On examples of open algebraic surfaces with $\bar{q}=1, \bar{P}_2=0$

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1 Introduction

In this note algebraic surfaces are assumed to be defined over the complex number field $\mathbb{C}$. In [6] the author proved that any nonsingular open algebraic surface S with $\bar{\kappa}(S) \geq 0$ and $\bar{q}(S) \geq 1$ satisfies $\bar{P}_{12}(S) \geq 1$ and gave some examples with $\bar{q}(S)=1, \bar{P}_2(S)=0$ which are rational and have structures of elliptic fiber spaces over $\mathbb{C}^*$, where $\bar{\kappa}(S)$ is the *logarithmic Kodaira dimension* of S , $\bar{q}(S)$ is the *logarithmic irregularity* of S and $\bar{P}_m(S)$ is the *logarithmic m-genus* of S. In this note we give some results on the classification of such surfaces using the theory of elliptic surfaces by Kodaira ([3][4][5]).

First we fix our notations:

S : a nonsingular algebraic surface over $\mathbb{C}$

$\bar{S}$: a nonsingular complete algebraic surface which contains S as a Zariski open subset and such that $D=\bar{S}\backslash S$ is a reduced simple normal crossing divisor on $\bar{S}$

$K_{\bar{S}}$: the *canonical divisor* of $\bar{S}$

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$p_g(\bar{S})$	:	the *geometric genus* of $\bar{S}$
$P_m(\bar{S})$	:	the *m-genus* of $\bar{S}$ for a positive integer m
$\bar{p}_g(S)$	:	the *logarithmic geometric genus* of S
$\bar{P}_m(S)$	:	the *logarithmic m-genus* of S
$\bar{\kappa}(S)$	:	the *logarithmic Kodaira dimension* of S
$q(\bar{S})$	:	the *irregularity* of $\bar{S}$
$\bar{q}(S)$	:	the *logarithmic irregularity* of S
$p_a(\bar{S})$	:	the arithmetic genus of $\bar{S}$

We assume that S is a non-singular rational surface with an elliptic fibration $\pi : S \to \mathbb{C}^*$ which is extended to an elliptic fibration $\bar{\pi} : \bar{S} \to \mathbb{P}^1_{\mathbb{C}}$ such that $D = redF_1 + redF_2$ where $F_i (i = 1, 2)$ are fibres of $\bar{\pi}$ and $redF_i$ $(i = 1, 2)$ are reduced fibres.

Remark 1 Such a situation can arise from the quasi-Albanese mapping when $\bar{q}(S) = 1$. (Iitaka[2]).

Theorem 1 *Let S be as above. Suppose that $\bar{P}_2(S) = \bar{P}_3(S) = 0$ and that the fibration $\bar{\pi} : \bar{S} \to \mathbb{P}^1_{\mathbb{C}}$ is free from multiple fibres. Then we have following three cases:*

Case 1: $\bar{\kappa}(S) = 1$ and $\bar{P}_4(S) = 1$ and the moduli of general fibres of the ellipic fibration π are not constant and $\pi : S \to \mathbb{C}^*$ is isomorphic to a fixed one as a fibration over $\mathbb{C}^*$.

Case 2: $\bar{\kappa}(S) = 0$ and $\bar{P}_4(S) = 1$ and the moduli of general fibres of π are constant and $\pi : S \to \mathbb{C}^*$ is isomorphic to a fixed one.

Case 3: $\bar{\kappa}(S) = 0$ and $\bar{P}_6(S) = 1$ and the moduli of general fibres of π are constant and $\pi : S \to \mathbb{C}^*$ is isomorphic to a fixed one.

Remark 2 An explicit construction of case 1 was given in [6] p.357. Since the argument in [6] p.354 neglected the case in which the moduli of general fibres of π are constant, the conclusion of [6] Proposition 5 shoud be corrected to $\bar{P}_4(S) \geq 1$ or $\bar{P}_6(S) \geq 1$. But [6] Theorem 1 and Theorem 2 hold.

We will give explicit constructions of above three cases and some other cases in section 4. We have also some results in the case of $\bar{P}_2(S) = 0$ and $\bar{P}_3(S) \geq 1$ which are stated in section 3.

2 Preliminaries

For the fundamental results on elliptic surfaces we refer Kodaira's original papers [2][3][4] or the book [1]. Here we summarize them as far as we need to state our results.

By an elliptic fibration of a complex analytic surface X we mean a proper connected holomorphic map $f : X \to \Delta$, such that the general fibre X_w ($w \in \Delta$) is a non-singular elliptic curve. Unless otherwise stated we shall always assume that f is minimal, i.e. all fibres are free of exceptional curves.

Fact 1 ([5]Theorem 6.2) *The singular fibres of an elliptic fibration over the unit disk are classified in following types:* ${}_m\mathrm{I}_b, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{I}_b^*, \mathrm{II}^*, \mathrm{III}^*, \mathrm{IV}^*$, *where* b *is an integer* ≥ 0.

The type ${}_1\mathrm{I}_b$ *is also denoted by* I_b.

A fibre of type I_0 *is a regular fibre i.e. a non-singular elliptic curve with multiplicity* 1.

A fibre of type I_1 *is a rational curve with one node with multiplicity* 1.

A fibre of type $\mathrm{I}_b (b \geq 2)$ *is a cycle of* b *non-singular rational curves with multiplicity* 1.

A fibre of type ${}_m\mathrm{I}_b$ $(m \geq 2)$ *is called a multiple fibre which is the* m*-ple of a fibre of type* I_b.

A fibre of type II *is a rational curve with one cusp with multiplicity* 1.

A fibre of type III *consists of two non-singular rational curves intersecting one point with intersection multiplicity* 2 *and each component has* *multiplicity* 1.

A fibre of type IV *consists of three non-singular rational curves intersecting one point and each component has multiplicity* 1.

A fibre of all the other types is a tree of non-singular rational curves in which some component has multiplicity ≥ 2 *and some component has multiplicity* 1.

Any non-singular rational curve which appears as an irreducible component of a singular fibre has the self-intersection number -2.

Let $f : X \to \Delta$ be an elliptic fibration over an algebraic curve Δ without multiple fibres. Let $\{a_\rho\}$ be a finite subset of Δ such that for $u \in \Delta' = \Delta \backslash \{a_\rho\}$, $f^{-1}(u)$ is a regular fibre. Kodaira defined the *functional invariant* $\mathcal{J} = \mathcal{J}(u)$ of f which is a meromorphic function on Δ and the *homological invariant* G belonging to $\mathcal{J}$ which is a sheaf on Δ extended from a localy constant sheaf on Δ'. We denote by $\mathcal{F}(\mathcal{J}, G)$ the set of all elliptic fibrations over Δ free from multiple fibres whose functional and homological invariants are $\mathcal{J}$ and G.

Fact 2 ([4]) *Any elliptic fibration over* Δ *free from multiple fibres belongs to some* $\mathcal{F}(\mathcal{J}, G)$.

Fact 3 ([5]) *If a meromorphic function* $\mathcal{J}$ *on* Δ *is given, there exist a finite number of homological invariants belonging to* $\mathcal{J}$.

Fact 4 ([5]Theorem 10.2) *For a meromorphic function* $\mathcal{J}$ *on* Δ *and the homological invariant* G *belonging to* $\mathcal{J}$, *there is a unique member* B *in* $\mathcal{F}(\mathcal{J}, G)$ *which possesses a global holomorphic section.* B *is called the basic member of* $\mathcal{F}(\mathcal{J}, G)$.

For $a_\rho \in \Delta$ we denote by C_{a_ρ} the fibre over a_ρ.

Fact 5 ([5]Theorem 9.1 or [1]p.159 Table 6) *The behaviour of the functional invariant $\mathcal{J}(s)$ is as follows:*

1. *If $\mathcal{J}(a_\rho) \neq 0, 1, \infty$ or $\mathcal{J}(s)$ has a zero of order $h \equiv 0(3)$ at a_ρ or $\mathcal{J}(s) - 1$ has a zero of order $h \equiv 0(2)$ at a_ρ, then the type of C_{a_ρ} is I_0 or I_0^*.*

2. *If $\mathcal{J}(s)$ has a pole of order b at a_ρ, then the type of C_{a_ρ} is I_b or $\mathrm{I}_b^*(b \geq 1)$.*

3. *If $\mathcal{J}(s)$ has a zero of order $h \equiv 1(3)$ at a_ρ, then the type of C_{a_ρ} is* II *or* IV*.

4. *If $\mathcal{J}(s)$ has a zero of order $h \equiv 2(3)$ at a_ρ, then the type of C_{a_ρ} is* II* *or* IV.

5. *If $\mathcal{J}(s) - 1$ has a zero of order $h \equiv 1(2)$ at a_ρ, then the type of C_{a_ρ} is* III *or* III*.

Remark 3 In the case of $\mathcal{J}(s) = c = constant$, if $c \neq 0, 1$ then the basic member has no singular fibres. But if $c = 0$ any of types II, IV*, II* or IV can occur, and if $c = 1$ then type III or III* can occur. (cf. Examples in section 4.)

For the compact analytic surface X with the elliptic fibration $f : X \to \Delta$, we denote by $\nu(\mathrm{T})$ the number of singular fibres of f of type T.

Fact 6 ([5]Theorem 12.2) *The arithmetic genus p_a of X is given by the formula*

$$\begin{aligned} 12(p_a + 1) &= \sum_b b\nu(\mathrm{I}_b) + \sum_b (b+6)\nu(\mathrm{I}_b^*) + 2\nu(\mathrm{II}) + 3\nu(\mathrm{III}) + 4\nu(\mathrm{IV}) \\ &\quad + 10\nu(\mathrm{II}^*) + 9\nu(\mathrm{III}^*) + 8\nu(\mathrm{IV}^*). \end{aligned} \tag{1}$$

Remark 4 If X is isomorphic to the basic member B of $\mathcal{F}(\mathcal{J}, G)$, then $\sum_b b(\nu(\mathrm{I}_b) + \nu(\mathrm{I}_b^*))$ is equal to the total multiplicity of the poles of the meromorphic function $\mathcal{J}$ which we denote by j.

3 Proof of Theorem 1 and some results

We assume that S is a non-singular rational surface with an elliptic fibration $\pi : S \to \mathbb{C}^*$ which is extended to an elliptic fibration $\bar{\pi} : \bar{S} \to \mathbb{P}^1_{\mathbb{C}}$ such that $D = redF_1 + redF_2$ where $F_i (i = 1, 2)$ are fibres of $\bar{\pi}$ and $redF_i$ are reduced fibres. Since we assume that D is a simple normal crossing divisor, $F_i (i = 1, 2)$ may contain some exceptional curves. By contracting them, we have a minimal elliptic fibration $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ and a proper birational morphism $\mu : \bar{S} \to \hat{S}$ such that $\bar{\pi} = \hat{\pi} \circ \mu$ and $\mu|_S$ is an isomorphism. We denote $\mu_* F_i = \hat{F}_i \ (i = 1, 2)$.

Remark 5 Under the above assumption there exists an exceptional curve on $\hat{S}$ which is not contained in a fibre of $\hat{\pi}$, because there is no irreducible irrational curve C with $C^2 \leq 0$ on $\mathbb{P}^2_{\mathbb{C}}$ or Hirzebruch surfaces Σ_n.

Lemma 1 *Under the above condition, the elliptic fibration $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ has at most one multiple fibre.*

Proof: We refer [6]p.353. Anyway it is an immediate consequence of the canonical bundle formula [3]. □

Lemma 2 *Under the above condition, if the elliptic fibration $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ has no multiple fibres, then it possesses a global section.*

Proof: Since there exists an exceptional curve E on $\hat{S}$, if $\hat{\pi}$ has no multiple fibres we have

$$-1 = E \cdot K_{\hat{S}} = -E \cdot \hat{F}$$

where $\hat{F}$ is a general fibre of $\hat{\pi}$. Thus E is a global section. □

Lemma 3 *Under the above condition, if $\bar{P}_2(S) = 0$ then* $\{$type of $\hat{F}_i \ | i = 1, 2\}$ *is* $\{\mathrm{III}^*, \mathrm{II}\}$ *or* $\{\mathrm{III}^*, \mathrm{III}\}$ *or* $\{\mathrm{II}^*, \mathrm{II}\}$ *or* $\{\mathrm{IV}^*, \mathrm{III}\}$ *or* $\{\mathrm{IV}^*, \mathrm{II}\}$ *or* $\{\mathrm{IV}^*, \mathrm{IV}\}$.

Proof: From [6]p.353 Lemma 4, we infer that one of $\hat{F}_i(i=1,2)$ is of type II* or III* or IV*. Since $\bar{p}_g(S)=0$ we infer from [6]p.345 Lemma 2 that D contains no fibres of type I_b or I_b^*, and from the assumption that $\hat{S}$ is rational we have $p_a(\hat{S})=0$. Combining these with the formula (1) we can easily deduce the conclusion. □

Remark 6 If we assume that $\hat{\pi}$ has no multiple fibres and that the moduli of general fibres of $\hat{\pi}$ are not constant i.e. the functional invariant $\mathcal{J}$ is not constant, then $j \geq 1$ and therefore only $\{\mathrm{III}^*, \mathrm{II}\}$, $\{\mathrm{IV}^*, \mathrm{III}\}$ and $\{\mathrm{IV}^*, \mathrm{II}\}$ are the cases. If {type of $\hat{F}_i$ $|i=1,2\}$ is $\{\mathrm{II}^*, \mathrm{II}\}$ or $\{\mathrm{IV}^*, \mathrm{IV}\}$ then $\mathcal{J}=0$, and if {type of $\hat{F}_i$ $|i=1,2\}$ is $\{\mathrm{III}^*, \mathrm{III}\}$ then $\mathcal{J}=1$.

Lemma 4 *Under the above condition, if $\bar{P}_2(S)=\bar{P}_3(S)=0$ then* {type of $\hat{F}_i$ $|i=1,2\}$ *is* $\{\mathrm{III}^*, \mathrm{II}\}$ *or* $\{\mathrm{III}^*, \mathrm{III}\}$ *or* $\{\mathrm{II}^*, \mathrm{II}\}$.

Proof: If {type of $\hat{F}_i$ $|i=1,2\}$ is $\{\mathrm{IV}^*, \mathrm{III}\}$ or $\{\mathrm{IV}^*, \mathrm{II}\}$ or $\{\mathrm{IV}^*, \mathrm{IV}\}$, then we infer that $\bar{P}_3(S) \geq 1$ by the argument in [6]p.355-356. □

Proof of Theorem 1: If the assumption of Theorem 1 holds, we infer from Lemma 2 and Fact 2 and Fact 4 that $\hat{S}$ is the basic member of some $\mathcal{F}(\mathcal{J}, G)$ where $\mathcal{J}$ is a meromorphic function on $\mathbb{P}^1_{\mathbb{C}}$. First we assume that $\mathcal{J}$ is not constant. Then we infer from Remark 6 and Lemma 4 that {type of $\hat{F}_i$ $|i=1,2\}$ is $\{\mathrm{III}^*, \mathrm{II}\}$. Then by the argument of [6]p.354 Case 1, we infer that $\bar{P}_4(S)=1$, $\bar{P}_{12}(S)=2$ and $\bar{\kappa}(S)=1$. On the other hand from the formula (1) we infer that $\hat{\pi}$ has only one singular fibre except for $\hat{F}_i(i=1,2)$ which is of type I_1. Hence by choosing a suitable inhomogeneous coordinate for $\mathbb{P}^1_{\mathbb{C}}$, we may assume that C_0 is of type II and C_1 is of type III* and C_∞ is of type I_1. Denoting this inhomogeneous coordinate by s we infer from Fact 5 that $\mathcal{J}(s)$ has only one pole with multiplicity 1 at $s=\infty$ and has a zero of order $h \equiv 1(3)$ at $s=0$ and that $\mathcal{J}(s)-1$ has a zero of order $h \equiv 1(2)$ at $s=1$. It is obvious that a meromorphic function $\mathcal{J}(s)$ on $\mathbb{P}^1_{\mathbb{C}}$

satisfying such conditions is only $\mathcal{J}(s) = s$. Since the homological invariant G belonging to $\mathcal{J}(s) = s$ which induces the above set of singular fibres is unique, $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ is isomorphic to the basic member of $\mathcal{F}(\mathcal{J}, G)$ which is uniquely determined upto isomorphisms induced by automorphisms of $\mathbb{P}^1_{\mathbb{C}}$. Thus we have the conclusion in case 1.

Next we assume that {type of $\hat{F}_i$ $|i = 1, 2\}$ is $\{\mathrm{III}^*, \mathrm{III}\}$. Then the functional invariant $\mathcal{J}$ is constant 1 and there are no singular fibres except for $\hat{F}_i(i = 1, 2)$.The elliptic fibre space $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ satisfying these conditions is uniquely determined upto isomorphisms induced by automorphisms of $\mathbb{P}^1_{\mathbb{C}}$ and therefore $\pi : S \to \mathbb{C}^*$ is isomorphic to fixed one. In this case by the similar argument as in [6]p.354 Case 1 we have an effective divisor which belongs to the linear system $|4(K_{\bar{S}} + D)|$ and whose intersection form is negative definite. Hence $\bar{P}_4(S) = 1$ and $\bar{\kappa}(S) = 0$. Thus we have the conclusion in case 2.

Now we assume that {type of $\hat{F}_i$ $|i = 1, 2\}$ is $\{\mathrm{II}^*, \mathrm{II}\}$. Then $\mathcal{J} = 0$ and there are no singular fibres except for $\hat{F}_i(i = 1, 2)$ and we infer that $\pi : S \to \mathbb{C}^*$ is isomorphic to a fixed one. In this case we have $\bar{P}_6(S) = 1$ and $\bar{\kappa}(S) = 0$. Thus we have the conclusion in case 3. Q. E. D.

Proposition 2 *If* {type of $\hat{F}_i|i = 1, 2\}$ *is* $\{\mathrm{IV}^*, \mathrm{III}\}$ *and* $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ *has no multiple fibres,then* $\pi : S \to \mathbb{C}^*$ *is isomorphic to a fixed one as a fibration over* $\mathbb{C}^*$ *and* $\bar{P}_4(S) = 1$ *and* $\bar{\kappa}(S) = 0$.

Proof. By the same argument in the proof of Theorem 1, we can apply the formula (1) to $\hat{S}$. Since $p_a(\hat{S}) = 0$ and $\nu(\mathrm{IV}^*) \geq 1$, $\nu(\mathrm{III}) \geq 1$, we infer that $\hat{\pi}$ has only one singular fibre except for $\hat{F}_i(i = 1, 2)$ which is of type I_1. Hence by choosing a suitable inhomogeneous coordinate for $\mathbb{P}^1_{\mathbb{C}}$, we may assume that C_0 is of type IV^* and C_1 is of type III and C_∞ is of type I_1. Denoting this inhomogeneous coordinate by s we infer that $\mathcal{J}(s)= s$ in the same way as in the proof of Theorem 1. Hence $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ is isomorphic to the basic member of $\mathcal{F}(\mathcal{J}, G)$ where G is the homological invariant determined by the singular fibres. On the other hand by the similar argument as in [6]p.354 Case 1, we have an effective divisor which belongs to the linear system $|4(K_{\bar{S}}+D)|$ and whose intersection form is negative definite. Hence $\bar{P}_4(S)=1$ and $\bar{\kappa}(S)=0$. Q. E. D.

Remark 7 The functional invariant in Proposition 2 is the same as in Theorem 1 but the homological invariant is different.

Remark 8 If $\hat{\pi}$ has a multiple fibre we have $\bar{P}_4(S)\geq 1$ and $\bar{\kappa}(S)=1$.

Proposition 3 *If* {type of $\hat{F}_i | i=1,2$} *is* {IV*, II} *then* $\bar{P}_3(S)\geq 1$ *and* $\bar{\kappa}(S)=1$.*Moreover if we assume that* $\hat{\pi}:\hat{S}\to\mathbb{P}^1_{\mathbb{C}}$ *has no multiple fibres, we have following three cases:*
Case 1: The functional invariant $\mathcal{J}$ *is not constant and* $\pi: S\to\mathbb{C}^*$ *is isomorphic to a fixed one as a fibration over* $\mathbb{C}^*$.
Case 2: The functional invariant $\mathcal{J}$ *is not constant and* $\pi: S\to\mathbb{C}^*$ *is isomorphic to a member of a family of elliptic fibrations over* $\mathbb{C}^*$ *parametrized by* $\mathbb{C}\backslash\{0,1,-1\}$.
Case 3: $\mathcal{J}=0$ *and* $\pi: S\to\mathbb{C}^*$ *is isomorphic to a fixed one as a fibration over* $\mathbb{C}^*$.

Proof. $\bar{P}_3(S)\geq 1$ was already stated in the proof of Lemma 4. By the argument in [6]p.355-356 we infer immediately that $\bar{P}_6(S)\geq 2$. Thus we have $\bar{\kappa}(S)=1$.

First we assume that $\mathcal{J}$ is not constant. Then $j\geq 1$ and we infer from formula (1) that $\nu(\mathrm{IV}^*)=\nu(\mathrm{II})=1$ and $j=2$. Thus we have two cases one of which is $\nu(\mathrm{I}_2)=1$, $\nu(\mathrm{I}_1)=0$ and the other is $\nu(\mathrm{I}_2)=0$, $\nu(\mathrm{I}_1)=2$ and there is no singular fibres except for above 3 or 4 fibres.

Case 1: Suppose that $\nu(\mathrm{I}_2)=1$. Then there is three singular fibres whose types are II and IV* and I_2 respectively. Hence by choosing a suitable inhomogeneous coordinate for $\mathbb{P}^1_{\mathbb{C}}$, we may assume that C_0 is of type IV* and

C_1 is of type II and C_∞ is of type I_2. Denoting this inhomogeneous coordinate by s we infer from Fact 5 that $\mathcal{J}(s)$ is a quadratic polynomial in s such that $\mathcal{J}(0) = \mathcal{J}(1) = 0$. Thus we have $\mathcal{J}(s) = cs(s-1)$ where c is a nonzero constant. Since $\hat{S}$ has no singular fibres of type III or III*, $\mathcal{J}(s) - 1$ has no simple zeros. Hence the quadratic equation $cs(s-1) = 1$ has a multiple root and therefore we have $c = -4$. Thus $\mathcal{J}(s) = -4s(s-1)$ and $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ is isomorphic to the basic member of $\mathcal{F}(\mathcal{J}, G)$ where $\mathcal{J}(s) = -4s(s-1)$ and G is the uniquely determined homological invariant by the set of singular fibres.

Case 2: Suppose that $\nu(\mathrm{I}_1) = 2$. In this case there are four singular fibres and we can prescribe the positions of any three singular fibres by a suitable automorphism of $\mathbb{P}^1_{\mathbb{C}}$. Hence by choosing a suitable inhomogeneous coordinate for $\mathbb{P}^1_{\mathbb{C}}$ we may assume that C_0 is of type IV* and C_1 is of type II and C_∞ is of type I_1. If we denote this inhomogeneous coordinate by s, the remaining singular fibre which is of type I_1 is defined by $s = \alpha$ where $\alpha \neq 0, 1, \infty$. Then we infer from Fact 5 that $\mathcal{J}(0) = \mathcal{J}(1) = 0$ and that $\lim_{s\to\alpha}(s-\alpha)\mathcal{J}(s)$ and $\lim_{s\to\infty} s^{-1}\mathcal{J}(s)$ are nonzero constants. Hence we have $\mathcal{J}(s) = cs(s-1)(s-\alpha)^{-1}$ where c is a nonzero constant. Since $\mathcal{J}(s) - 1$ has no simple zeros, the quadratic equation $cs(s-1) = s - \alpha$ has a multiple root. Thus we have $(c+1)^2 = 4c\alpha$ and

$$\mathcal{J}(s) = \frac{4c^2 s(s-1)}{4cs - (c+1)^2} \tag{2}$$

Since $\alpha \neq 0, 1, \infty$ we have $c \neq 0, 1 - 1$. For $c \in \mathbb{C}\backslash\{0, 1, -1\}$ we denote by $\bar{B}_c$ the basic member of $\mathcal{F}(\mathcal{J}, G)$ where $\mathcal{J}(s)$ is given by (2) and G is the homological invariant determined by the singular fibres. We denote by $\bar{\mathcal{B}}$ the family $\{\bar{B}_c | c \in \mathbb{C}\backslash\{0, 1, -1\}\}$. Since every member $\bar{B}_c$ of $\bar{\mathcal{B}}$ has fixed fibres C_0 of type IV* and C_1 of type II, we put $\bar{B}_c \backslash C_0 \cup C_1 = B_c$ and denote by $\mathcal{B}$ the family $\{B_c | c \in \mathbb{C}\backslash\{0, 1, -1\}\}$. Then $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ is isomorphic to a member of the family $\bar{\mathcal{B}}$ and $\pi : S \to \mathbb{C}^*$ is isomorphic to a member of the family $\mathcal{B}$.

Case 3: Finally we assume that $\mathcal{J}$ is constant. Then $\mathcal{J}=0$ since there are singular fibres of type IV* and II. Then we infer from formula (1) that $\nu(\mathrm{IV}^*)=1$ and $\nu(\mathrm{II})=2$ and there are no other singular fibres. Since we can prescribe the positions of these three singular fibres by an automorphism of $\mathbb{P}^1_{\mathbb{C}}$, we infer that $\pi : S \to \mathbb{C}^*$ is isomorphic to a fixed one.

Q.E.D.

Proposition 4 *If* $\{\text{type of } \hat{F}_i | i=1,2\}$ *is* $\{\mathrm{IV}^*, \mathrm{IV}\}$ *and* $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ *has no multiple fibres, then* $\bar{P}_3(S)=1$ *and* $\bar{\kappa}(S)=0$ *and* $\pi : S \to \mathbb{C}^*$ *is isomorphic to a fixed one as a fibration over* $\mathbb{C}^*$*.*

Proof. From the assumption we have $\mathcal{J}=0$ and there are no singular fibres except for $\hat{F}_i (i=1,2)$. Thus we infer that $\pi : S \to \mathbb{C}^*$ is isomorphic to a fixed one. On the other hand by the similar argument as in [6]p.354 Case 1, we have an effective divisor which belongs to the linear system $|3(K_{\bar{S}}+D)|$ and whose intersection form is negative definite. Hence $\bar{P}_3(S)=1$ and $\bar{\kappa}(S)=0$. Q.E.D.

Remark 9 If $\hat{\pi}$ has a multiple fibre we have $\bar{P}_3(S) \geq 1$ and $\bar{\kappa}(S)=1$.

By Theorem 1 and Proposition 2-4 we have classified ellipic fibre spaces $\pi : S \to \mathbb{C}^*$ which are extended to fibre spaces $\bar{\pi} : \bar{S} \to \mathbb{P}^1_{\mathbb{C}}$ with no multiple fibres such that $\bar{S}$ are rational and $\bar{P}_2(S)=0$. In the next section we give explicit constructions of examples of each cases.

4 Constructions of examples

In this section we denote by $(X:Y:Z)$ a homogeneous coodinate for $\mathbb{P}^2_{\mathbb{C}}$.

Example 1 Let C be a rational curve of degree 3 with one cusp in $\mathbb{P}^2_{\mathbb{C}}$ and L be a line which is tangent to C at one non-singular point and intersects simply with C at one another point. Removing the base points of the linear system generated by $3L$ and C, we have an elliptic fibre space $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ with singular fibres of type III* and II which correspond to $3L$ and C respectively. For example we put $C = \{Y^2Z - X^3 = 0\}$ and $L = \{-3X + 2Y + Z = 0\}$. Then a fibre which does not correspond to $3L$ is defined by $\{Y^2Z - X^3 - w(-3X + 2Y + Z)^3 = 0\}$ where $w \in \mathbb{C}$. We infer that when $w = 108^{-1}$ the fibre has one node at $(X : Y : Z) = (-1 : 1 : 1)$. Thus this is the case we treated in Theorem 1 Case 1.

Remark 10 The above construction is different from the one in [6]p.357.

Example 2 Let C be a conic in $\mathbb{P}^2_{\mathbb{C}}$ and L_1 and L_2 be two different tangent lines of C. Then by the linear system generated by $3L_1$ and $C + L_2$ we have an elliptic fibre space $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ with singular fibres of type III* and III which correspond to $3L_1$ and $C + L_2$ respectively. For example we put $C = \{YZ - X^2 = 0\}$ and $L_1 = \{Z = 0\}$ and $L_2 = \{Y = 0\}$. Then a fibre which does not correspond to $3L_1$ is defined by $\{(YZ - X^2)Y - wZ^3 = 0\}$ where $w \in \mathbb{C}$. For $w \neq 0$ they are all nonsingular and isomorphic to each other. Thus this is the case in Theorem 1 Case 2.

Example 3 Let C be a rational curve of degree 3 with one cusp in $\mathbb{P}^2_{\mathbb{C}}$ and L be the tangent line of C at the nonsingular inflection point. Then by the linear system generated by $3L$ and C we have an elliptic fibre space $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ with singular fibres of type II* and II which correspond to $3L$ and C respectively. For example we put $C = \{YZ^2 - X^3 = 0\}$ and $L = \{Y = 0\}$. Then a fibre which does not correspond to $3L$ is defined by $\{YZ^2 - X^3 - wY^3 = 0\}$ where $w \in \mathbb{C}$. For $w \neq 0$ they are all nonsingular and isomorphic to each other. Thus this is the case in Theorem 1 Case 3.

Example 4 Let C be a conic in $\mathbb{P}^2_{\mathbb{C}}$ and L_1 be a line intersecting with C simply at two point p, q and L_2 a tangent line of C at a point different from p and q. Then by the linear system generated by $3L_1$ and $C + L_2$ we have an elliptic fibre space $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ with singular fibres of type IV* and

III which correspond to $3L_1$ and $C+L_2$ respectively. For example we put $C=\{YZ-X^2=0\}$ and $L_1=\{Y-Z=0\}$ and $L_2=\{Y=0\}$. Then a fibre which does not correspond to $3L_1$ is defined by $\{(YZ-X^2)Y-w(Y-Z)^3=0\}$ where $w\in\mathbb{C}$. We infer that when $w=-4/27$ the fibre has a node at $(X:Y:Z)=(0:-2:1)$. Thus this is the case in Proposition 2.

Example 5 Let C be a rational curve of degree 3 with one cusp in $\mathbb{P}^2_{\mathbb{C}}$ and L be a line which intersects simply with C at three points. Then by the linear system generated by C and $3L$ we have an elliptic fibre space $\hat{\pi}:\hat{S}\to\mathbb{P}^1_{\mathbb{C}}$ with singular fibres of type IV* and II which correspond to $3L$ and C respectively. If we put $C=\{Y^2Z-X^3=0\}$ and $L=\{X-Z=0\}$, then a fibre which does not correspond to $3L$ is defined by $\{Y^2Z-X^3-w(X-Z)^3=0\}$ where $w\in\mathbb{C}$. Then by calculation we infer that for $w=-1$ the corresponding fibre is of type I_2, and for $w\neq 0,-1$ the corresponding fibre is regular. Thus this is the case in Proposition 3 Case 1.

Example 6 In the above example if we put $C=\{Y^2Z-X^3=0\}$ and $L=\{3a^2X+uY-Z=0\}$, where $u,a\in\mathbb{C}$ such that $au(u+2a^3)(u-2a^3)\neq 0$, then a fibre which does not correspond to $3L$ is defined by $\{Y^2Z-X^3-w(3a^2X+uY-Z)^3=0\}$ where $w\in\mathbb{C}$. Then by calculation we infer that for two different nonzero values of w the fibre has one node. In fact for $w=w_1=-(4/27)(u+2a^3)^{-2}$ the corresponding fibre has a node at $(X:Y:Z)=(a:1:(-1/2)u)$, and for $w=w_2=-(4/27)(u-2a^3)^{-2}$ the corresponding fibre has a node at $(X:Y:Z)=(-a:1:(-1/2)u)$. Thus this is the case in Proposition 3 Case 2.

Example 7 In the above example if we put $C=\{Y^2Z-X^3=0\}$ and $L=\{Y-Z=0\}$, then a fibre which does not correspond to $3L$ is defined by $\{Y^2Z-X^3-w(Y-Z)^3=0\}$ where $w\in\mathbb{C}$. Then we infer that for $w=-4/27$ the fibre has one cusp at $(X:Y:Z)=(0:-2:1)$ and for

$w \neq 0, -4/27$ the fibres are all non-singular and isomorphic to each other. Thus this is the case in Proposition 3 Case 3.

Example 8 Let L_i $(i = 1, 2, 3)$ be three different lines in $\mathbb{P}^2_{\mathbb{C}}$ intersecting one point p and L_0 be a line which does not contain p. Then by the linear system generated by $3L_0$ and $L_1 + L_2 + L_3$ we can construct an elliptic fibre space $\hat{\pi} : \hat{S} \to \mathbb{P}^1_{\mathbb{C}}$ with singular fibres of type IV^* and IV which correspond to $3L_0$ and $L_1 + L_2 + L_3$ respectively. For example we put $L_0 = \{Z = 0\}$, $L_1 = \{X = 0\}$, $L_2 = \{Y = 0\}$ and $L_3 = \{X + Y = 0\}$. Then a fibre which does not correspond to $3L_0$ is defined by $\{XY(X + Y) - wZ^3 = 0\}$ where $w \in \mathbb{C}$. Then for $w \neq 0$ all fibres are non-singular and isomorphic to each other. Thus this is the case in Proposition 4.

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