ON THE WEAK CONVERGENCE OF MEASURES (I)

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§ 1. **Introduction.** In this paper we shall investigate the weak convergence of measures. Yu. V. Prohorov, and Robert Bartoszynski have shown many conditions for the weak convergence of measures in separable and complete metric spaces, which are expressed in terms of convergence of measures generated in finite dimentional Euclidean spaces. In what follows R will denote a complete separable metric space and $C(R)$ the totality of functionals $f(x)$ which are continuous and bounded on R. Denote by $M(R)$ the space of all finite measures defined on the Borel σ-field of subsets of R. A sequence μ_n of element of $M(R)$ will be called weakly convergent to $\mu \in M(R)$ if for every bounded and continuous function $f(x)$ on R

(1,1) $$\lim_{n\to\infty}\int_R f(x)\mu_n(dx) = \int_R f(x)\mu(dx).$$

Weak convergence of μ_n to μ is denoted by the symbol $\mu_n \Rightarrow \mu$. Let F be any closed set. Denote by F^ε the open set $S(F,\varepsilon)$. Define the numbers $\varepsilon_{1,2}$, $\varepsilon_{2,1}$ as the greatest lower bound of those ε for which, for all closed set $F \subset R$, $\mu_1(F) \leqq \mu_2(F^\varepsilon)+\varepsilon$,
respectively $\mu_2(F) \leqq \mu_1(F^\varepsilon)+\varepsilon$. Let

(1.2) $$L(\mu_1,\mu_2) = \max(\varepsilon_{1,2},\varepsilon_{2,1}).$$

The following Theorem can be found in [1],[2].

THEOREM. The function L, defined by (1.2), is a metric in the space $M(R)$, and the conditions $\mu_n \Rightarrow \mu_0$ and $L(\mu_n,\mu_0)\to 0$ are equivalent. Moreover, $M(R)$ with the metric L is a complete separable space.

We shall first introduce some elementary facts.

§ 2. Somc elementary facts

THEOREM 1. *Let* $\overset{(k)}{\mu}_n \in M(R)$ (k=1,2,……, m; n=0,1,2,……). *If* $\lim_{n\to\infty} L(\overset{(k)}{\mu}_n, \overset{(k)}{\mu}_0)=0$

and $\{c_1,c_2,\cdots\cdots,c_m\}$ *is a finite set of positive real numbers, then,* $\lim_{n\to\infty} L(\sum_{k=1}^{m} C_k \overset{(k)}{\mu}_n, \sum_{k=1}^{m} C_k \mu_0{}^{(k)}=0$.

PROOF. Let max $(L(\overset{(1)}{\mu}_n, \overset{(1)}{\mu}_0), L(\overset{(2)}{\mu}_n, \overset{(2)}{\mu}_0), \cdots\cdots, L(\overset{(m)}{\mu}_n, \overset{(m)}{\mu}_0))=\varepsilon$, then, for all closed set $F \subset R$

$(\sum_{k=1}^{m} C_k \overset{(k)}{\mu}_n)(F) = \sum_{k=1}^{m} C_k \overset{(k)}{\mu}_n(F) \leqq \sum_{k=1}^{m} C_k (\overset{(k)}{\mu}_0(F^\varepsilon)+\varepsilon) = \sum_{k=1}^{m} C_k \overset{(k)}{\mu}_0(F^\varepsilon) + \varepsilon\sum_{k=1}^{m} C_k$ and $(\sum_{k=1}^{m} C_k \overset{(k)}{\mu}_0)(F) \leqq \sum_{k=1}^{m} C_k \overset{(k)}{\mu}_n$
$(F^\varepsilon) + \varepsilon \sum_{k=1}^{m} C_k$,

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on the other hand we have $(\sum_{k=1}^{m} C_k \overset{(k)}{\mu_n}(F) \leqq (\sum_{k=1}^{m} C_k \overset{(k)}{\mu_0})(F^{\varepsilon'}) + \varepsilon' = \sum_{k=1}^{m} C_k \overset{(k)}{\mu_0}(F^{\varepsilon'}) + \varepsilon'$

$$\text{and} \quad (\sum_{k=1}^{m} C_k \overset{(k)}{\mu_0})(F) \leqq \sum_{k=1}^{m} C_k \overset{(k)}{\mu_n}(F^{\varepsilon''}) + \varepsilon''.$$

If we write ε_0' = the greatest lower bound of those ε',

and ε_0'' = the greatest lower bound of those ε'',

then, $L(\sum_{k=1}^{m} C_k \overset{(k)}{\mu_n}, \sum_{k=1}^{m} C_k \overset{(k)}{\mu_0}) = \max(\varepsilon_0', \varepsilon_0'') \leqq \max(\varepsilon, \varepsilon \sum_{k=1}^{m} C_k)$,

hence $\lim L(\sum_{k=1}^{m} C_k \overset{(k)}{\mu_n}, \sum_{k=1}^{m} C_k \overset{(k)}{\mu_0}) = 0.$ |

COROLLARY. *If* $\lim_{n\to\infty} L(\mu_n, \mu_0) = 0$ *and* $\lim_{n\to\infty} L(\nu_n, \nu_0) = 0$, *then* $\lim_{n\to\infty} L(\mu_n + \nu_n, \mu_0 + \nu_0) = 0$. (*That is to say* $\mu_n + \nu_n \Rightarrow \mu_0 + \nu_0$).

THEOREM 2. *If* $\sum_{n=1}^{\infty} L(\mu_n, \mu_0) < \infty$, *then* $\mu_n \Rightarrow \mu_0$.

PROOF. *If* $\sum L(\mu_n, \mu_0) < \infty$, then $L(\mu_n, \mu_0) \to 0$; the desired result follows from Theorem of § 1. |

THEOREM 3. *If* $\sum_{n=1}^{\infty} L(\mu_n, \mu) < \infty$ *and* $\sum_{n=1}^{\infty} L(\mu_n, \nu) < \infty$ *then* $\mu \equiv \nu$.

To prove this Theorem we need the following lemma. [1].

LEMMA. Let μ_1, μ_2 and μ_3 be elements of $M(R)$. Then

(a) $L(\mu_1, \mu_2) \geqq 0$, $L(\mu_1, \mu_1) = 0$, *and* $L(\mu_1, \mu_2) = L(\mu_2, \mu_1)$.

(b) $L(\mu_1, \mu_3) \leqq L(\mu_1, \mu_2) + L(\mu_2, \mu_3)$.

(c) *If* $L(\mu_1, \mu_2) = 0$, then $\mu_1 \equiv \mu_2$.

PROOF of THEOREM 3. If $\sum L(\mu_n, \mu) < \infty$, and $\sum L(\mu_n, \nu) < \infty$, then $L(\mu_n, \mu) \to 0$ and $L(\mu_n, \nu) \to 0$, respectively.

By Lemma $L(\mu, \nu) \leqq L(\mu, \mu_n) + L(\mu_n, \nu)$,

hence $L(\mu, \nu) \leqq \lim L(\mu, \mu_n) + \lim L(\mu_n, \nu) = 0$,

$L(\mu, \nu) = 0$, $\mu \equiv \nu$. |

THEOREM 4. *If* $\sum_{n=1}^{\infty} L(\mu_n, \mu) < \infty$ *and* $L(\mu, \nu) = 0$, *then* $\sum_{n=1}^{\infty} L(\mu_n, \nu) < \infty$.

PROOF. Since, for every positive integers n,

$$L(\mu_n, \nu) \leqq L(\mu_n, \mu) + L(\mu, \nu) = L(\mu_n, \mu),$$

therefore $\sum L(\mu_n, \nu) \leqq \sum L(\mu_n, \mu) < \infty$. |

From Theorem2, we may investigate the convergence of $\sum_{n=1}^{\infty} L(\mu_n, \mu_0)$ instead of $\mu_n \Rightarrow \mu_0$. We shall describe some theorems below and the proofs of these theorems are omitted, since they can be found in books of series, for instance [5].

THEOREM 5. *If* $\sum_{n=1}^{\infty} L(\mu_n, \mu_0)$ *is convergent series, then so is* $\sum_{n=1}^{\infty} \alpha_n L(\mu_n, \mu_0)$, *if the* *factors* α_n *satisfy the inequalities* $0<\alpha_n\leqq k$ *for every* n.

THEOREM 6. *Let* $\sum_{n=1}^{\infty} L(\mu_n, \mu_0)$ *and* $\sum_{n=1}^{\infty} L(\nu_n,\nu_0)$ *be two series. If* $\sum_{n=1}^{\infty} L(\mu_n, \mu_0)$ *and* $\sum_{n=1}^{\infty} L(\nu_n, \nu_0)$ *satisfy, for every* $n>$ *a number* m,

(a) *the condition* $L(\mu_n, \mu_0)\leqq L(\nu_n, \nu_0)$,

(b) *and speaking generally* $L(\mu_n, \mu_0) \leqq c\, L(\nu_n, \nu_0)$ *(c; positive constant) then, the series* $\sum_{n=1}^{\infty} L(\mu_n, \mu_0)$ *is convergent, when the series* $\sum_{n=1}^{\infty} L(\nu_n, \nu_0)$ *is convergent.*

THEOREM 7. *The two series* $\sum_{n=1}^{\infty} L(\mu_n, \mu_0)$ *and* $\sum_{n=1}^{\infty} L(\nu_n, \nu_0)$ *are either both convergent or both divergent provided* $\lim_{n\to\infty}\frac{L(\mu_n, \mu_0)}{L(\nu_n, \nu_0)} \neq 0,\infty$ *exists.*

THEOREM 8. $\sum_{n=1}^{\infty} L(\mu_n, \mu_0)$ *is convergent when* $\overline{\lim_{n\to\infty}}\sqrt{L(\mu_n, \mu_0)}$ *or* $\overline{\lim_{n\to\infty}}\frac{L(\mu_{n+1}, \mu_0)}{L(\mu_n, \mu_0)}<1$.

§3. Equivalence relations and continuous mappings

In order to cotinue our study, we introduce the follwing definition.

DEFINITION 1. *Two sequences* $\{\mu_n\}$ *and* $\{\nu_n\}$ *of point of* $M(R)$ *are equivalent if* $\sum_{n=1}^{\infty} L(\mu_n, \nu_n)<\infty$.

THEOREM 9. *Let* $M(R)$ *be a metric space with the metric* L, *then*

(1) *Any sequence of* $M(R)$ *is equivalent to itself.*

(2) *If* $\{\mu_n\}$ *and* $\{\nu_n\}$ *are sequences of* $M(R)$, *then* $\{\mu_n\}$ *is equivalent to* $\{\nu_n\}$ *if and only if* $\{\nu_n\}$ *is eqivalent to* $\{\mu_n\}$.

(3) *Let* $\{\mu_n\}$, $\{\nu_n\}$, $\{\lambda_n\}$ *be sequences of* $M(R)$. *If* $\{\mu_n\}$ *is equivalent to* $\{\nu_n\}$ *and* $\{\nu_n\}$ *is equivalent to* $\{\lambda_n\}$, *then* $\{\mu_n\}$ *is equivalent to* $\{\lambda_n\}$.

PROOF. From the definition $L(\mu_n, \nu_n) = \max(\varepsilon_{\mu_n,\nu_n}, \varepsilon_{\nu_n,\mu_n})$
where $\varepsilon_{\mu_n,\nu_n}$ = the greatest lower bound of ε, that for all closed $F\subset R$, we have

$$\mu_n(F)\leqq \nu_n(F^{\varepsilon})+\varepsilon,$$

and $\varepsilon_{\nu_n,\mu_n}$ = the greatest lower bound of ε, that for all closed $F\subset R$, we have

$$\nu_n(F)\leqq \mu_n(F^{\varepsilon})+\varepsilon,$$

hence (1) $L(\mu_n, \mu_n)=0$ (n=1,2,……), therefore $\sum_{n=1}^{\infty} L(\mu_n, \mu_n)=0$.

(2) If $\sum L(\mu_n, \nu_n)<\infty$, then $\sum L(\nu_n, \mu_n) = \sum L(\mu_n, \nu_n)<\infty$, and conversely.

(3) If $\sum L(\mu_n, \nu_n)<\infty$ and $\sum L(\nu_n, \lambda_n)<\infty$, then, by the triangle property $L(\mu_n, \lambda_n)\leqq L(\mu_n, \nu_n)+L(\nu_n, \lambda_n)$ we have
$\sum L(\mu_n, \lambda_n)\leqq \sum L(\mu_n, \nu_n)+\sum L(\nu_n, \lambda_n)<\infty$. ∎

THEOREM 10. *Let* $M(R)$ *be a metric space with the metric* L *and* $\mu\in M(R)$. *If* $\sum_{n=1}^{\infty} L(\mu_n, \mu)<\infty$, *then* $\sum_{n=1}^{\infty} L(\nu_n, \mu)<\infty$ *if and only if* $\sum_{n=1}^{\infty} L(\mu_n, \nu_n)<\infty$.

PROOF. If $\sum L(\mu_n, \mu)<\infty$ and $\sum L(\nu_n, \mu)<\infty$, then
$\sum L(\mu_n, \nu_n)\leqq \sum L(\mu_n, \mu)+\sum L(\nu_n, \mu)<\infty$,

conversely, if $\sum L(\mu_n, \nu_n)<\infty$, $\sum L(\mu_n,\mu)<\infty$,
then $\sum L(\nu_n, \mu)\leqq \sum L(\nu_n, \mu_n) + \sum L(\mu_n,\mu) <\infty$. ∎

THEOREM 11. *Let $M(R)$ be a metric space with the metric L and $\mu \in M(R)$. If $\{\mu_n\}$ is equivalent to $\{\nu_n\}$, then $\mu_n\Rightarrow\mu$ if and ouly if $\nu_n\Rightarrow\mu$.*

PROOF. If $\mu_n\Rightarrow\mu$, then, by Theorem of §1 $L(\mu_n, \mu)\to 0$.
From hypothesis, $\sum L(\mu_n, \nu_n)<\infty$, therefore,
$L(\nu_n, \mu)\leqq L(\nu_n, \mu_n)+ L(\mu_n, \mu)$,
$\lim L(\nu_n, \mu) \leqq\lim L(\nu_n, \mu_n) + \lim L(\mu_n, \mu) = 0$,
hence $\nu_n\Rightarrow \mu$.
conversely, if $\nu_n\Rightarrow \mu$, then
$L(\mu_n, \mu)\leqq L(\mu_n, \nu_n) + L(\nu_n, \mu)$
$\lim L(\mu_n, \mu) \leqq\lim L(\mu_n, \nu_n) + \lim L(\nu_n,\mu) = 0$,
hence $\mu_n\Rightarrow\mu$. ∎

Let R^* be a complete separable metric space and let $\mu \in M(R)$. If f is a continuous function mapping R into R^*, then, the condition $\mu^f(A)=\mu\{f^{-1}(A)\}$ for the μ-measurable $f^{-1}(A)$ defines the measure $\mu^f \in M(R^*)$. PROHOROV has introduced the following theorem.[1]. (We shall write BARTOSZYNSKI' s form [2]).

THEOREM. The condition $\mu_n\Rightarrow\mu_0$ holds if and only if for every real μ-almost everywhere continuous function f on R we have $\mu_m^f\Rightarrow \mu_0^f$.

We shall introduce the notion of a continuity in the sense of the weak convergence of measures.

DEFINITION 2. *Let f: $R\to R^*$ be a continuous mapping, and μ_0 a point of $M(R)$. Then f is continuous (type 1) in the sense of the weak convergence of measures at the point μ_0 if and only if given any sequence $\{\mu_n\}$ of points of $M(R)$ satisfying $\sum_{n=1}^{\infty} L(\mu_n, \mu_0) <\infty$, the sequence $\{\mu_n^f\}$ satisfies $\sum_{n=1}^{\infty} L(\mu_n^f, \mu_0^f)<\infty$. The mapping f is continuous in the sense of the weak convergence of measures if and only if f is continuousat at the point μ for every μ in $M(R)$.*

Form the definition of metric L, for f is continuous in the sense of the weak convergence of measures, it is sufficient that, for all closed set $A\subset R^*$, $(f^{-1}(A))^{\varepsilon}\subset f^{-1}(A^{\cdot})$.

Example. The mapping f: $R\to R$ defined by $f(x) = x$ for every $x \in R$ is clearly continuous (type 1) in the sense of the weak convergence of measures.

THEOREM 12. *Let f: $R\to R^*$ be a continuous mapping (type 1) in the sense of the weak convergence. If $\sum_{n=1}^{\infty} L(\mu_n, \mu_0)<\infty$, and $\sum_{n=1}^{\infty} L(\nu_n, \mu_0)<\infty$, then $\{\mu_n^f\}$ and $\{\nu_n^f\}$ are equivalent.*

PROOF. If $\sum L(\mu_n, \mu_0)<\infty$, $\sum L(\nu_n, \mu_0)<\infty$ and f is a continuous (type 1) in the sense of the weak convergence of measures, then $\sum L(\mu_n^f, \mu_0^f)<\infty$ and $\sum L(\nu_n^f, \mu_0^f) <\infty$. An application of Theorem 10 completes the proof. ∎

DEFINITION 3. *Let f: $R\to R^*$ be a continuous mapping, and μ_0 a point of $M(R)$. Then*

f is continuous (type 2) in the sense of the weak convergence of measures at point μ_0 if and only if, given any sequence $\{\mu_n\}$ of points of $M(R)$ satisfying $\lim_{n\to\infty} L(\mu_n, \mu_0) = 0$, the sequence $\{\mu_n^f\}$ satisfies $\sum_{n=1}^{\infty} L(\mu_n^f, \mu_0^f) < \infty$. The mapping f is continuous (type 2) in the sense of the weak convergence of measures if and only if f is continuous at the point μ for every μ in $M(R)$.

THEOREM 13. *Let f: $R \to R^*$ be a continuous mapping (type 2) in the sense of the weak convergence of measures and μ_0 a point of $M(R)$. Then, if $L(\mu_n, \mu_0) \to 0$, $\mu_n^f \Rightarrow \mu_0^f$.*

PROOF. If $L(\mu_n, \mu_0) \to 0$, then by hypothesis, $\sum L(\mu_n^f, \mu_0^f) < \infty$, and therefore, $L(\mu_0^f, \mu_0^f) \to 0$, by Theorem of §1, $\mu_n^f \Rightarrow \mu_0^f$. ∎

THEOREM 14. *If f is a continuous mapping (type 2), then f is a continuous mapping (type 1).*

PROOF. If $\sum L(\mu_n, \mu_0) < \infty$ then $\lim L(\mu_n, \mu_0) = 0$, by hypothesiis, $\lim L(\mu_n, \mu_0) = 0$ implies $\sum L(\mu_n^f, \mu_0^f) = 0$. This implies that f is a continuous mapping (type 1). ∎

DEFINITION 4. *Two continuous mappings f and g are called equivalent in the sense of the weak convergence of measures if $\sum_{n=1}^{\infty} L(\mu_n^f, \mu_n^g) < \infty$* for any sequence $\{\mu_n\}$.

THEOREM 15. *Let f, g and h are continuous mappings, then*

(1) *f is equivalent to itself.*

(2) *f is equivalent to g if and only if g is equivalent to f.*

(3) *If f is equivalent to g and g is equivalent to h, then f is equivalent to h.*

PROOF. (1) and (2) are obious. To prove (3), we denote that

$L(\mu_n^f, \mu_n^h) \leqq L(\mu_n^f, \mu_n^g) + L(\mu_n^g, \mu_n^h)$. From the definition

$L(\mu_n^f, \mu_n^g) = \max(\varepsilon_{f,g}, \varepsilon_{g,f})$, where

$\varepsilon_{f,g}$ = the greatest lower bound of those ε, that for all closed set $F \subset R$, we have

$$\mu_n^f(F) \leqq \mu_n^g(F^\varepsilon) + \varepsilon,$$

and $\varepsilon_{g,f}$ = the greatest lower bound of those ε, that for all closed set $F \subset R$, we have

$$\mu_n^g(F) \leqq \mu_n^f(F^\varepsilon) + \varepsilon.$$

Similarly, $L(\mu_n^g, \mu_n^h) = \max(\varepsilon_{g,h}, \varepsilon_{h,g})$, where, $\varepsilon_{g,h}$ and $\varepsilon_{h,g}$ denote the greatest lower bound of those ε, that for every closed set $F \subset R$ we have $\mu_n^g(F) \leqq \mu_n^h(F^\varepsilon) + \varepsilon$ and $\mu_n^h(F) \leqq \mu_n^g(F^\varepsilon) + \varepsilon$ respectively, and moreover $L(\mu_n^f, \mu_n^h) = \max(\varepsilon_{f,h}, \varepsilon_{h,f})$, where $\varepsilon_{f,h}$ and $\varepsilon_{h,f}$ denote the greatest lower bound of those ε, that for every closed set $F \subset R$ we have $\mu^f(F) \leqq \mu_n^h(F^\varepsilon) + \varepsilon$ and $\mu_n^h(F) \leqq \mu_n^f(F^\varepsilon) + \varepsilon$ respectively.

Hence $\mu_n^f(F) \leqq \mu_n^g(F^{\varepsilon_{f,g}}) + \varepsilon_{f,g} \leqq \mu_n^g(\overline{F^{\varepsilon_{f,g}}}) + \varepsilon_{f,g} \leqq \mu_n^h(\overline{F^{\varepsilon_{f,g}}})^{\varepsilon_{g,h}} + \varepsilon_{f,g} + \varepsilon_{g,h}$

$$\leqq \mu_n^h(F^{\varepsilon_{f,g}+\varepsilon_{g,h}}) + \varepsilon_{f,g} + \varepsilon_{gh},$$

$$\varepsilon_{f,h} \leqq \varepsilon_{f,g} + \varepsilon_{g,h},$$

similarly $\varepsilon_{h,f} \leqq \varepsilon_{g,f} + \varepsilon_{h,g}$,

From those equations, we have

$$\max(\varepsilon_{f,h},\ \varepsilon_{h,f}) \leqq \max(\varepsilon_{f,g},\ \varepsilon_{g,f}) + \max(\varepsilon_{g,h},\ \varepsilon_{h,g})$$

namely, $L(\mu_n^f,\ \mu_n^h) \leqq L(\mu_n^f, \mu_n^g) + L(\mu_n^g,\ \mu_n^h)$

and hence $\sum L(\mu_n^f,\ \mu^h) \leqq \sum L(\mu_n^f,\ \mu_n^g) + \sum L(\mu_n^g,\ \mu_n^h) < \infty$. ▮

THEOREM 16. *If f and g are continuous (type 1 or type 2) at $\mu_0 \in M(R)$, and equivalent, then $\mu_0^f = \mu_0^g$.*

PROOF. By hypothesis, if $\sum L(\mu_n,\ \mu_0) < \infty$ (or $\lim L(\mu_n, \mu_0) = 0$),

then $\sum L(\mu_n^f,\ \mu_0^f) < \infty$, $\sum L(\mu_n^g,\ \mu_0^g) < \infty$, and $\sum L(\mu_n^f,\ \mu_n^g) < \infty$;

hence $L(\mu_0^f,\ \mu_0^g) \leqq L(\mu_n^f,\ \mu_0^f) + L(\mu_n^f,\ \mu_n^g) + L(\mu_n^g,\ \mu^g) \rightarrow 0$

$L(\mu_0^f,\ \mu_0^g) = 0$, we have $\mu_0^f = \mu_0^g$. ▮

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