(2) \quad y(x_{n+1}) = y(x_n) + hf(x_{n+1}, y(x_{n+1})) + T_n \left( T_n = \frac{1}{2} h^2 y''(x_n + \theta_n h), 0 \leq \theta_n \leq 1 \right),$$

$$y_{n+1}^{(i+1)} = y_n + hf(x_{n+1}, y_{n+1}^{(i)}), \quad (n = 0, 1, \dots)$$

where we denote the truncation error of  $n$ -step by  $T_n$ .

And the calculated value of  $y_n$  will be given by the formula:

$$(3) \quad Y_{n+1}^{(i+1)} = Y_n + hf(x_{n+1}, Y_{n+1}^{(i)}) - R_n^{(i+1)},$$

where  $R_n^{(i+1)}$  is the round-off error of the  $(i+1)$ -th iteration of  $n$ -step

And if the difference

$$L_n^{(i+1)} = Y_n^{(i+1)} - Y_n^{(i)}$$

is smaller than the constant  $L$ :

$$|L_n^{(i+1)}| \leq L,$$

then we set

$$Y_{n+1} = Y_{n+1}^{(i+1)}.$$

From the equations (2), (3), we obtain the relation

$$\begin{aligned} y(x_{n+1}) - Y_{n+1}^{(i+1)} &= y(x_n) - Y_n + hf(x_{n+1}, y(x_{n+1})) \\ &\quad - hf(x_{n+1}, Y_{n+1}^{(i)}) + T_n + R_n^{(i+1)}. \end{aligned}$$

And if  $\frac{\partial f}{\partial y}(x, y)$  exists, we have the relations

$$f(x_{n+1}, \eta(x_{n+1})) - f(x_{n+1}, Y_{n+1}^{(i)}) = \frac{\partial f}{\partial y}(x_{n+1}, \eta_{n+1})(y(x_{n+1}) - Y_{n+1}^{(i)}),$$

for some  $\eta_{n+1}$  which lies between  $Y_{n+1}^{(i)}$  and  $y(x_{n+1})$ ,  
and

$$Y_{n+1}^{(i+1)} - y(x_{n+1}) = Y_{n+1} - y(x_{n+1}).$$

If we set

$$e_n = y(x_n) - Y_n, \quad e_0 = 0,$$

$$W_n^{(i)} = T_n + R_n^{(i)} + h \frac{\partial f}{\partial y}(x_{n+1}, \eta_{n+1})(Y_n - Y_n^{(i)}),$$

then we obtain the relation

$$(4) \quad e_{n+1} = e_n + h \frac{\partial f}{\partial y}(x_{n+1}, \eta_{n+1}) e_{n+1} + W(x + (n+1)h)^{(i)}.$$

And by using the backward difference operator, the above equation may be written in the form (for some constant  $\rho$ )

$$(5) \quad \nabla e_{n+1} = \rho e_{n+1} + \left( h \frac{\partial f}{\partial y}(x_{n+1}, \eta_{n+1}) - \rho \right) e_{n+1} + W_{n+1}^{(i)}.$$

Here we discuss the asymptotic behavior of the difference equations (4), (5).

**THEOREM 1** *Under the following assumptions,*

$$(I) \quad |f_y(x_0 + \nu h, \eta_\nu)| \leq \Phi(x) \quad (-\infty < y < +\infty)$$

where  $\Phi(x)$  is a continuous function satisfying the following condition

$$^1K > 0, \quad ^1h > 0 \quad \sum_{\nu=1}^{\infty} h \Phi(x_0 + \nu h) \leq K \quad \text{for } 0 < h \leq h_0, \quad \text{and } \Phi(x) \leq M,$$

$$(II) \quad \sum_{\nu=1}^{\infty} |W_\nu^{(i)}| \leq E \quad \text{for } 0 < h \leq h_1,$$

we have

$$|e_n| \leq C \quad \text{for } 0 < h \leq \min \left\{ h_0, h_1, \frac{C - CK - E}{CM} \right\}.$$

**PROOF.**

The proof is derived by mathematical induction.

Let us assume

$$|e_\nu| \leq C \quad (\nu = 1, 2, 3, \dots, n-1),$$

and we shall show

$$|e_n| \leq C.$$

From (4), we have the inequality

$$|(1 - hf_y(x_0 + nh, \eta_n)) e_n| < h \sum_{\nu=1}^{n-1} |f_y(x_0 + \nu h, \eta_\nu) e_\nu| \\ + \sum_{\nu=1}^n |W_\nu^{(i)}|,$$

and taking the constant  $h$  small, we have

$$|e_n| \leq C.$$

Q.E.D.

Next we shall investigate the propagation of error more explicitly.

LEMMA [1]

*The solution of the equation*

$$(6) \quad \nabla z(x_0 + nh) = A_n z(x_0 + nh) + B(x_0 + nh) z(x_0 + nh) + W(x_0 + nh) \quad (n = 1, 2, 3, \dots), \\ z(x_0) = z_0, \quad (A_n \neq 1 : n = 1, 2, \dots),$$

is

$$z(x_0 + nh) = z_0 Y(x_0 + nh) + Y(x_0 + nh) \sum_{\nu=1}^{n-1} \{B(x_0 + \nu + 1) h\} Y^{-1}(x_0 + \nu h) z(x_0 + (\nu + 1) h) \\ + Y(x_0 + nh) \sum_{\nu=0}^{n-1} Y^{-1}(x_0 + \nu h) W(x_0 + (\nu + 1) h),$$

where  $Y(x)$  is the solution of the following difference equation

$$\begin{cases} \nabla Y(x_0 + \nu h) = A_\nu Y(x_0 + \nu h) & (\nu = 1, 2, \dots), \\ Y(x_0) = 1. \end{cases}$$

PROOF.

The proof is derived by the well-known method, namely the variation of parameters. Let

$$z(x_0 + nh) = Y(x_0 + nh) u(x_0 + nh),$$

then

$$\begin{aligned} \nabla z(x_0 + nh) &= \nabla \{Y(x_0 + nh) u(x_0 + nh)\} \\ &= (\nabla Y(x_0 + nh)) u(x_0 + nh) + Y(x_0 + (n-1)h) \nabla u(x_0 + nh) \\ &= A_n Y(x_0 + nh) u(x_0 + nh) + Y(x_0 + (n-1)h) \nabla u(x_0 + nh) \\ &= A_n z(x_0 + nh) + B(x_0 + nh) z(x_0 + nh) + W(x_0 + nh). \end{aligned}$$

Thus

$$\nabla u(x_0 + nh) = Y^{-1}(x_0 + (n-1)h) \{B(x_0 + nh) z(x_0 + nh) + W(x_0 + n)h\},$$

and hence

$$u(x_0 + nh) = u(x_0) + \sum_{\nu=1}^n Y^{-1}(x_0 + (\nu-1)h) B(x_0 + \nu h) z(x_0 + \nu h) \\ + \sum_{\nu=1}^n Y^{-1}(x_0 + (\nu-1)h) W(x_0 + \nu h),$$

where

$$z(x_0) = Y(x_0) u(x_0) \\ = u(x_0).$$

Thus we have the solution

$$z(x_0 + nh) = Y(x_0 + nh) u(x_0 + nh) \\ = z(x_0) Y(x_0 + nh) + Y(x_0 + nh) \sum_{\nu=0}^{n-1} \{ Y^{-1}(x_0 + \nu h) B(x_0 + (\nu+1)h) z(x_0 + (\nu+1)h) \} \\ + Y(x_0 + nh) \sum_{\nu=0}^{n-1} Y^{-1}(x_0 + \nu h) W(x_0 + (\nu+1)h).$$

Q.E.D.

#### LEMMA [2]

In the equation (6), if there exist constants  $\rho_n$ ,  $C$ ,  $\lambda_\nu$ ,  $L_1$ ,  $L_2$ , which satisfy the following conditions

$$(I) \quad |Y(x_0 + \nu h)| \leq C \quad (C; \text{constant}),$$

where  $Y(x)$  is a solution of the difference equation

$$\begin{cases} \nabla Y(x_0 + \nu h) = \rho_\nu Y(x_0 + \nu h), & (\rho_\nu \neq 1). \\ Y(x_0) = 1, \end{cases}$$

$$(II) \quad \sum_{\nu=0}^{\infty} |B(x_0 + \nu h)| \leq L_1,$$

$$(III) \quad |W(x_0 + \nu h)| \leq a_\nu e^{\lambda_1 + \lambda_2 + \dots + \lambda_{\nu+1}} \quad (\nu = 0, 1, 2, \dots),$$

where  $\sum_{\nu=0}^{\infty} a_\nu \leq L_2$ ,  $L_1 + L_2 < |1 - \rho_n - B(x_0 + nh)|$  ( $n = 1, 2, \dots$ ) and

$$|1 - \rho_n|^{-1} \leq e^{\lambda_{n+1}} \quad (n = 1, 2, 3, \dots),$$

then we have

$$|z(x_0 + nh)| \leq g(x_0) e^{\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n} \quad (n = 1, 2, 3, \dots).$$

#### PROOF.

The proof is derived by mathematical induction.

Let us assume

$$|z(x_0 + mh)| \leq g(x_0) e^{\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_m} \quad (m = 1, 2, \dots, n-1),$$

and we shall show that the above inequality holds for  $m=n$ .

From Lemma [1], we have

$$\begin{aligned}
& \{1 - Y(x_0 + nh) Y^{-1}(x_0 + (n-1)h) B(x_0 + nh)\} z(x_0 + nh) \\
&= z(x_0) Y(x_0 + nh) + Y(x_0 + nh) \sum_{\nu=1}^{n-2} B(x_0 + (\nu+1)h) Y^{-1}(x_0 + \nu h) z(x_0 + (\nu+1)h) \\
&\quad + Y(x_0 + nh) \sum_{\nu=1}^{n-1} Y^{-1}(x_0 + \nu h) W(x_0 + (\nu+1)h).
\end{aligned}$$

And hence we have the inequality

$$|(1 - \rho_n - B(x_0 + nh)) z(x_0 + nh)| \leq g(x_0) e^{\lambda_1 + \lambda_2 + \dots + \lambda_n} (L_1 + L_2),$$

and from the condition (3), we have

$$|z(x_0 + nh)| \leq g(x_0) e^{\lambda_1 + \lambda_2 + \dots + \lambda_n}.$$

Q.E.D.

From the above Lemma we may derive the next theorem.

**THEOREM 2** Consider the difference equation (5). If the following conditions are satisfied:

$$(I) \quad |W_v^{(i)}| \leq a_v e^{\lambda_1 + \lambda_2 + \dots + \lambda_{v+1}} \quad \text{for } 0 < h \leq h_1 \quad (v = 1, 2, \dots),$$

$$\text{where} \quad \sum_{\nu=1}^{\infty} |a_\nu| \leq \tilde{L}_1, \quad \sum_{\nu=1}^{\infty} |\rho_\nu| \leq \tilde{L}_2, \quad |(1 - \rho_n)^{-1}| \leq e^{\lambda_{n+1}}$$

$$\text{and} \quad \tilde{L}_1 + \tilde{L}_2 < 1,$$

$$(II) \quad |f_y(x_0 + \nu h, \eta_\nu)| \leq \Phi(x) \quad (-\infty < y < +\infty),$$

where  $\Phi(x)$  is continuous satisfying following conditions

$$\sum_{\nu=0}^{\infty} h \Phi(x_0 + \nu h) \leq 1 - \tilde{L}_1 - \tilde{L}_2 \quad \text{for } 0 < h \leq h_0 \quad \text{and} \quad \Phi(x) \leq K$$

then we have

$$|e_n| \leq g(x_0) e^{\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n} \quad \text{for } 0 < h \leq \min \left\{ h_0, h_1, \frac{1}{K} \right\}.$$

### References

- [1] R. Bellman: On the boundedness of solutions of nonlinear differential and difference equations. Trans. Amer. Math. Soc. 46, 357-386 (1948).
- [2] E.A. Goddington and N. Levinson: Theory of Ordinary Differential Equations. New York, McGraw-Hill 1955.
- [3] F.B. Hildebrand: Introduction to Numerical Analysis. New York, McGraw-Hill 1956.
- [4] M. Lotkin: Propagation of error in numerical integrations. Proc. Amer. Math. Soc. 5, 869-887 (1954).
- [5] M. Nakashima: On the Propagation of Error in Numerical Integration. Proc. Japan Acad. 48, 484-488 (1972).
- [6] M. Nakashima: A Correction to: On the Propagation of Error in Numerical Integration. (to appear)