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By

Masao HASHIGUCHI*

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Abstract

This article is a revised note of the lecture given by the author on May 16, 1983 on the occasion of his visit to University of Debrecen in Hungary. In this lecture the author talked about generalized Berwald spaces, and presented ten problems to be considered on these spaces.

§1. Generalized Berwald spaces.

In these several years I have been interested in generalized Berwald spaces. In this lecture I would like to talk about these spaces, and present some problems to ask your cooperations. For details of these spaces, refer to our synthesized paper [6], which was jointly written with Prof. Ichijyō on the occasion of this visit.

Let M be an n-dimensional differentiable manifold, and TM be the tangent bundle. A coordinate system (x^i) in M induces a canonical coordinate system (x^i, y^i) in TM. And we shall express a Finsler connection $F\Gamma$ by its coefficients $(F^i_{jk}, N^i_k, C^i_{jk})$. If $F^i_{jk}-F^i_{kj}=0$, then $F\Gamma$ is called *symmetric*, and if $F^i_{jk}-F^i_{kj}=\delta^i_j\sigma_k-\delta^i_k\sigma_j$ for some covariant vector field σ_k, then $F\Gamma$ is called *semi-symmetric*. Especially, if σ_k is a gradient vector field $\sigma_k=\partial\sigma(x)/\partial x^k$, we say $F\Gamma$ to be *σ-semi-symmetric*. A Finsler connection is called *linear*, if F^i_{jk} depend on position alone.

In a Finsler space $F^n=(M, L)$, where L is the fundamental function, the geodesics are expressed in the form

$$d^2x^i/ds^2+G^i_{jk}(x,\ dx/ds)(dx^j/ds)(dx^k/ds)=0,$$

where s is the arc-length. Then we have a canonical Finsler connection $B\Gamma=(G^i_{jk}, y^jG^i_{jk}, 0)$ named the *Berwald connection*. If $B\Gamma$ is linear, then the space F^n is called a *Berwald space*.

Let g_{ij} be the fundamental metric tensor of F^n. Putting

$$\Gamma^{*i}_{jk}=\frac{1}{2}g^{im}(\delta g_{im}/\delta x^k+\delta g_{km}/\delta x^j-\delta g_{jk}/\delta x^m),\ g^i_{jk}=\frac{1}{2}g^{im}\partial g_{jm}/\partial y^k,$$

we have another canonical Finsler connection $C\Gamma=(\Gamma^{*i}_{jk}, y^j\Gamma^{*i}_{jk}, g^i_{jk})$ named the *Cartan connection*. We know that $B\Gamma$ is linear if and only if $C\Gamma$ is linear, and then it holds

* Department of Mathematics, Faculty of Science, Kagoshima University, Kagoshima, Japan.

$\Gamma^{*}{}^{i}_{jk}=G^{i}_{jk}$. So we can define a Berwald space in terms of the Cartan connection.

Suggested by Prof. Wagner [18] (1943), Hashiguchi [3] (1975) generalized the notion of Berwald space, and defined a *generalized Berwald space* in general dimensions as a Finsler space such that we can introduce a *linear generalized Cartan connection* $GC\Gamma$, that is, a Finsler connection $(F^{i}_{jk}(x), y^{j}F^{i}_{jk}(x), g^{i}_{jk})$ satisfying $g_{ij|k}=0$. If $GC\Gamma$ is symmetric, then $GC\Gamma=C\Gamma$ by Prof. Matsumoto's axioms [10] (1966), and the space is a Berwald space.

By Prof. Matsumoto [11] (1981) a generalized Berwald space is also defined as a Finsler space such that we can introduce a *linear generalized Berwald connection* $GB\Gamma$, that is, a Finsler connection $(F^{i}_{jk}(x), y^{j}F^{i}_{jk}(x), 0)$ satisfying $L_{|k}=0$. If $GB\Gamma$ is symmetric, then $GB\Gamma=B\Gamma$ by Prof. Okada's axioms [15] (1982), and the space is a Berwald space.

By the definition of a generalized Berwald space, a Finsler space has the possibility that it becomes various generalized Berwald spaces. Because we might introduce various linear $GB\Gamma$ in a fixed Finsler space. So we have

Problem 1. *To discuss this possibility.*

We can also consider this problem under some conditions. For example, Aikou-Hashiguchi [1] (1981) showed

Theorem 1. *Let F^n be a generalized Berwald space by a linear $GB\Gamma=(F^{i}_{jk}(x), y^{j}F^{i}_{jk}(x), 0)$. If the paths with respect to a linear connection $F^{i}_{jk}(x)$ coincide with the geodesics of F^n, then F^n is a Berwald space. If in a Berwald space F^n we can find a tensor $T^{i}_{jk}(x)$ satisfying*

$$(\delta^{i}_{s}\delta^{r}_{j}+g_{sj}g^{ir}+2g^{i}_{sj}y^{r})T^{s}_{rk}=0,$$

then F^n becomes a generalized Berwald space by the linear $GB\Gamma=(G^{i}_{jk}+\frac{1}{2}T^{i}_{jk}, y^{j}(G^{i}_{jk}+\frac{1}{2}T^{i}_{jk}), 0)$ such that the paths coincide with the geodesics.

If we can find such a $T^{i}_{jk}(x)\neq 0$, a Berwald space becomes a non-trivial generalized Berwald space. So we have

Problem 2. *Is there a tensor $T^{i}_{jk}(x)\neq 0$ stated in Theorem 1?*

On the other hand, we have

Problem 3. *To find an example of a Finsler space which cannot become a generalized Berwald space.*

§2. Wagner spaces.

1°) As a special generalized Berwald space, we have a *Wagner space*, which is defined as a Finsler space such that we can introduce a *linear Wagner connection* $W\Gamma$, that is , a semi-symmetric linear $GC\Gamma$. Prof. Wagner [18] introduced such a space in the two-dimensional case, and showed that a space (M, L) with a so-called cubic metric $L=(a_{ijk}(x)y^{i}y^{j}y^{k})^{1/3}$ is an example. The following problem is due to Prof. Matsumoto:

Problem 4. *In general dimensions, to find a condition that a Finsler space with a cubic metric be a generalized Berwald space.*

In their paper [13] (1979), Professors Matsumoto and Numata gave a condition that such a space be a Berwald space. This result should be generalized to a result of a generalized Berwald space.

We have a lot of results on Berwald spaces. So we have generally

Problem 5. *To generalize results on Berwald spaces to the corresponding results on generalized Berwald spaces.*

2°) As another example of a two-dimensional Wagner space, Prof. Matsumoto [12] (1983) gave a Kropina space (M, L) with $L=a_{ij}(x)y^iy^j/b_i(x)y^i$, where a_{ij} is a Riemannian metric tensor field and b_i is a covariant vector field. And also, he showed that a two-dimensional Wagner space is a Berwald space if it is Landsberg. Therefore, a Kropina space is a Berwald space, if it is Landsberg.

By the result of Berwald [2], in a two-dimensional Berwald space the main scalar I is constant, if the curvature K does'nt vanish. However, the main scalar I of a Kropina space is given by

$$I=3/(2(b_1y^1+b_2y^2/b_1y^2-b_2y^1)^2+1)^{1/2},$$

when we express L in the form $L=((y^1)^2+(y^2)^2)/(b_1y^1+b_2y^2)$ using an isothermal coordinate for a_{ij} (Hashiguchi-Hōjō-Matsumoto [4] (1973)). Since the I is not constant, K vanishes. Therefore we have

Theorem 2. *A two-dimensional Kropina space is locally Minkowski, if it is Landsberg.*

A *locally Minkowski space* is, by the original definition, a Finsler space such that there exists a coordinate sysem (x^i) in which L is a function of y^i alone, and is characterized as a Berwald space whose curvature R^2 vanishes. So we hope to solve

Problem 6. *To give a direct proof for Theorem 2 based on the original definition.*

It is noted that a two-dimensional Kropina space is Landsberg if and only if $b_1+\sqrt{-1}b_2$ is a complex analytic function of the variable $x_1+\sqrt{-1}x_2$. Moreover, we have

Problem 7. *To discuss the above considerations for Kropina spaces in the case of general dimensions.*

3°) As a special Wagner space we have a *σ-Wagner space,* which is defined as a Finsler space such that we can introduce a *linear σ-Wagner connection $\Sigma W\Gamma$,* that is, a σ-semi-symmetric linear $GC\Gamma$.

Such a space plays an important role in the conformal theory of Finsler metrics. Hashiguchi-Ichijyō [5] (1977) showed

Theorem 3. *A Finsler space F^n is conformal to a Berwald space if and only if F^n is a σ-Wagner space.*

Theorem 4. *A Finsler space F^n is conformal to a locally Minkowski space, if and only if F^n is a σ-Wagner space with respect to a $\Sigma W\Gamma$ whose curvature R^2 vanishes.*

Theorem 4 has been also proved directly by the original definition by Prof. Tamắssy and Prof. Matsumoto [6] (1979).

These theorems show that if we know a result on a Berwald space or a locally Minkowski space, we can directly obtain a result on a space conformal to a Berwald space or a locally Minkowski space in terms of a σ-Wagner space. For example, Prof. Kắntor

(nee Varga) [17] (1978) showed that *if an $n(\geq 3)$-dimensional Finsler space F^n is a Berwald space of scalar curvature K, then it is a Riemannian space of constant curvature K or a locally Minkowski space, according as $K \neq 0$ or $K=0$.*
(This was also obtained in Numata [14].)

Let F^n be a Wagner space. If $K=R_{ijk}y^j X^i X^k / L^2 h_{ik} X^i X^k$ does'nt depend on X^i, the space is called *of W-scalar curvature.* The theorem of Professors Kãntor and Numata was generalized by Hashiguchi-Kãntor [7] (1979) as follows:

Theorem 5. *If an $n(\geq 3)$-dimensional Finsler space is a σ-Wagner space of W-scalar curvature K, then it is conformal to a Riemannian space of constant curvature K, or conformal to a locally Minkowski space, according as $K \neq 0$ or $K=0$.*

Generally, we have

Problem 8. *To obtain theorems on σ-Wagner spaces, corresponding the other theorems on Berwald spaces or locally Minkowski spaces.*

Especially, we hope to pay attention to results on the spaces of scalar curvature.

§3. $\{V, H\}$-manifolds.

Lastly, we shall give a few words from the global standpoint. Prof. Ichijyō [8, 9] (1976) gave an interesting theory. Let V be an n-dimensional linear space with a fixed base $\{e_a\}$. A global coordinate system (v^a) is introduced on V by $v=v^a e_a$.

Let $f(v)$ be a positive-valued differentiable function defined on $V-\{0\}$, which satisfies the following condition:

(1) $f(\lambda v)=\lambda f(v)$ for $\lambda>0$,

(2) (g_{ab}) is positive-definite, where $g_{ab}=\frac{1}{2}\partial^2 f^2/\partial v^a \partial v^b$.

Then the set $G=\{T \in GL(V) | f(Tv)=f(v)$ for any $v \in V\}$ is a closed subgroup of the general linear group $GL(V)$, and so becomes a Lie group.

Let H be a Lie subgroup of G, and let M admit the H-structure in the sense of G-structure. Let $\{U\}$ be a coordinate system and $z=\{z_a\}$ be a linear frame adapted to the H-structure. Then any tangent vector y at x is expressed as $y=v^a z_a$, to which $v=v^a e_a$ corresponds. If we define a function $L(x, y)$ on $TM-\{0\}$ by $L(x, y)=f(v)$, it is shown that L does'nt depend on $\{U\}$, $\{z_a\}$. Thus we have a Finsler space (M, L), which is called a *$\{V, H\}$-manifold.*

Now, if we take a G-connection $F^i_{jk}(x)$ relative to the H-structure, it is shown that $(F^i_{jk}, y^j F^i_{jk}, 0)$ satisfies $L_{|k}=0$. A $\{V, H\}$-manifold is just a generalized Berwald space.

Conversely, let (M, L) be a generalized Berwald space with respect to a linear $GB\Gamma$ $=(F^i_{jk}, y^j F^i_{jk}, 0)$. If we take the holonomy group H of the linear connection $F^i_{jk}(x)$, it is shown that (M, L) is a $\{V, H\}$-manifold under some conditions.

We said "under some conditions". M should be connected. The other conditions are concerned with the fundamental function L. L should be defined on $TM-\{0\}$, and (g_{ij}) should be positive-definite. However, such conditions are too strong. So we have

Problem 9. *To define a $\{V, H\}$-manifold under somewhat weaker conditions.*

The theory of Prof. Ichijyō is significant in the sense that global treatments are possible in generalized Berwald spaces. So we have

Problem 10. *To consider the above problems from the standpoint of G-structure.*

But the most important problem is to classify all the generalized Berwald spaces. We should find much more interesting examples.

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