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A POSTERIORI ERROR ESTIMATES FOR TWO POINT BOUNDARY VALUE PROBLEMS

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Abstract

We consider error estimates for the Galerkin approximations of two point boundary value problems. The error formulas are asymptotically expressed in terms of *a posteriori* errors.

1. Introduction

In this paper we consider error estimates for the Galerkin approximations of the following two point boundary value problems:

$$\begin{aligned} &-(a(x)u')'+b(x)u=f(x), \qquad x\in I,\\ &u(0)=u(1)=0 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &-u''+a(x)u'+b(x)u=f(x), \qquad x\in I,\\ &u(0)=u(1)=0. \end{aligned} \tag{1.2}$$

Already, by Babuška and Rheinboldt, error formulas and optimal partitions have been published in the case of the piecewise linear approximation for (1.1) ([1]). In this paper we employ the piecewise polynomials of degree more than 2. The error formulas in [1] were considered under the conditions:

$$u_0^{(r+1)}(x)\neq 0, \qquad x\in I$$

and

$$u_0^{(r+1)}(\mu_k)=0, \quad u_0^{(r+2)}(\mu_k)\neq 0, \quad k=1, \dots, q, \quad 0\leq \mu_1<\mu_2<\dots<\mu_q\leq 1, \tag{1.3}$$

where u_0 is the solution of (1.1).

The main object of this paper is to introduce error formulas under more general condition than (1.3):

$$u_0^{(r+1)}(\mu_k)=0, \qquad k=1, \dots, q, \qquad 0\leq \mu_1<\mu_2<\dots<\mu_q\leq 1.$$

First, in Section 3, we consider the following simple problem:

$$\begin{aligned} &-u''=f(x), \qquad x\in I,\\ &u(0)=u(1)=0. \end{aligned}$$

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For this problem we consider the properties of the error and error estimates. And, based on these properties and error formulas, we consider error formulas for (1.1) and (1.2) in Sections 4 and 5, respectively. The error formulas are asymptotically expressed in terms of *a posteriori* errors.

The results in this paper may be generalized for error estimates under other norms than we use here. Also, by using the results of Sections 4 and 5, we shall mention optimal partitions ([3]).

2. Notations

Let $I=[0, 1]$. On I we consider partitions

$$\Delta : 0=x_0<x_1<x_2<\dots<x_{m-1}<x_m=1$$

and introduce the notations

$$\left.\begin{array}{l} I_j=[x_{j-1}, x_j] \\ h_j=x_j-x_{j-1} \end{array}\right\}, \qquad j=1, \dots, m,$$

$$\overline{h}=\max_{1\le j\le m} h_j, \qquad \underline{h}=\min_{1\le j\le m} h_j.$$

All partitions Δ which for fixed $\lambda>0$, $\varkappa\ge1$ satisfy

$$\underline{h}\ge\lambda\overline{h}^{\varkappa}$$

are said to be $(\lambda, \varkappa)$-regular.

On an interval J $(J\subseteq I)$ we define

$$(u, v)_J=\int_J uv\, dx.$$

If $P_r(J)$ denote the collection of all polynomials of degree not greater than r, then continuous piecewise polynomial space $\mathscr{M}_\Delta^r$ is defined as usual by

$$\mathscr{M}_\Delta^r=\{v\in C^0(I) \mid v|_{I_j}\in P_r(I_j),\ j=1, \dots, m\,;\ v(0)=v(1)=0\}.$$

And $P_r^0(J)$ consists of the polynomials which belong to $P_r(J)$ and vanish at the endpoints of J.

Also let $\eta_1^r, \eta_2^r, \dots, \eta_{r-1}^r$ be the different zero points of the Jacobi polynomial

$$J_r(x)=\frac{1}{x(1-x)}\frac{d^{r-1}}{dx^{r-1}}[(x(1-x))^r]$$

with weight function $x(1-x)$ and we define

$$x_{ij}^r=x_{i-1}+h_i\eta_j^r, \qquad i=1, \dots, m, \qquad j=1, \dots, r-1.$$

From now on, let $r\ge2$ and C be a generic constant independent of any partition.

3. *A posteriori* error estimates—Part I

In this section we consider the following two point boundary value problem:

$$(3.1) \qquad \begin{array}{ll} Lu\equiv-u''=f(x), & x\in I, \\ u(0)=u(1)=0, & \end{array}$$

where we assume that $f\in C^r(I)$.

The solution u_0 of (3.1) belongs to $C^{r+2}(I)$. Let $z_{\Delta,r}\in\mathscr{M}_\Delta^r$ be the Galerkin approximation to u_0 determined by the relation

$$(z_{\Delta,r}', v')_I=(f, v)_I, \qquad \forall v\in\mathscr{M}_\Delta^r.$$

Set

$$z = u_0 - z_{\Delta,r}.$$

Then the following result is well known:

LEMMA 3.1. *For all partitions* Δ *the error* z *satisfies at the knots*

$$z(x_j)=0, \qquad j=0, \ldots, m.$$

Proof. The Green's function $G(x, \xi)$ for (3.1) is given by

$$G(x, \xi)=\begin{cases} x(1-\xi), & 0\leq x\leq \xi, \\ \xi(1-x), & \xi\leq x\leq 1. \end{cases}$$

In particular, at the knots it follows that

$$G(x_j, \cdot)\in \mathscr{M}_\Delta^r, \qquad j=0, \ldots, m. \tag{3.2}$$

By using $G(x, \cdot)$ we have

$$\begin{aligned} u(x)&=(Lu, G(x, \cdot))_I \\ &=\left(u', \frac{\partial G}{\partial \xi}(x, \cdot)\right)_I \end{aligned}$$

This representation holds for $u\in H_0^1(I)$ so that it can be applied to z. Since

$$(z', v')_I=0, \qquad \forall v\in \mathscr{M}_\Delta^r,$$

we have

$$\begin{aligned} z(x_j)&=\left(z', \frac{\partial G}{\partial \xi}(x_j, \cdot)\right)_I \\ &=\left(z', \frac{\partial G}{\partial \xi}(x_j, \cdot)-v'\right)_I, \qquad \forall v\in \mathscr{M}_\Delta^r, \end{aligned} \tag{3.3}$$

from which follows

$$|z(x_j)|\leq \|z'\|_{L^2(I)} \inf_{v\in u_\Delta^r} \left\|\frac{\partial G}{\partial \xi}(x_j, \cdot)-v'\right\|_{L^2(I)}.$$

From (3.2) it follows that

$$z(x_j)=0, \qquad j=0, \ldots, m.$$

This completes the proof of Lemma 3.1.

Note that this lemma holds for all continuous piecewise polynomials which are the Galerkin approximations to u_0. Next lemma shows the relation between $z_{\Delta,r}$ and $z_{\Delta,r+1}$ at the knots and the Jacobi points.

LEMMA 3.2. *For all partitions* Δ, *at the knots and the Jacobi points we have*

$$\begin{aligned} &z_{\Delta,r+1}(x_i)-z_{\Delta,r}(x_i)=0, \qquad i=0, \ldots, m, \\ &z_{\Delta,r+1}(x_{ij}^r)-z_{\Delta,r}(x_{ij}^r)=0, \qquad i=1, \ldots, m,\ j=1, \ldots, r-1. \end{aligned}$$

Proof. It follows from Lemma 3.1 that

$$\left.\begin{aligned} u_0(x_i)-z_{\Delta,r}(x_i)=0 \\ u_0(x_i)-z_{\Delta,r+1}(x_i)=0 \end{aligned}\right\}, \qquad i=0, \ldots, m.$$

Hence

$$z_{\Delta,r+1}(x_i)-z_{\Delta,r}(x_i)=0, \qquad i=0, \ldots, m.$$

Since

$$\left.\begin{aligned} (z_{\Delta,r}', w')_{I_i}=(f, w)_{I_i} \\ (z_{\Delta,r+1}', w')_{I_i}=(f, w)_{I_i} \end{aligned}\right\}, \qquad \forall w\in P_r^0(I_i), \qquad i=1, \ldots, m,$$

we have

$$(3.4)\qquad (z'_{\Delta,r+1}-z'_{\Delta,r},\ w')_{I_i}=0,\qquad \forall w\in P_r^0(I_i),\qquad i=1,\ \dots,\ m.$$

We take $w_j\in P_r^0(I_i)$ which satisfies $w_j''(x_{ik}^r)=\delta_{jk}$ as w. Since $z_{\Delta,r+1}-z_{\Delta,r}\in P_{r+1}^0(I_i)$ and $w_j''\in P_{r-2}(I_i)$, it follows from the property of Jacobi points that there are positive constants ω_k with $1\le k\le r-1$ such that

$$\begin{aligned}|(z'_{\Delta,r+1}-z'_{\Delta,r},\ w'_j)_{I_i}| &= |(z_{\Delta,r+1}-z_{\Delta,r},\ w''_j)_{I_i}|\\ &=\left|h_i\sum_{k=1}^{r-1}\omega_k\frac{(z_{\Delta,r+1}(x_{ik}^r)-z_{\Delta,r}(x_{ik}^r))w_j''(x_{ik}^r)}{\eta_k(1-\eta_k)}\right|\\ &=h_i\frac{\omega_j}{\eta_j(1-\eta_j)}|z_{\Delta,r+1}(x_{ij}^r)-z_{\Delta,r}(x_{ij}^r)|.\end{aligned}$$

Hence it follows from (3.4) that

$$z_{\Delta,r+1}(x_{ij}^r)-z_{\Delta,r}(x_{ij}^r)=0.$$

This completes the proof of Lemma 3.2.

Also it follows from Lemma 3.1 that for each subinterval I_j the following estimate holds independent of every other subinterval.

LEMMA 3.3. *For all partitions* Δ *there are constants* C *such that*

$$\|z^{(k)}\|_{L^\infty(I_j)}\le C\|u_0^{(r+1)}\|_{L^\infty(I_j)}h_j^{r+1-k},\qquad k=0,\ \dots,\ r,\qquad j=1,\ \dots,\ m,$$

where the constants C *depend on* r *and* k *but not on* h_j.

Proof. Note that

$$(z',\ v')_{I_j}=0,\qquad \forall v\in P_r^0(I_j),\qquad j=1,\ \dots,\ m.$$

Let $\hat{u}_0$ be the Lagrange interpolation of degree r to u_0 on I_j. Then it follows from Lemma 3.1 that $z_{\Delta,r}-\hat{u}_0\in P_r^0(I_j)$, and, therefore,

$$\begin{aligned}(z',\ z')_{I_j}&=(z',\ z'+(z_{\Delta,r}-\hat{u}_0)')_{I_j}\\ &=(z',\ u'_0-\hat{u}'_0)_{I_j}\\ &\le C\|z'\|_{L^2(I_j)}\|u_0^{(r+1)}\|_{L^\infty(I_j)}h_j^{r+1/2},\end{aligned}$$

i.e.,

$$\|z'\|_{L^2(I_j)}\le C\|u_0^{(r+1)}\|_{L^\infty(I_j)}h_j^{r+1/2}.$$

Hence we have

$$\begin{aligned}\|z\|_{L^\infty(I_j)}&\le C\|z'\|_{L^2(I_j)}h_j^{1/2}\\ &\le C\|u_0^{(r+1)}\|_{L^\infty(I_j)}h_j^{r+1}.\end{aligned}$$

Also we have

$$\begin{aligned}\|z_{\Delta,r}-\hat{u}_0\|_{L^\infty(I_j)}&\le\|z\|_{L^\infty(I_j)}+\|u_0-\hat{u}_0\|_{L^\infty(I_j)}\\ &\le C\|u_0^{(r+1)}\|_{L^\infty(I_j)}h_j^{r+1},\end{aligned}$$

which together with Markoff's inequality implies that

$$\|z_{\Delta,r}^{(k)}-\hat{u}_0^{(k)}\|_{L^\infty(I_j)}\le C\|u_0^{(r+1)}\|_{L^\infty(I_j)}h_j^{r+1-k},\qquad k=0,\ \dots,\ r.$$

On the other hand,

$$\|u_0^{(k)}-\hat{u}_0^{(k)}\|_{L^\infty(I_j)}\le C\|u_0^{(r+1)}\|_{L^\infty(I_j)}h_j^{r+1-k},\qquad k=0,\ \dots,\ r.$$

Hence it follows that

$$\begin{aligned}\|z^{(k)}\|_{L^\infty(I_j)}&\le\|u_0^{(k)}-\hat{u}_0^{(k)}\|_{L^\infty(I_j)}+\|\hat{u}_0^{(k)}-z_{\Delta,r}^{(k)}\|_{L^\infty(I_j)}\\ &\le C\|u_0^{(r+1)}\|_{L^\infty(I_j)}h_j^{r+1-k},\qquad k=0,\ \dots,\ r.\end{aligned}$$

This completes the proof of Lemma 3.3.

Now set

$$\left.\begin{array}{l}\rho_j(x)=u_0''(x)-z_{\Delta,r}''(x)\\ \phi_j(x)=z_{\Delta,r+1}''(x)-z_{\Delta,r}''(x)\end{array}\right\},\qquad x\in I_j,\qquad j=1,\ \dots,\ m.$$

Then we obtain the following lemma:

LEMMA 3.4. *Suppose that*

$$u_0^{(r+1)}(x)\neq 0,\qquad \forall x\in I.$$

Then, for all partitions Δ *there are constants* $C(r)$ *and* $\tilde{C}(r)$ *such that*

$$\|z'\|_{L^2(I)}=C(r)\left[\sum_{j=1}^{m}\|\phi_j\|_{L^2(I_j)}^2h_j^2\right]^{1/2}(1+O(\bar{h}))\quad\text{as}\quad \bar{h}\to 0,\tag{3.5}$$

and

$$\|z'\|_{L^2(I)}=\tilde{C}(r)\left[\sum_{j=1}^{m}\|\phi_j^{(r-1)}\|_{L^2(I_j)}^2h_j^{2r}\right]^{1/2}(1+O(\bar{h}))\quad\text{as}\quad \bar{h}\to 0,\tag{3.6}$$

where the constants in the bounds of the O-terms depend on f and r but not on Δ *and the constants* $C(r)$ *and* $\tilde{C}(r)$ *are uniquely determined by r.*

Proof. Set

$$\left.\begin{array}{l}\sigma_j(x)=\rho_j(x)-\phi_j(x)\\ \psi_{1,j}(x)=z_{\Delta,r+1}(x)-z_{\Delta,r}(x)\\ \psi_{2,j}(x)=u_0(x)-z_{\Delta,r+1}(x)\end{array}\right\},\qquad \forall x\in I_j,\qquad j=1,\ \dots,\ m.$$

Then obviously

$$\left.\begin{array}{ll}(3.7)\quad & z(x)=\psi_{1,j}(x)+\psi_{2,j}(x)\\ (3.8)\quad & \sigma_j(x)=u_0''(x)-z_{\Delta,r+1}''(x)\end{array}\right\},\qquad \forall x\in I_j,\qquad j=1,\ \dots,\ m,$$

$$(\psi_{2,j}',\ v')_{I_j}=-(\sigma_j,\ v)_{I_j},\qquad \forall v\in H_0^1(I_j),\qquad j=1,\ \dots,\ m.\tag{3.9}$$

Also set

$$\rho_0=\min\{|u_0^{(r+1)}(x)|,\ x\in I\},$$

$$\left.\begin{array}{l}\bar{\rho}_j=\max\{|u_0^{(r+1)}(x)|,\ x\in I_j\}\\ \bar{\phi}_j=\max\{|\phi_j^{(r-1)}(x)|,\ x\in I_j\}\end{array}\right\},\qquad j=1,\ \dots,\ m.$$

By the assumption we have

$$\frac{|\rho_j^{(r-1)}(x)|}{\rho_0}\geq 1,\qquad \forall x\in I_j,\qquad j=1,\ \dots,\ m.$$

and, hence, it follows from Lemma 3.3 and (3.8) that

$$|\sigma_j^{(r-1)}(x)|=|u_0^{(r+1)}(x)-z_{\Delta,r+1}^{(r+1)}(x)|\leq Ch_j\leq C\frac{|\rho_j^{(r-1)}(x)|}{\rho_0}h_j$$

This implies that

$$\begin{aligned}\phi_j^{(r-1)}(x)&=\rho_j^{(r-1)}(x)-\sigma_j^{(r-1)}(x)\\ &=\rho_j^{(r-1)}(x)(1+O(h_j))\quad\text{as}\quad h_j\to 0.\end{aligned}\tag{3.10}$$

Therefore, for all j with $1\leq j\leq m$,

$$\begin{aligned}|\sigma_j(x)|&\leq Ch_j^r\leq C\frac{\bar{\rho}_j}{\rho_0}h_j^r\\ &\leq C\frac{\bar{\phi}_j}{\rho_0}h_j^r(1+O(h_j))\quad\text{as}\quad h_j\to 0,\qquad x\in I_j.\end{aligned}\tag{3.11}$$

On the other hand, it follows from $\phi_j\in P_{r-1}(I_j)$ that

$$\bar{\phi}_j \leq C \| \phi_j^{(r-1)} \|_{L^2(I_j)} h_j^{-1/2}$$
$$\leq C \| \phi_j^{(r-2)} \|_{L^2(I_j)} h_j^{-3/2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\leq C \| \phi_j \|_{L^2(I_j)} h_j^{-r+1/2}.$$

Combining this inequality with (3.11), we obtain

$$| \sigma_j(x) | \leq C \| \phi_j \|_{L^2(I_j)} h_j^{1/2} (1 + O(h_j))$$

and

$$| \sigma_j(x) | \leq C \| \phi_j^{(r-1)} \|_{L^2(I_j)} h_j^{r-1/2} (1 + O(h_j)),$$

which imply

(3.12) $$\| \sigma_j \|_{L^2(I_j)} \leq C \| \phi_j \|_{L^2(I_j)} h_j (1 + O(h_j)) \quad \text{as} \quad h_j \to 0$$

and

(3.13) $$\| \sigma_j \|_{L^2(I_j)} \leq C \| \phi_j^{(r-1)} \|_{L^2(I_j)} h_j^r (1 + O(h_j)) \quad \text{as} \quad h_j \to 0.$$

Also, from (3.9) and $\psi_{2,j} \in H_0^1(I_j)$ we have

$$\| \psi_{2,j}' \|_{L^2(I_j)}^2 \leq \| \psi_{2,j} \|_{L^2(I_j)} \| \sigma_j \|_{L^2(I_j)}$$
$$\leq C \| \psi_{2,j}' \|_{L^2(I_j)} \| \sigma_j \|_{L^2(I_j)} h_j$$

i.e.,

(3.14) $$\| \psi_{2,j}' \|_{L^2(I_j)} \leq C \| \sigma_j \|_{L^2(I_j)} h_j,$$

which together with (3.12) and (3.13) gives

(3.15) $$\| \psi_{2,j}' \|_{L^2(I_j)} \leq C \| \phi_j \|_{L^2(I_j)} h_j^2 (1 + O(h_j))$$

and

(3.16) $$\| \psi_{2,j}' \|_{L^2(I_j)} \leq C \| \phi_j^{(r-1)} \|_{L^2(I_j)} h_j^{r+1} (1 + O(h_j)).$$

Moreover, it follows from Lemma 3.2 that

$$\psi_{1,j}(x_{j-1}) = \psi_{1,j}(x_{j1}^r) = \ldots = \psi_{1,j}(x_{jr-1}^r) = \psi_{1,j}(x_j) = 0.$$

Let s_{r+1} be the polynominal of degree $r+1$ on I so that

$$s_{r+1}(0) = s_{r+1}(\eta_1^r) = \ldots = s_{r+1}(\eta_{r-1}^r) = s_{r+1}(1) = 0$$

and $s_{r+1}^{(r+1)}(x) = 1$.Then we have

$$\psi_{1,j}(x) = \psi_{1,j}^{(r+1)}(x) h_j^{r+1} s_{r+1}\left(\frac{x - x_{j-1}}{h_j}\right), \quad x \in I_j, \qquad j = 1, \ldots, m,$$

where $\psi_{1,j}^{(r+1)}(x)$ is a constant. We denote

$$C(r) = \frac{\| s_{r+1}' \|_{L^2(I)}}{\| s_{r+1}'' \|_{L^2(I)}}, \qquad \tilde{C}(r) = \frac{\| s_{r+1}' \|_{L^2(I)}}{\| s_{r+1}^{(r+1)} \|_{L^2(I)}}.$$

From the representation of $\psi_{1,j}$ we obtain

$$\| \psi_{1,j}' \|_{L^2(I_j)} = C(r) \| \psi_{1,j}'' \|_{L^2(I_j)} h_j = \tilde{C}(r) \| \psi_{1,j}^{(r+1)} \|_{L^2(I_j)} h_j^r,$$

i.e.,

(3.17) $$\| \psi_{1,j}' \|_{L^2(I_j)} = C(r) \| \phi_j \|_{L^2(I_j)} h_j = \tilde{C}(r) \| \phi_j^{(r-1)} \|_{L^2(I_j)} h_j^r.$$

It follows from (3.7), (3.15) and (3.17) that with some α

$$\| z' \|_{L^2(I_j)}^2 = (\psi_{1,j}' + \psi_{2,j}', \ \psi_{1,j}' + \psi_{2,j}')_{I_j}$$
$$= \| \psi_{1,j}' \|_{L^2(I_j)}^2 + 2\alpha \| \psi_{1,j}' \|_{L^2(I_j)} \| \psi_{2,j}' \|_{L^2(I_j)} + \| \psi_{2,j}' \|_{L^2(I_j)}^2$$
$$= C(r)^2 \| \phi_j \|_{L^2(I_j)}^2 h_j^2 (1 + O(h_j))$$

and

$$\|z'\|_{L^2(I)}=C(r)\left[\sum_{j=1}^{m}\|\phi_j\|^2_{L^2(I_j)}h_j^2\right]^{1/2}(1+O(\bar{h}))\quad \text{as}\quad \bar{h}\to 0.$$

Similarly, by (3.16) and (3.17) we obtain

$$\|z'\|_{L^2(I)}=\tilde{C}(r)\left[\sum_{j=1}^{m}\|\phi_j^{(r-1)}\|^2_{L^2(I_j)}h_j^{2r}\right]^{1/2}(1+O(\bar{h}))\quad \text{as}\quad \bar{h}\to 0$$

This completes the proof of this lemma.

From now on, let the constants $C(r)$ and $\tilde{C}(r)$ be the values in Lemma 3.4.

By Lemma 3.4 we obtain the following result:

THEOREM 3.5. *On the assumption of Lemma* 3.4 *we have*

$$\|z'\|_{L^2(I)}=C(r)\left[\sum_{j=1}^{m}\|\rho_j\|^2_{L^2(I_j)}h_j^2\right]^{1/2}(1+O(\bar{h}))\quad \text{as}\quad \bar{h}\to 0. \tag{3.18}$$

and

$$\|z'\|_{L^2(I)}=\tilde{C}(r)\left[\sum_{j=1}^{m}\|u_0^{(r+1)}\|^2_{L^2(I_j)}h_j^{2r}\right]^{1/2}(1+O(\bar{h}))\quad \text{as}\quad \bar{h}\to 0, \tag{3.19}$$

where the constants in the bounds of the O-terms depend on f and r but not on Δ.

Proof. By (3.12) we have

$$\begin{aligned}\|\rho_j\|_{L^2(I_j)}&=\|\phi_j+\sigma_j\|_{L^2(I_j)}\\ &=\|\phi_j\|_{L^2(I_j)}(1+O(h_j))\end{aligned} \tag{3.20}$$

which together with (3.5) gives (3.18).

Also, by (3.10) we have

$$\|\phi_j^{(r-1)}\|_{L^2(I_j)}=\|\rho_j^{(r-1)}\|_{L^2(I_j)}(1+O(h_j))$$

Since $\rho_j^{(r-1)}=u_0^{(r+1)}$, we have

$$\|\phi_j^{(r-1)}\|_{L^2(I_j)}=\|u_0^{(r+1)}\|_{L^2(I_j)}(1+O(h_j)).$$

Hence, by (3.6) we obtain (3.19).

In Lemma 3.4 and Theorem 3.5 we assume that $u_0^{(r+1)}(x)\neq 0$ for all $x\in I$. Clearly, the assumption is very severe. But, actually, the results are largely valid also when $u_0^{(r+1)}$ has zeros in I. In order to show these we prove the following lemma and theorem:

LEMMA 3.6. *Suppose that*

$$u_0^{(r+1)}(\mu_k)=0,\qquad k=1,\ \dots,\ q,\qquad 0\le\mu_1<\mu_2<\dots<\mu_q\le 1.$$

For any $(\lambda,\ \varkappa)$*-regular partition* Δ *with* $1\le\varkappa<\dfrac{r+1}{r}$ *we have*

$$\|z'\|_{L^2(I)}=C(r)\left[\sum_{j=1}^{m}\|\phi_j\|^2_{L^2(I_j)}h_j^2\right]^{1/2}(1+O(\bar{h}^{\varepsilon}))\quad \text{as}\quad \bar{h}\to 0 \tag{3.21}$$

and

$$\|z'\|_{L^2(I)}=\tilde{C}(r)\left[\sum_{j=1}^{m}\|\phi_j^{(r-1)}\|^2_{L^2(I_j)}h_j^{2r}\right]^{1/2}(1+O(\bar{h}^{\varepsilon}))\quad \text{as}\quad \bar{h}\to 0, \tag{3.22}$$

where $\varepsilon=r+1-r\varkappa$ *and the constants in the bounds of the O-terms depend on f and r but not on* Δ.

Proof. For any $\delta>0$ we introduce the sets

$$I_\delta=\{x\in I\,|\,|x-\mu_k|<\delta\ \text{for some}\ \mu_k\},\qquad I_\delta^c=I\setminus I_\delta,$$
$$J_\delta=\{j=1,\ \dots,\ m\,;\ I_j\cap I_\delta\neq\phi\},\qquad J_\delta^c=\{1,\ \dots,\ m\}\setminus J_\delta.$$

We assume that $\delta_0\le(8q)^{-1}$ and, hence, that

$$\sum_{j\in J_{\delta_0}} h_j \leq 2(\delta_0+\bar{h})q \leq 4\delta_0 q \leq \frac{1}{2}, \qquad \text{for} \quad \bar{h}\leq\delta_0. \tag{3.23}$$

Since $\min\{|u_0^{(r+1)}(x)|,\ x\in I_{\delta_0}^c\}=\rho_0>0$, for the subintervals I_j with $j\in J_{\delta_0}^c$ we have

$$\|z'\|_{L^2(I_j)}^2=\tilde{C}(r)^2\|u_0^{(r+1)}\|_{L^2(I_j)}^2 h_j^{2r}(1+O(h_j)).$$

Hence, for $\bar{h}\leq\delta_0$ it follows from (3.23) that

$$\begin{aligned}\|z'\|_{L^2(I)}^2 &\geq \sum_{j\in J_{\delta_0}^c}\|z'\|_{L^2(I_j)}^2\\ &=\tilde{C}(r)^2\left[\sum_{j\in J_{\delta_0}^c}\|u_0^{(r+1)}\|_{L^2(I_j)}^2 h_j^{2r}\right](1+O(\bar{h}))\\ &\geq \tilde{C}(r)^2\rho_0^2\lambda^{2r}\bar{h}^{2rx}\left(\sum_{j\in J_{\delta_0}^c} h_j\right)(1+O(\bar{h}))\\ &\geq C\bar{h}^{2rx}(1+O(\bar{h})).\end{aligned} \tag{3.24}$$

On the other hand, by Lemma 3.3 we have

$$|\sigma_j(x)|\leq Ch_j^r, \qquad j=1,\ \dots,\ m$$

from which follows by (3.14)

$$\begin{aligned}\|\psi_{2,j}'\|_{L^2(I_j)}^2 &\leq Ch_j^{2r+3}\\ &\leq C\bar{h}^{2(r+1-rx)}\bar{h}^{2rx}h_j\\ &\leq C\|z'\|_{L^2(I)}^2\bar{h}^{2\varepsilon}h_j(1+O(\bar{h})) \quad \text{as} \quad \bar{h}\to 0, \qquad j=1,\ \dots,\ m.\end{aligned}$$

Then there are some constants α, β_1 and β_2 such that

$$\begin{aligned}\|z'\|_{L^2(I)}^2 &= \sum_{j=1}^m(\psi_{1,j}'+\psi_{2,j}',\ \psi_{1,j}'+\psi_{2,j}')_{I_j}\\ &=\left(\sum_{j=1}^m\|\psi_{1,j}'\|_{L^2(I_j)}^2\right)+2\alpha\left(\sum_{j=1}^m\|\psi_{1,j}'\|_{L^2(I_j)}^2\right)^{1/2}\left(\sum_{j=1}^m\|\psi_{2,j}'\|_{L^2(I_j)}^2\right)^{1/2}\\ &\quad+\left(\sum_{j=1}^m\|\psi_{2,j}'\|_{L^2(I_j)}^2\right)\\ &=\left(\sum_{j=1}^m\|\psi_{1,j}'\|_{L^2(I_j)}^2\right)+\beta_1\left(\sum_{j=1}^m\|\psi_{1,j}'\|_{L^2(I_j)}^2\right)^{1/2}\|z'\|_{L^2(I)}\bar{h}^{\varepsilon}(1+O(\bar{h}))\\ &\quad+\beta_2\|z'\|_{L^2(I)}^2\bar{h}^{2\varepsilon}(1+O(\bar{h})).\end{aligned}$$

Hence we have

$$\|z'\|_{L^2(I_j)}^2=\left(\sum_{j=1}^m\|\psi_{1,j}'\|_{L^2(I_j)}^2\right)(1+O(\bar{h}^{\varepsilon})) \quad \text{as} \quad \bar{h}\to 0,$$

where the values $\|\psi_{1,j}'\|_{L^2(I_j)}^2$ with $1\leq j\leq m$ may be computed in the same way as in Lemma 3.4

THEOREM 3.7. *On the assumption of Lemma* 3.6 we have

$$\|z'\|_{L^2(I)}=C(r)\left[\sum_{j=1}^m\|\rho_j\|_{L^2(I_j)}^2 h_j^2\right]^{1/2}(1+O(\bar{h}^{\varepsilon})) \quad \text{as} \quad \bar{h}\to 0 \tag{3.25}$$

and

$$\|z'\|_{L^2(I)}=\tilde{C}(r)\left[\sum_{j=1}^m\|u_0^{(r+1)}\|_{L^2(I_j)}^2 h^{2r}\right]^{1/2}(1+O(\bar{h}^{\varepsilon})) \quad \text{as} \quad \bar{h}\to 0, \tag{3.26}$$

where the constants in the bounds of the O-terms depend on f and r but not on Δ.

Proof. It follows from Lemma 3.3 and (3.24) that

$$\left.\begin{aligned}&\|\sigma_j\|_{L^2(I_j)}^2 h_j^2\\ &\|\sigma_j^{(r-1)}\|_{L^2(I_j)}^2 h_j^{2r}\end{aligned}\right\}\leq Ch_j^{2r+3}\leq C\|z'\|_{L^2(I)}^2\bar{h}^{2\varepsilon}h_j(1+O(\bar{h})). \tag{3.27}$$

Hence, by (3.21) there are some constants β_1, β_2 and α_j with $1\leq j\leq m$ such that

$$C(r)^2\Big(\sum_{j=1}^{m}\|\rho_j\|^2_{L^2(I_j)}h_j^2\Big)=C(r)^2\Big(\sum_{j=1}^{m}\|\phi_j+\sigma_j\|^2_{L^2(I_j)}h_j^2\Big)$$
$$=C(r)^2\sum_{j=1}^{m}(\|\phi_j\|^2_{L^2(I_j)}+2\alpha_j\|\phi_j\|_{L^2(I_j)}\|\sigma_j\|_{L^2(I_j)}+\|\sigma_j\|^2_{L^2(I_j)})h_j^2$$
$$=C(r)^2\sum_{j=1}^{m}\|\phi_j\|^2_{L^2(I_j)}h_j^2$$
$$+\beta_1\Big(\sum_{j=1}^{m}\|\phi_j\|^2_{L^2(I)}h_j^2\Big)^{1/2}\|z'\|_{L^2(I)}\bar{h}^{\varepsilon}(1+O(\bar{h}))$$
$$+\beta_2\|z'\|^2_{L^2(I)}\bar{h}^{2\varepsilon}(1+O(\bar{h}))$$
$$=\|z'\|^2_{L^2(I)}(1+O(\bar{h}^{\varepsilon}))\quad\text{as}\quad\bar{h}\to 0,$$

which implies that (3.25) holds.

By (3.22) and (3.27) we obtain (3.26) in the same way as in the proof of (3.25).

Here we remark that (3.18) and (3.25) in Theorems 3.5 and 3.7 are *a posteriori* computable error estimates.

Moreover let

$$\delta_1\geq\alpha_j\geq\delta_2>0,\qquad j=1,\ \dots,\ m.$$

Then, similarly as in the proof of Lemmas 3.4 and 3.6, Theorems 3.5 and 3.7, we easily obtain the following result:

THEOREM 3.8. *If*

$$u_0^{(r+1)}(x)\neq 0,\qquad \forall x\in I.$$

Then, for all partitions Δ, *we have*

$$\Big[\sum_{j=1}^{m}\alpha_j\|z'\|^2_{L^2(I_j)}\Big]^{1/2}=C(r)\Big[\sum_{j=1}^{m}\alpha_j\|\rho_j\|^2_{L^2(I_j)}h_j^2\Big]^{1/2}(1+O(\bar{h}))\quad\text{as}\quad\bar{h}\to 0 \tag{3.28}$$

and

$$\Big[\sum_{j=1}^{m}\alpha_j\|z'\|^2_{L^2(I_j)}\Big]^{1/2}=\tilde{C}(r)\Big[\sum_{j=1}^{m}\alpha_j\|u_0^{(r+1)}\|^2_{L^2(I_j)}h_j^{2r}\Big]^{1/2}(1+O(\bar{h}))\quad\text{as}\quad\bar{h}\to 0, \tag{3.29}$$

where the constants in the bounds of the O-terms depend on f *and* r *but not on* Δ.

Also if

$$u_0^{(r+1)}(\mu_k)=0,\qquad k=1,\ \dots,\ q,\qquad 0\leq\mu_1<\mu_2<\dots<\mu_q\leq 1,$$

then, for any $(\lambda,\ \varkappa)$*-regular partitions* Δ *with* $1\leq\varkappa<\dfrac{r+1}{r}$, *we have*

$$\Big[\sum_{j=1}^{m}\alpha_j\|z'\|^2_{L^2(I_j)}\Big]^{1/2}=C(r)\Big[\sum_{j=1}^{m}\alpha_j\|\rho_j\|^2_{L^2(I_j)}h_j^2\Big]^{1/2}(1+O(\bar{h}^{\varepsilon}))\quad\text{as}\quad\bar{h}\to 0 \tag{3.30}$$

and

$$\Big[\sum_{j=1}^{m}\alpha_j\|z'\|^2_{L^2(I_j)}\Big]^{1/2}=\tilde{C}(r)\Big[\sum_{j=1}^{m}\alpha_j\|u_0^{(r+1)}\|^2_{L^2(I_j)}h_j^{2r}\Big]^{1/2}(1+O(\bar{h}^{\varepsilon}))\quad\text{as}\quad\bar{h}\to 0, \tag{3.31}$$

where $\varepsilon=r+1-r\varkappa$ *and the constants in the bounds of the O-terms depend on* f *and* r *but not on* Δ.

These results shall play important parts in Section 4.

The following Table I shows some values of the constants $C(r)$ and $\tilde{C}(r)$.

In the following sections, we consider the more general two point boundary value problems.

TABLE I

r	$C(r)$	$\tilde{C}(r)$
2	$\frac{1}{2\sqrt{15}}$	$\frac{1}{12\sqrt{5}}$
3	$\frac{1}{2\sqrt{42}}$	$\frac{1}{120\sqrt{7}}$
4	$\frac{1}{120}$	$\frac{1}{5040}$

4. *A posteriori* error estimates—Part II

In this section we consider the following two point boundary value problem:

$$Lu \equiv -(a(x)u')' + b(x)u = f(x), \qquad x \in I, \tag{4.1}$$
$$u(0) = u(1) = 0,$$

where we assume that $a \in C^{r+1}(I)$, b, $f \in C^r(I)$ and

$$a(x) \geq \underline{a} > 0, \qquad b(x) \geq 0, \qquad x \in I.$$

It is well known that the solution u_0 of (4.1) belongs to $C^{r+2}(I)$. Let $u_{\Delta,r} \in \mathscr{M}_\Delta^r$ be the Galerkin approximation to u_0 determined by the relation

$$(au'_{\Delta,r}, v')_I + (bu_{\Delta,r}, v)_I = (f, v)_I, \qquad \forall v \in \mathscr{M}_\Delta^r,$$

and $z_{\Delta,r} \in \mathscr{M}_\Delta^r$ be the solution of equations

$$(z'_{\Delta,r}, v')_I = (-u''_0, v)_I, \qquad \forall v \in \mathscr{M}_\Delta^r.$$

Note that $z_{\Delta,r}$ is the Galerkin approximation for (3.1) whose the solution is exactly u_0.

Set

$$\mathrm{e} = u_0 - u_{\Delta,r}$$
$$z = u_0 - z_{\Delta,r}$$

Obviously the error z satisfies the properties in Section 3. For $r=1$, Babuška and Rheinboldt have analyzed the error e ([1]). Here we analyze it for $r \geq 2$.

Now we introduce the norm

$$\| u \|_{E(I)} = \left[\int_I (au'^2 + bu^2) dx \right]^{1/2}$$

on $H_0^1(I)$. If

$$\| u \|_{L^2(I)} \leq C \| u' \|_{L^2(I)} \bar{h},$$

then

$$\| \sqrt{a}\, u' \|_{L^2(I)} = \| u \|_{E(I)} (1 + O(\bar{h}^2)) \quad \text{as} \quad \bar{h} \to 0. \tag{4.2}$$

First we prove the following lemma:

LEMMA 4.1. *For each subinterval I_j of a given partition Δ there is a constant C such that*

$$| e(x_{j-1}) - e(x_j) | \leq C \| e' \|_{L^2(I)} \bar{h}^{\,r-1/2} h_j, \qquad j = 1, \ldots, m, \tag{4.3}$$

where the constant C depends on a and b but not on Δ.

Proof. Let u_1 and u_2 be the solutions of the initial value problems:

$$Lu=0, \qquad u(0)=0, \qquad u'(0)=-1,$$
$$Lu=0, \qquad u(1)=0, \qquad u'(1)=-1,$$

respectively.

Set

$$F(\xi)=\frac{1}{-a(\xi)(u_1(\xi)u_2'(\xi)-u_1'(\xi)u_2(\xi))}.$$

Then the Green's function for (4.1) is represented by

$$G(x,\xi)=\begin{cases} u_1(x)u_2(\xi)F(\xi), & 0\leq x\leq \xi, \\ u_1(\xi)u_2(x)F(\xi), & \xi\leq x\leq 1. \end{cases}$$

In the same way as in (3.3) we have

$$\begin{aligned} e(x_{j-1})-e(x_j) &= (ae', \frac{\partial}{\partial\xi}(G(x_{j-1},\cdot)-G(x_j,\cdot))_I \\ &\quad +(be, G(x_{j-1},\cdot)-G(x_j,\cdot))_I \\ &= (ae', \frac{\partial}{\partial\xi}(G(x_{j-1},\cdot)-G(x_j,\cdot))-v')_I \\ &\quad +(be, G(x_{j-1},\cdot)-G(x_j,\cdot)-v)_I, \qquad \forall v\in \mathscr{M}_\Delta^r, \end{aligned}$$

from which follows

$$(4.4) \qquad |e(x_{j-1})-e(x_j)|\leq C\|e'\|_{L^2(I)} \inf_{v\in\mathscr{M}_\Delta^r}\|\frac{\partial}{\partial\xi}(G(x_{j-1},\cdot)-G(x_j,\cdot))-v'\|_{L^2(I)}$$

Hereon

$$\begin{aligned} &\inf_{\substack{v\in\mathscr{M}_\Delta^r,\\ v(x_k)=G(x_{j-1},x_k)\\ -G(x_j,x_k)\\ k=j-1,j}} \|\frac{\partial}{\partial\xi}(G(x_{j-1},\cdot)-G(x_j,\cdot))-v'\|_{L^2(I\setminus I_j)}^2 \\ &= \inf_{\substack{v\in\mathscr{M}_\Delta^r\\ v(x_k)=G(x_{j-1},x_k)\\ -G(x_j,x_k)\\ k=j-1,j}} \Big[\sum_{i=1}^{j-1}\|(u_2(x_{j-1})-u_2(x_j))(Fu_1)'-v'\|_{L^2(I_i)}^2 \\ &\qquad + \sum_{i=j+1}^{m}\|(u_1(x_{j-1})-u_1(x_j))(Fu_2)'-v'\|_{L^2(I_i)}^2\Big] \\ &\leq C\bar{h}^{2r}h_j^2 \end{aligned}$$

and

$$\inf_{\substack{v\in P_r(I_j)\\ v(x_k)=G(x_{j-1},x_k)\\ -G(x_j,x_k)\\ k=j-1,j}} \|\frac{\partial}{\partial\xi}(G(x_{j-1},\cdot)-G(x_j,\cdot))-v'\|_{L^2(I_j)}^2\leq Ch_j^{2r+1}.$$

Therefore we obtain

$$\inf_{v\in\mathscr{M}_\Delta^r}\|\frac{\partial}{\partial\xi}(G(x_{j-1},\cdot)-G(x_j,\cdot))-v'\|_{L^2(I)}\leq C\bar{h}^{r-1/2}h_j.$$

which together with (4.4) gives (4.3).

Also we obtain the following relation between e and z:

LEMMA 4.2. *Let e and z be the errors associated with* (4.1) *and* (3.1) *which have the same solution u_0, respectively. Then*

$$(4.5) \qquad \|e'\|_{L^2(I)}=\|z'\|_{L^2(I)}(1+O(\bar{h}^2)) \quad \text{as} \quad \bar{h}\to 0,$$

where the constant in the bound of the O-term depends on a, b and r but not on Δ.

Proof. By the definition of $z_{\Delta,r}$ we have

$$(z', z')_I=(z', e')_I$$

and

$$\begin{aligned}\| e'-z' \|^2_{L^2(I)}&=(e'-z', e'-z')_I\\&=(e', e')_I-2(e', z')_I+(z', z')_I\\&=(e', e')_I-(z', z')_I\\&=\| e' \|^2_{L^2(I)}-\| z' \|^2_{L^2(I)},\end{aligned}$$

i.e.,

$$\| e' \|^2_{L^2(I)}-\| z' \|^2_{L^2(I)}=\| e'-z' \|^2_{L^2(I)}. \tag{4.6}$$

Also

$$(z', v')_{I_j}=0, \qquad \forall v\in P_r^0(I_j), \qquad j=1, \dots, m. \tag{4.7}$$

Let ν be the piecewise linear function so that

$$\nu(x_j)=e(x_j), \qquad j=1, \dots, m.$$

It follows from the property of a that

$$\begin{aligned}\| e'-z' \|^2_{L^2(I)}&\leq\frac{1}{\underline{a}}(a(e'-z'), e'-z')_I\\&=\frac{1}{\underline{a}}\{(a(e'-z'), e'-z'-\nu')_I+(a(e'-z'), \nu')_I\}\\&\leq\frac{1}{\underline{a}}\{|(be, e-z-\nu)_I|+|(az', e'-z'-\nu')_I|+|(a(e'-z'), \nu')_I|\}\end{aligned}$$

Now let $a_j=a\left(\dfrac{x_{j-1}+x_j}{2}\right)$, then

$$|a(x)-a_j|\leq Ch_j, \qquad x\in I_j, \qquad j=1, \dots, m. \tag{4.8}$$

Also, since $\| e-z-\nu \|_{L^2(I_j)}\leq C \| e'-z' \|_{L^2(I_j)}h_j$ and

$$\| e \|_{L^2(I)}\leq C \| e' \|_{L^2(I)}\overline{h}, \tag{4.9}$$

we have

$$\begin{aligned}\| e'-z'-\nu' \|^2_{L^2(I)}&\leq C \sum_{j=1}^{m} h_j^{-2} \| e-z-\nu \|^2_{L^2(I_j)}\\&\leq C \sum_{j=1}^{m} h_j^{-2} \| e'-z' \|^2_{L^2(I_j)}h_j^2\\&\leq C \| e'-z' \|^2_{L^2(I)}\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}|(be, e-z-\nu)_I|&\leq C \| e \|_{L^2(I)} \| e-z-\nu \|_{L^2(I)}\\&\leq C\| e' \|_{L^2(I)} \| e'-z' \|_{L^2(I)}\overline{h}^2.\end{aligned}$$

It follows from (4.6), (4.7), (4.8) and (4.10) that

$$\begin{aligned}|(az', e'-z'-\nu')_I|&\leq\left|\sum_{j=1}^{m} ((a-a_j)z', e'-z'-\nu')_{I_j}\right|\\&\quad+\left|\sum_{j=1}^{m} a_j(z', e'-z'-\nu')_{I_j}\right|\\&\leq C \| z' \|_{L^2(I)} \| e'-z' \|_{L^2(I)}\overline{h}\\&\leq C \| e' \|_{L^2(I)} \| e'-z' \|_{L^2(I)}\overline{h}.\end{aligned}$$

Moreover it follows from (4.3) that

$$|(a(e'-z'), \nu')_I|\leq C \| e'-z' \|_{L^2(I)} \| e' \|_{L^2(I)}\overline{h}^{r-1/2}.$$

Therefore

$$\| e'-z' \|^2_{L^2(I)} \leq C \| e' \|_{L^2(I)} \| e'-z' \|_{L^2(I)} \bar{h}$$

and

$$\| e'-z' \|_{L^2(I)} \leq C \| e' \|_{L^2(I)} \bar{h}. \tag{4.11}$$

Hence, from (4.6) we have

$$0 \leq \| e' \|^2_{L^2(I)} - \| z' \|^2_{L^2(I)} \leq C \| e' \|^2_{L^2(I)} \bar{h}^2,$$

which gives (4.5).

Now set

$$\left.\begin{aligned} r_j(x) &= (Lu_{\Delta,r}-f)(x) \\ &= a(x)e''(x)+a'(x)e'(x)-b(x)e(x), \qquad x \in I_j \\ a_j &= a\left(\frac{x_{j-1}+x_j}{2}\right) \end{aligned}\right\}, \quad j=1, \ldots, m.$$

Obviously

$$|a(x)-a_j| \leq Ch_j \leq C\frac{a_j}{\underline{a}}h_j, \qquad x \in I_j, \qquad j=1, \ldots, m$$

which implies

$$a(x)=a_j(1+O(h_j)) \quad \text{as} \quad h_j \to 0, \qquad x \in I_j, \qquad j=1, \ldots, m. \tag{4.12}$$

Using r_j and a_j with $1 \leq j \leq m$, the following error formulas hold:

THEOREM 4.3. *Suppose that*

$$u_0^{(r+1)}(x) \neq 0, \qquad x \in I.$$

Then

$$\| e \|_{E(I)} = C(r)\left[\sum_{j=1}^m \frac{\| r_j \|^2_{L^2(I_j)}}{a_j} h_j^2\right]^{1/2}(1+O(\bar{h})) \quad \text{as} \quad \bar{h} \to 0 \tag{4.13}$$

and

$$\| e \|_{E(I)} = \tilde{C}(r)\left[\sum_{j=1}^m \| \sqrt{a}\, u_0^{(r+1)} \|^2_{L^2(I_j)} h_j^{2r}\right]^{1/2}(1+O(\bar{h})) \quad \text{as} \quad \bar{h} \to 0, \tag{4.14}$$

where the constants in the bounds of the O-terms depend on a, b, f and r but not on Δ.

Proof. Set

$$\left.\begin{aligned} \tau_j(x) &= z''_{\Delta,r}(x) - u''_{\Delta,r}(x) \\ \tilde{r}_j(x) &= u''_0(x) - z''_{\Delta,r}(x) \end{aligned}\right\}, \ x \in I_j, \ j=1, \ldots, m,$$

$$d_1 = \left(\sum_{j=1}^m \frac{\| a\tilde{r}_j \|^2_{L^2(I_j)}}{a_j} h_j^2\right)^{1/2},$$

$$d_2 = \left(\sum_{j=1}^m \| \tau_j \|^2_{L^2(I_j)} h_j^2\right)^{1/2}.$$

It follows that

$$\begin{aligned} \| \sqrt{a}\, z' \|^2_{L^2(I)} &= \sum_{j=1}^m (az', z')_{I_j} \\ &= \left(\sum_{j=1}^m a_j \| z' \|^2_{L^2(I_j)}\right)(1+O(\bar{h})), \end{aligned}$$

which together with (3.28) and (4.12) implies that

$$\|\sqrt{a}z'\|_{L^2(I)}=C(r)\Big[\sum_{j=1}^{m}a_j\|\tilde{r}_j\|^2_{L^2(I_j)}h_j^2\Big]^{1/2}(1+O(\overline{h}))$$
$$=C(r)\Big[\sum_{j=1}^{m}\frac{\|a\tilde{r}_j\|^2_{L^2(I_j)}}{a_j}h_j^2\Big]^{1/2}(1+O(\overline{h}))\quad\text{as}\quad\overline{h}\to0.$$

Also, by using (4.6) and (4.11), we have

$$\begin{aligned}&|\|\sqrt{a}e'\|^2_{L^2(I)}-\|\sqrt{a}z'\|^2_{L^2(I)}|\\&=|(a(e'+z'),(e'-z'))_I|\\&\le C\|e'+z'\|_{L^2(I)}\|e'-z'\|_{L^2(I)}\\&\le C\|e'\|^2_{L^2(I)}\overline{h}\\&\le C\|\sqrt{a}e'\|^2_{L^2(I)}\overline{h}.\end{aligned}$$

Thus

$$\|\sqrt{a}e'\|_{L^2(I)}=\|\sqrt{a}z'\|_{L^2(I)}(1+O(\overline{h}))\quad\text{as}\quad\overline{h}\to0 \tag{4.15}$$

and

$$C(r)d_1=\|\sqrt{a}e'\|_{L^2(I)}(1+O(\overline{h}))\quad\text{as}\quad\overline{h}\to0.$$

It follows from (4.11) that

$$\begin{aligned}d_2&\le C\|u'_{\Delta,r}-z'_{\Delta,r}\|_{L^2(I)}\\&\le C\|e'-z'\|_{L^2(I)}\\&\le C\|e'\|_{L^2(I)}\overline{h}\\&\le C\|e\|_{E(I)}\overline{h}.\end{aligned}$$

Hence, there are some constants α_j with $1\le j\le 9$ such that

$$\begin{aligned}C(r)^2\sum_{j=1}^{m}\frac{\|r_j\|^2_{L^2(I_j)}}{a_j}h_j^2&=C(r)^2\sum_{j=1}^{m}\frac{h_j^2}{a_j}\int_{I_j}(a(x)\tilde{r}_j(x)+a(x)\tau_j(x)\\&\qquad+a'(x)e'(x)-b(x)e(x))^2dx\\&=C(r)^2d_1^2+\alpha_1d_2^2+\alpha_2\|e'\|^2_{L^2(I)}\overline{h}^2+\alpha_3\|e\|^2_{L^2(I)}\overline{h}^2\\&\quad+\alpha_4d_1d_2+\alpha_5d_1\|e'\|_{L^2(I)}\overline{h}+\alpha_6d_1\|e\|_{L^2(I)}\overline{h}\\&\quad+\alpha_7d_2\|e'\|_{L^2(I)}\overline{h}+\alpha_8d_2\|e\|_{L^2(I)}\overline{h}\\&\quad+\alpha_9\|e'\|_{L^2(I)}\|e\|_{L^2(I)}\overline{h}^2\\&=\|\sqrt{a}e'\|^2_{L^2(I)}(1+O(\overline{h}))\quad\text{as}\quad\overline{h}\to0.\end{aligned}$$

Since (4.9) holds, from (4.2) this implies

$$\|e\|_{E(I)}=C(r)\Big[\sum_{j=1}^{m}\frac{\|r_j\|^2_{L^2(I_j)}}{a_j}h_j^2\Big]^{1/2}(1+O(\overline{h}))\quad\text{as}\quad\overline{h}\to0.$$

Moreover, it follows from (3.29) that

$$\|\sqrt{a}z'\|_{L^2(I)}=\tilde{C}(r)\Big[\sum_{j=1}^{m}a_j\|u_0^{(r+1)}\|^2_{L^2(I_j)}h_j^{2r}\Big]^{1/2}(1+O(\overline{h}))$$
$$=\tilde{C}(r)\Big[\sum_{j=1}^{m}\|\sqrt{a}u_0^{(r+1)}\|^2_{L^2(I_j)}h_j^{2r}\Big]^{1/2}(1+O(\overline{h}))\quad\text{as}\quad\overline{h}\to0,$$

where together with (4.2) and (4.15) gives (4.14).

Also we obtain the following theorem:

THEOREM 4.4. *Suppose that*

$$u_0^{(r+1)}(\mu_k)=0,\qquad k=1,\dots,q,\qquad 0\le\mu_1<\mu_2<\dots<\mu_q\le1.$$

For any $(\lambda,\varkappa)$*-regular partition* Δ *with* $1\le\varkappa<\dfrac{r+1}{r}$, *we have*

$$(4.16)\qquad \|e\|_{E(I)}=C(r)\Big[\sum_{j=1}^{m}\frac{\|r_j\|^2_{L^2(I_j)}}{a_j}h_j^2\Big]^{1/2}(1+O(\bar{h}^{\varepsilon}))\quad\text{as}\quad \bar{h}\to 0$$

and

$$(4.17)\qquad \|e\|_{E(I)}=\tilde{C}(r)\Big[\sum_{j=1}^{m}\|\sqrt{a}\,u_0^{(r+1)}\|_{L^2(I_j)}h_j^{2r}\Big]^{1/2}(1+O(\bar{h}^{\varepsilon}))\quad\text{as}\quad \bar{h}\to 0,$$

where the constants in the bounds of the O-terms depend on a, b, f and r but not on Δ.

Proof. It follows from (3.30) that

$$\begin{aligned}\|\sqrt{a}\,z'\|_{L^2(I)}&=\Big[\sum_{j=1}^{m}a_j\|z'\|^2_{L^2(I_j)}\Big]^{1/2}(1+O(\bar{h}))\\ &=C(r)\Big[\sum_{j=1}^{m}a_j\|\tilde{r}_j\|^2_{L^2(I_j)}h_j^2\Big]^{1/2}(1+O(\bar{h}^{\varepsilon}))\\ &=C(r)\Big[\sum_{j=1}^{m}\frac{\|a\tilde{r}_j\|^2_{L^2(I_j)}}{a_j}h_j^2\Big]^{1/2}(1+O(\bar{h}^{\varepsilon}))\quad\text{as}\quad \bar{h}\to 0.\end{aligned}$$

Also, it follows from (3.31) that

$$\begin{aligned}\|\sqrt{a}\,z'\|_{L^2(I)}&=\tilde{C}(r)\Big[\sum_{j=1}^{m}a_j\|u_0^{(r+1)}\|^2_{L^2(I_j)}h_j^{2r}\Big]^{1/2}(1+O(\bar{h}^{\varepsilon}))\\ &=\tilde{C}(r)\Big[\sum_{j=1}^{m}\|\sqrt{a}\,u_0^{(r+1)}\|^2_{L^2(I_j)}h_j^{2r}\Big]^{1/2}(1+O(\bar{h}^{\varepsilon}))\quad\text{as}\quad \bar{h}\to 0.\end{aligned}$$

After this, on the same proof as Theorem 4.3, we obtain (4.16) and (4.17).

We remark that (4.13) and (4.16) in Theorems 4.3 and 4.4 are *a posteriori* computable error estimates. Also, (4.14) and (4.17) will play the important parts in the discussion of optimal partitions ([3]).

5. *A posteriori* error estimates—Part III

In this section we consider the following two point boundary value problem:

$$(5.1)\qquad \begin{aligned}&Lu\equiv -u''+a(x)u'+b(x)u=f(x),\qquad x\in I,\\ &u(0)=u(1)=0,\end{aligned}$$

where we assume that $a,\ b,\ f\in C^r(I)$.

It is well known that the solution u_0 of (5.1) belongs to $C^{r+2}(I)$. Let $u_{\Delta,r}\in\mathscr{M}_\Delta^r$ be the Galerkin approximation to u_0 determined by the relation

$$(u'_{\Delta,r},\ v')_I+(au'_{\Delta,r}+bu_{\Delta,r},\ v)_I=(f,\ v)_I,\qquad \forall v\in\mathscr{M}_\Delta^r$$

and $z_{\Delta,r}\in\mathscr{M}_\Delta^r$ be the solution of equations

$$(z'_{\Delta,r},\ v')_I=(-u''_0,\ v)_I,\qquad \forall v\in\mathscr{M}_\Delta^r.$$

Note that the Galerkin approximation $u_{\Delta,r}$ exists for $\bar{h}$ sufficiently small and that $z_{\Delta,r}$ is the Galerkin approximation for (3.1) whose the solution is exactly u_0.

Set

$$\begin{aligned}e&=u_0-u_{\Delta,r},\\ z&=u_0-z_{\Delta,r}.\end{aligned}$$

Obviously the error z satisfies the properties in Section 3. The following relation holds between e and z:

LEMMA 5.1. *Let e and z be the errors associated with* (5.1) *and* (3.1) *which have the same solution* u_0, *respectively. Then*

$$(5.2)\qquad \|e'\|_{L^2(I)}=\|z'\|_{L^2(I)}(1+O(\bar{h}^2))\quad\text{as}\quad \bar{h}\to 0,$$

where the constant in the bound of the O-term depends on a, b and r but not on Δ.

Proof. By the definition of the Galerkin approximation we have

$$(e', e')_I+(ae'+be, e)_I=(e', z')_I+(ae'+be, z)_I,$$
$$(z', z')_I=(z', e')_I.$$

By Theorem 8 of [2], we have

$$\|e\|_{L^2(I)}\leq C\|e'\|_{L^2(I)}\bar{h},$$
$$\|z\|_{L^2(I)}\leq C\|z'\|_{L^2(I)}\bar{h}.$$

Moreover let ν be the piecewise linear function so that

$$\nu(x_j)=e(x_j), \qquad j=0, \dots, m.$$

Also let $G(x, \xi)$ be the Green's function for (5.1). Then in the same way as in (3.3), we have

$$e(x_j)=\left(e', \frac{\partial G}{\partial \xi}(x_j, \cdot)-v'\right)_I+(ae'+be, G(x_j, \cdot)-v)_I, \qquad \forall v\in \mathscr{M}_\Delta^r.$$

from which follows

$$|e(x_j)|\leq C\|e'\|_{L^2(I)}\inf_{v\in \mathscr{M}_\Delta^r}\left\|\frac{\partial G}{\partial \xi}(x_j, \cdot)-v'\right\|_{L^2(I)}.$$

Therefore

$$\|\nu\|_{L^2(I)}\leq C\|e'\|_{L^2(I)}\bar{h}^r$$

and

$$\begin{aligned}\|e-z\|_{L^2(I)}&\leq\|e-z-\nu\|_{L^2(I)}+\|\nu\|_{L^2(I)}\\&\leq C(\|e'-z'\|_{L^2(I)}\bar{h}+\|e'\|_{L^2(I)}\bar{h}^r).\end{aligned}\tag{5.3}$$

On the other hand, we have

$$\begin{aligned}\|e'-z'\|^2_{L^2(I)}&=(e'-z', e'-z')_I\\&=(e', e')_I-2(e', z')_I+(z', z')_I\\&=(e', e')_I-(z', z')_I\\&=\|e'\|^2_{L^2(I)}-\|z'\|^2_{L^2(I)}.\end{aligned}\tag{5.4}$$

Hence it follows from (5.3) and (5.4) that

$$\begin{aligned}0&\leq\|e'\|^2_{L^2(I)}-\|z'\|^2_{L^2(I)}=(ae'+be, z-e)_I\\&\leq C(\|e'\|_{L^2(I)}+\|e\|_{L^2(I)})\|z-e\|_{L^2(I)}\\&\leq C\|e'\|_{L^2(I)}\left(\sqrt{\|e'\|^2_{L^2(I)}-\|z'\|^2_{L^2(I)}}\,\bar{h}+\|e'\|_{L^2(I)}\bar{h}^r\right)\end{aligned}$$

i.e.,

$$\sqrt{\|e'\|^2_{L^2(I)}-\|z'\|^2_{L^2(I)}}\leq C\|e'\|_{L^2(I)}\bar{h}.$$

From above we obtain

$$0\leq\|e'\|^2_{L^2(I)}-\|z'\|^2_{L^2(I)}\leq C\|e'\|^2_{L^2(I)}\bar{h}^2,\tag{5.5}$$

which implies

$$\|e'\|_{L^2(I)}=\|z'\|_{L^2(I)}(1+O(\bar{h}^2)) \quad \text{as} \quad \bar{h}\to 0.$$

Now set

$$\begin{aligned}r_j(x)&=(Lu_{\Delta,r}-f)(x)\\&=e''(x)-a(x)e'(x)-b(x)e(x), \qquad x\in I_j, \quad j=1, \dots, m.\end{aligned}$$

Then from Theorem 3.5 we obtain the following theorem:

THEOREM 5.2. *Suppose that*

$$u_0^{(r+1)}(x) \neq 0, \qquad x \in I.$$

Then

$$\| e' \|_{L^2(I)} = C(r)\left[\sum_{j=1}^{m} \| r_j \|^2_{L^2(I_j)} h_j^2\right]^{1/2} (1+O(\bar{h})) \quad \text{as} \quad \bar{h} \to 0 \tag{5.6}$$

and

$$\| e' \|_{L^2(I)} = \tilde{C}(r)\left[\sum_{j=1}^{m} \| u_0^{(r+1)} \|^2_{L^2(I_j)} h_j^{2r}\right]^{1/2} (1+O(\bar{h})) \quad \text{as} \quad \bar{h} \to 0, \tag{5.7}$$

where the constants in the bounds of the O-terms depend on a, b, f and r but not on Δ.

Proof. Set

$$\left.\begin{aligned} \tau_j(x) &= z''_{\Delta,r}(x) - u''_{\Delta,r}(x) \\ \tilde{r}_j(x) &= u''_0(x) - z''_{\Delta,r}(x) \end{aligned}\right\}, \quad x \in I_j, \quad j=1, \dots, m,$$

$$d_1 = \left(\sum_{j=1}^{m} \| \tilde{r}_j \|^2_{L^2(I_j)} h_j^2\right)^{1/2},$$

$$d_2 = \left(\sum_{j=1}^{m} \| \tau_j \|^2_{L^2(I_j)} h_j^2\right)^{1/2}.$$

From (5.4) and (5.5) we have

$$\begin{aligned} \| z'_{\Delta,r} - u'_{\Delta,r} \|^2_{L^2(I)} &= \| e' - z' \|^2_{L^2(I)} \\ &= \| e' \|^2_{L^2(I)} - \| z' \|^2_{L^2(I)} \\ &\leq C \| e' \|^2_{L^2(I)} \bar{h}^2 \end{aligned}$$

and it follows from $(z_{\Delta,r} - u_{\Delta,r})|_{I_j} \in P_r(I_j)$ that

$$\begin{aligned} d_2 &\leq C\left(\sum_{j=1}^{m} \| z'_{\Delta,r} - u_{\Delta,r}' \|^2_{L^2(I_j)}\right)^{1/2} \\ &= C \| z'_{\Delta,r} - u'_{\Delta,r} \|_{L^2(I)} \\ &\leq C \| e' \|_{L^2(I)} \bar{h}. \end{aligned} \tag{5.8}$$

Also there are some constants a_i with $1 \leq i \leq 9$ such that

$$\begin{aligned} C(r)^2 \sum_{j=1}^{m} \| r_j \|^2_{L^2(I_j)} h_j^2 &= C(r)^2 \sum_{j=1}^{m} h_j^2 \int_{I_j} (\tilde{r}_j(x) + \tau_j(x) - a(x)e'(x) - b(x)e(x))^2 dx \\ &= C(r)^2 d_1^2 + a_1 d_2^2 + a_2 \| e' \|^2_{L^2(I)} \bar{h}^2 + a_3 \| e \|^2_{L^2(I)} \bar{h}^2 + a_4 d_1 d_2 \\ &\quad + a_5 d_1 \| e' \|_{L^2(I)} \bar{h} + a_6 d_1 \| e \|_{L^2(I)} \bar{h} + a_7 d_2 \| e' \|_{L^2(I)} \bar{h} \\ &\quad + a_8 d_2 \| e \|_{L^2(I)} \bar{h} + a_9 \| e' \|_{L^2(I)} \| e \|_{L^2(I)} \bar{h}^2 \end{aligned} \tag{5.9}$$

It follows from (3.18) and (5.2) that

$$\begin{aligned} C(r) d_1 &= \| z' \|_{L^2(I)} (1+O(\bar{h})) \\ &= \| e' \|_{L^2(I)} (1+O(\bar{h})) \quad \text{as} \quad \bar{h} \to 0, \end{aligned}$$

which together with (5.8) and (5.9) gives

$$C(r)^2 \sum_{j=1}^{m} \| r_j \|^2_{L^2(I_j)} h_j^2 = \| e' \|^2_{L^2(I)} (1+O(\bar{h})) \quad \text{as} \quad \bar{h} \to 0.$$

Moreover, from (3.19) and (5.2) we obtain the error formula (5.7).

Also from Theorem 3.7 we obtain the following theorem:

THEOREM 5.3. *Suppose that*

$$u_0^{(r+1)}(\mu_k) = 0, \qquad k=1, \dots, q, \qquad 0 \leq \mu_1 < \mu_2 < \dots < \mu_q \leq 1.$$

For any $(\lambda, \varkappa)$*-regular partition* Δ *with* $1 \leq \varkappa < \frac{r+1}{r}$ *we have*

$$(5.10)\qquad \|e'\|_{L^2(I)}=C(r)\left[\sum_{j=1}^{m}\|r_j\|^2_{L^2(I_j)}h_j^2\right]^{1/2}(1+O(\overline{h}^{\,\varepsilon}))\quad \text{as}\quad \overline{h}\to 0$$

and

$$(5.11)\qquad \|e'\|_{L^2(I)}=\tilde{C}(r)\left[\sum_{j=1}^{m}\|u_0^{(r+1)}\|^2_{L^2(I_j)}h_j^{2r}\right]^{1/2}(1+O(\overline{h}^{\,\varepsilon}))\quad \text{as}\quad \overline{h}\to 0,$$

where the constants in the bounds of the O-terms depend on a, b, f and r but not on Δ.

Proof. It follows from (3.25) and (5.2) that

$$\begin{aligned}C(r)d_1&=\|z'\|_{L^2(I)}(1+O(\overline{h}^{\,\varepsilon}))\\&=\|e'\|_{L^2(I)}(1+O(\overline{h}^{\,\varepsilon}))\quad \text{as}\quad \overline{h}\to 0,\end{aligned}$$

which together with (5.8) and (5.9) implies that

$$C(r)^2\sum_{j=1}^{m}\|r_j\|^2_{L^2(I_j)}h_j^2=\|e'\|^2_{L^2(I)}(1+O(\overline{h}^{\,\varepsilon}))\quad \text{as}\quad \overline{h}\to 0.$$

Hence (5.10) is given.

Also, from (3.26) and (5.2) we obtain the error formula (5.11).

We remark that (5.6) and (5.10) in Theorems 5.2 and 5.3 are *a posteriori* computable error estimates. Also, (5.7) and (5.11) will play the important parts in the discussion of optimal partitions ([3]).

In this paper we consider the error estimates for $r\geq 2$. But, the proofs of the lemmas and the theorems in Sections 3, 4 and 5 apply to the case of $r=1$. Hence similar results are given for $r=1$. Then we obtain

$$C(1)=\tilde{C}(1)=\frac{1}{2\sqrt{3}}\,.$$

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