Construction of concave shoulder surfaces

ISOKAWA Yukinao*

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Abstract

A general method of constructing concave shoulder surfaces is presented, and a process of development of these surfaces is described explicitly.

keywords: shoulder, developable surface, concavity

1 Introduction

An excellent review paper [1] explains the shoulder as follows:

> The shoulder of packaging machine is a developable surface that guides the packing material without stretching or tearing. The shoulder is traditionally manufactured by bending a flexible plate along a given bending curve, also without stretching or tearing.

As the shoulder is a developable surface, it is a ruled surface, that is, it can be completely specified by generating lines.

In this paper, we first review a general method of constructing shoulder surfaces in section 2, and show several examples of shoulders constructed by the method in section 3.

In the section 4 we make a new attempt to describe the process of development of shoulder surfaces. Further, in section 5, we study the possibility of constructing concave shouder surfaces.

2 Method of constructing shoulder surfaces

Consider a curve $C_0 = \{\mathbf{r}_0(\phi) : 0 \leqq \phi \leqq \pi\}$ which lies on the lateral face S_0 of a prism, whose base is defined by $\{r(\phi) : 0 \leqq \phi \leqq \pi\}$ in the polar coordinates. Thus the curve C_0 can be expressed as

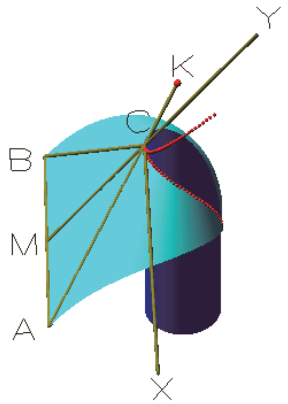
$$\mathbf{r}_0(\phi) = (r(\phi)\cos\phi, r(\phi)\sin\phi, z(\phi)). \tag{1}$$

Now we will develop the curve C_0 onto a plane S_1 without changing both the length of the curve and the distance of any point on the curve from the fixed point K in space. Denote the developed curve on the plane S_1 by $C_1 = \{\mathbf{r}_1(\phi) : 0 \leqq \phi \leqq \pi\}$, which can be expressed as

$$\mathbf{r}_1(\phi) = (X(\phi), Y(\phi)) \tag{2}$$

in the Cartesian coordinates system in the plane S_1.

* Professor of Kagoshima University, Faculty of Education



The first requirement that the length of the curve is preserved in the course of development, can be formulated as

$$r'(\phi)^2 + r(\phi)^2 + z'(\phi)^2 = X'(\phi)^2 + Y'(\phi)^2. \tag{3}$$

The second requirement that the distance of any point on the curve from K is preserved in the course of development, can be formulated as

$$(r(\phi)\cos\phi - k_x)^2 + (r(\phi)\sin\phi - k_y)^2 + (z(\phi) - k_z)^2 = (X(\phi) - k_X)^2 + (Y(\phi) - k_Y)^2, \tag{4}$$

where we suppose that $\mathrm{K} = (k_x, k_y, k_z)$ in space and $\mathrm{K} = (k_X, k_Y)$ in the plane S_1. In the following we set

$$\mathrm{K} = (k_x, k_y, k_z) = (r_0 - e\cos\alpha, -e\tan\beta, h + e\sin\alpha)$$

and

$$\mathrm{K} = (k_X, k_Y) = (-e\tan\beta, e).$$

Suppose that the lateral face S_0 (of the prism), the plane S_1, and the fixed point K are given. Moreover suppose that the start points of the two curves C_0 and C_1, ie., the points for $\phi = 0$, coincide each other. In spite of these restrictions, by using (3) and (4), we can not determine C_0 and C_1 uniquely. In fact there are infinitely many candidates of the two curves that satisfy both the fundamental equations.

However there is a simple method of constructing the two curves. Suppose that

$$z(\phi) = h - Y(\phi). \tag{5}$$

Then the fundamental equation (3 implies $X'(\phi)^2 = r'(\phi)^2 + r(\phi)^2$. Hence we get

$$X(\phi) = \int_0^\phi \sqrt{r'(\phi)^2 + r(\phi)^2}\, d\phi. \tag{6}$$

Then, substituting (5) and (6) into the fundamental equation (4), we obtain

$$\begin{aligned} Y(\phi) \ &= \ \lambda \, \{-k_x r(\phi)(1-\cos\phi) - k_y(X(\phi) - r(\phi)\sin\phi) \\ &\quad + \frac{1}{2}X(\phi)^2 - r(\phi)(r(\phi) - r_0)\Big\}, \end{aligned} \tag{7}$$

where we put $1/\lambda = e(1 + \sin\alpha)$.

Now we reveal several restrictions on parameters h, e, α, β. First we require the condition $z(\pi) = 0$, which implies that

$$e = \frac{\frac{1}{2}X(\pi)^2 - r(\pi)(r_0 + r(\pi))}{h(1+\sin\alpha) - 2r(\pi)\cos\alpha - X(\pi)\tan\beta}. \tag{8}$$

Consider a shoulder surface which consists of straight lines, each of which emanates from any point P on the curve C_0 and has the direction specified by the vector $\overrightarrow{\mathrm{KP}}$. Suppose that the shoulder surface does not overlap with the prism whose base defined by $r(\phi)$. Then it is necessary that $-r(\pi) < k_x < r(0)$, which is equivalent to

$$0 < e < \frac{r(0) + r(\pi)}{\cos\alpha}.$$

Hence, using (8), we can derive

$$h > \frac{1}{1+\sin\alpha}\left\{X(\pi)\tan\beta + r(\pi)\cos\alpha + \frac{X(\pi)^2\cos\alpha}{2(r(0)+r(\pi))}\right\}. \tag{9}$$

Therefore our method to set parameters is as follows:

1. Set α and β arbitrarily so that $0 < \alpha, \beta < \pi/2$.
2. Set h so that it satisfies (9).
3. Set e by (8).

3 Several examples

3.1 Shoulder surface based on a circle

Suppose that a base curve is a circle of radius r_0. Then the developed curve $C_1 = (X(\phi), Y(\phi))$ has a simple form,

$$X(\phi) = r_0\phi,\ Y(\phi) = \lambda\left\{-k_x r_0(1-\cos\phi) - k_y r_0(\phi - \sin\phi) + \frac{1}{2}r_0^2\phi^2\right\}.$$

Note that the curve C_1 looks like a parabola in human eye, but it is really not.

3.2 Shoulder surface besed on a kite

Suppose that a base curve is a kite with round vertices. A kite can be parametrized by a, b, ω, ϵi, where ω may be positive. A rhombus is a particular case of a kite such that $\omega = 0$.

The equation $r(\phi)$ of a kite for $0 \leqq \phi \leqq \frac{\pi}{2} - \omega$ is given as follows:

$$r(\phi) = \begin{cases} a\cos\phi + \sqrt{\epsilon^2 - a^2\sin^2\phi} & \text{if } 0 \leqq \phi \leqq \delta_1 \\ \dfrac{l1}{\cos(\phi - \gamma_1)} & \text{if } \delta_1 < \phi < \frac{\pi}{2} - \omega - \delta_2 \\ b\cos\phi' + \sqrt{\epsilon^2 - b^2\sin^2\phi'} & \text{if } \frac{\pi}{2} - \omega - \delta_2 \leqq \phi \leqq \frac{\pi}{2} - \omega \end{cases}, \tag{10}$$

where we put $\phi' = \frac{\pi}{2} - \omega - \phi$ and

$$\begin{aligned} &\tan\gamma_1 = \frac{a - b\sin\omega}{b\cos\omega},\ \tan\gamma_2 = \frac{b - a\sin\omega}{a\cos\omega}, \\ &l_1 = a\cos\gamma_1 + \epsilon = b\cos\gamma_2 + \epsilon, \\ &\tan\delta_1 = \frac{\epsilon\sin\gamma_1}{a + \epsilon\cos\gamma_1},\ \tan\delta_2 = \frac{\epsilon\sin\gamma_2}{b + \epsilon\cos\gamma_2}. \end{aligned}$$

Note that, since a kite is convex, it is necessary that $a - b\sin\omega > 0,\ b - a\sin\omega > 0$. If we denote the function (10) by $r(\phi; a, \omega)$, then the equation $r(\phi)$ for $\frac{\pi}{2} - \omega < \phi \leqq \frac{\pi}{2}$ can be given by $r(\pi - \phi; a, -\omega)$.

Using (6) we get for $0 \leqq \phi \leqq \frac{\pi}{2} - \omega$

$$X(\phi) = \begin{cases} \epsilon\phi + \epsilon \arcsin\left(\dfrac{a}{\epsilon}\sin\phi\right) & \text{if } 0 \leqq \phi \leqq \delta_1 \\ X(\delta_1) + l_1\{\tan(\phi - \gamma_1) + \tan(\gamma_1 - \delta_1)\} & \text{if } \delta_1 < \phi < \frac{\pi}{2} - \omega - \delta_2 \\ X(\pi/2 - \omega - \delta_2) - \epsilon\phi' - \epsilon \arcsin\left(\dfrac{b}{\epsilon}\sin\phi'\right) & \\ \qquad + \epsilon\delta_2 + \epsilon \arcsin\left(\dfrac{b}{\epsilon}\sin\delta_2\right) & \text{if } \frac{\pi}{2} - \omega - \delta_2 \leqq \phi \leqq \frac{\pi}{2} - \omega \end{cases}, \qquad (11)$$

Denote the function (11) by $X(\phi; a, \omega)$. Let us introduce a variable $\psi = \pi - \phi$ and a function $\tilde{X}(\psi) = X(\pi) - X(\phi)$. Then we can see that $\tilde{X}(\psi) = X(\psi; a, -\omega)$, where we have to replace $\gamma_1, \gamma_2, l_1, \delta_1, \delta_2, \omega$ in (11) by $\gamma_3, \gamma_4, l_2, \delta_3, \delta_4, -\omega$ respectively, i.e., $\psi' = \frac{\pi}{2} + \omega - \psi$ and

$$\begin{aligned} &\tan\gamma_3 = \frac{a + b\sin\omega}{b\cos\omega},\ \tan\gamma_4 = \frac{b + a\sin\omega}{a\cos\omega}, \\ &l_2 = a\cos\gamma_3 + \epsilon = b\cos\gamma_4 + \epsilon, \\ &\tan\delta_3 = \frac{\epsilon\sin\gamma_3}{a + \epsilon\cos\gamma_3},\ \tan\delta_4 = \frac{\epsilon\sin\gamma_4}{b + \epsilon\cos\gamma_4}. \end{aligned}$$

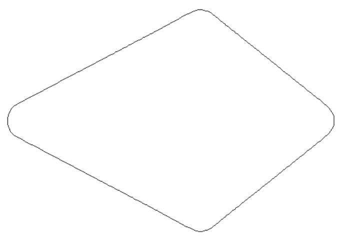
Therefore we see

$$X(\phi) = X(\pi) - \tilde{X}(\pi - \phi) \quad \text{for } \phi > \frac{\pi}{2} - \omega,$$

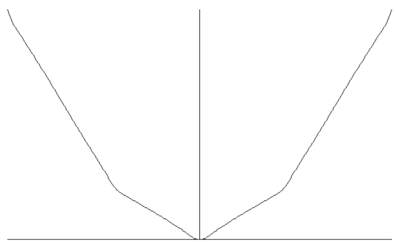
where $X(\pi)$ can be computed by

$$X(\pi) = X\left(\frac{\pi}{2} - \omega\right) + \tilde{X}\left(\frac{\pi}{2} + \omega\right).$$

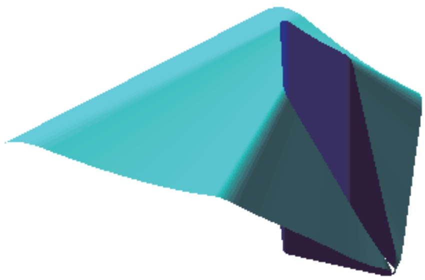
When the parameters of the base curve are $a = 3, b = 2, \omega = \pi/5, \epsilon = 0.4$, it looks like



If the additional parameters are $\alpha = \beta = \pi/5, h = 10$, then $e \approx 2.32$ and the developped curve is drawn as



Then the shoulder surface looks like



3.3 Shoulder surface based on a trapezoid

Suppose that a base curve is a trapezoid with round vertices. The equation of the trapezoid is:

$$r(\phi) = \begin{cases} \dfrac{c+\epsilon}{\cos\phi} & \text{if } 0 \leqq \phi \leqq \gamma_1 - \delta_1 \\ l_1 \cos(\phi - \gamma_1) + \sqrt{\epsilon^2 - l_1^2 \sin^2(\phi - \gamma_1)} & \text{if } \gamma_1 - \delta_1 < \phi < \gamma_1 + \delta_2 \\ \dfrac{l_2}{-\cos(\phi + \gamma_2)} & \text{if } \gamma_1 + \delta_2 \leqq \phi \leqq \pi - \gamma_3 - \delta_3 \\ -l_3 \cos(\phi + \gamma_3) + \sqrt{\epsilon^2 - l_3^2 \sin^2(\phi + \gamma_3)} & \text{if } \pi - \gamma_3 - \delta_3 < \phi < \pi - \gamma_3 + \delta_4 \\ \dfrac{c+\epsilon}{-\cos\phi} & \text{if } \pi - \gamma_3 + \delta_4 \leqq \phi \leqq \pi \end{cases} \tag{12}$$

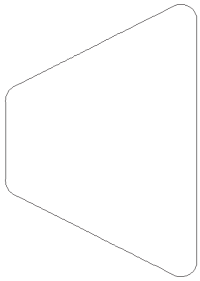
where $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3, \delta_4$ are defined by

$$\begin{aligned} &\tan\gamma_1 = \frac{a}{c},\ \tan\gamma_2 = \frac{2c}{a-b},\ \tan\gamma_3 = \frac{b}{c}, \\ &l_1 = \frac{c}{\cos\gamma_1},\ l_2 = a\sin\gamma_2 - c\cos\gamma_2,\ l_3 = \frac{c}{\cos\gamma_3}, \\ &\tan(\gamma_1 - \delta_1) = \frac{a}{c+\epsilon},\ \tan(\gamma_1 + \delta_2) = \frac{a + \epsilon\sin\gamma_2}{c - \epsilon\cos\gamma_2}, \\ &\tan(\gamma_3 + \delta_3) = \frac{b + \epsilon\sin\gamma_2}{c + \epsilon\cos\gamma_2},\ \tan(\gamma_3 - \delta_4) = \frac{b}{c+\epsilon}. \end{aligned}$$

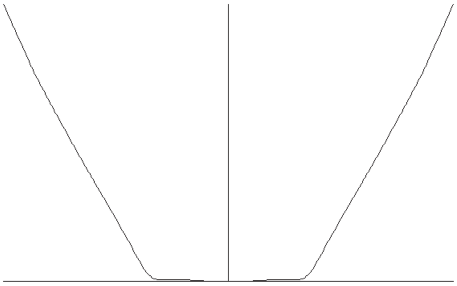
Hence we have

$$X(\phi) = \begin{cases} (c+\epsilon)\tan\phi & \text{if } 0 \leqq \phi \leqq \gamma_1 - \delta_1 \\ X(\gamma_1 - \delta_1) + \epsilon\delta_1 + \epsilon \arcsin\left(\dfrac{l_1}{\epsilon}\sin\delta_1\right) & \\ \quad +\epsilon(\phi - \gamma_1) + \epsilon\arcsin\left(\dfrac{l_1}{\epsilon}\sin(\phi - \gamma_1)\right) & \text{if } \gamma_1 - \delta_1 < \phi < \gamma_1 + \delta_2 \\ X(\gamma_1 + \delta_2) - l_2 \tan(\gamma_1 + \gamma_2 + \delta_2) & \\ \quad +l_2 \tan(\phi + \gamma_2) & \text{if } \gamma_1 + \delta_2 \leqq \phi \leqq \pi - \gamma_3 - \delta_3 \\ X(\pi - \gamma_3 - \delta_3) - \epsilon(\pi - \delta_3) + \epsilon\arcsin\left(\dfrac{l_3}{\epsilon}\sin\delta_3\right) & \\ \quad +\epsilon(\phi + \gamma_3) - \epsilon\arcsin\left(\dfrac{l_3}{\epsilon}\sin(\phi + \gamma_3)\right) & \text{if } \pi - \gamma_3 - \delta_3 < \phi < \pi - \gamma_3 + \delta_4 \\ X(\pi - \gamma_3 + \delta_4) + (c+\epsilon)\tan(\gamma_3 - \delta_4) & \\ \quad +(c+\epsilon)\tan\phi & \text{if } \pi - \gamma_3 + \delta_4 \leqq \phi \leqq \pi \end{cases} \tag{13}$$

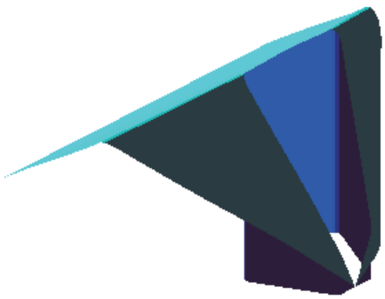
When the parameters of the base curve are $a = 3, b = 1, c = 2, \epsilon = 0.4$, it looks like



If the additional parameters are $\alpha = \beta = \pi/5, h = 12$, then $e \approx 4.42$ and the developped curve is drawn as



Then the shoulder surface looks like



3.4 Shoulder surface besed on a regular p-gon

Suppose that a base curve is a regular p-gon with round vertices. For $0 \leqq \phi \leqq \pi/p$, its equation is given by:

$$r(\phi) = \begin{cases} \dfrac{c+\epsilon}{\cos\phi} & \text{if } 0 \leqq \phi \leqq \delta_1 \\ \dfrac{l_1}{\cos(\phi - \gamma_1} & \text{if } \delta_1 < \phi < 2\gamma_1 - \delta_1 \\ r(2\gamma_1 - \phi) & \text{if } 2\gamma_1 - \delta_1 \leqq \phi \leqq 2\gamma_1, \end{cases} \tag{14}$$

where

$$\gamma_1 = \frac{\pi}{p},\ l_1 = a\cos\gamma_1 + \epsilon,\ \tan\delta_1 = \frac{\epsilon\sin\gamma_1}{a + \epsilon\cos\gamma_1}.$$

For $\phi > \pi/p$, the equation $r(\phi)$ is extended periodically with period $2\gamma_1$, i.e., since there is an positive integer k such that

$$\phi = \phi_1 + 2\gamma_1 k,\ 0 \leqq \phi < 2\gamma_1,$$

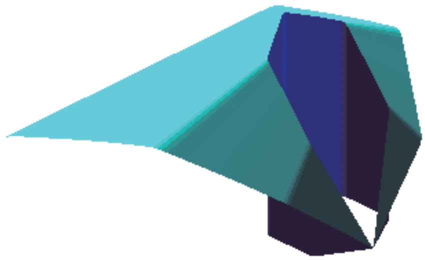
we define $r(\phi) = r(\phi_1)$.

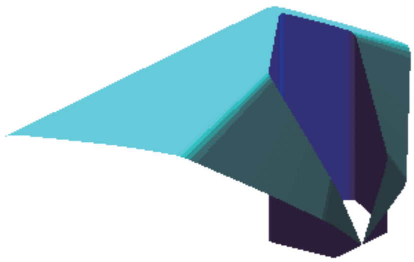
Hence, for $0 \leqq \phi \leqq \pi/p$, we have

$$X(\phi) = \begin{cases} \epsilon\phi + \epsilon \arcsin\left(\frac{a}{\epsilon}\sin\phi\right) & \text{if } 0 \leqq \phi \leqq \delta_1 \\ X(\delta_1) + l_1\{\tan(\phi - \gamma_1) + \tan(\gamma_1 - \delta_1)\} & \text{if } \delta_1 < \phi < 2\gamma_1 - \delta_1 \\ X(2\gamma_1 - \delta_1) - \epsilon\phi' - \epsilon \arcsin\left(\frac{a}{\epsilon}\sin\phi'\right) & \\ \qquad + \epsilon\delta_1 + \epsilon \arcsin\left(\frac{a}{\epsilon}\sin\delta_1\right) & \text{if } 2\gamma_1 - \delta_1 \leqq \phi \leqq 2\gamma_1, \end{cases} \tag{15}$$

For $\phi > \pi/p$, the equation $X(\phi)$ is extended periodically with period $2\gamma_1$.

The following figures are shoulder surfaceis whose base curves are a hexagon and a pentagon respectively:





3.5 Shoulder surface besed on an ellipse

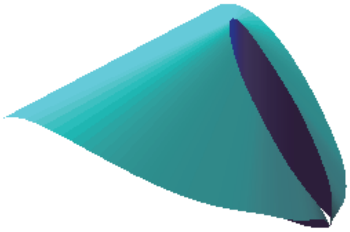
Suppose that a base curve is an ellipse. For $0 \leqq \phi \leqq \pi$, its equation is given by

$$r(\phi) = \frac{ab}{\sqrt{b^2\cos^2\phi + a^2\sin^2\phi}}. \tag{16}$$

Hence

$$X(\phi) = ab\int_0^{\phi} \frac{\sqrt{b^4\cos^2\phi + a^4\sin^2\phi}}{(b^2\cos^2\phi + a^2\sin^2\phi)^{3/2}}\,d\phi. \tag{17}$$

Unfortunately it is impossible to express the above integral in an elementary way, and thus we need to integrate numerically. The following figure shows a shoulder surface where $a = 3, b = 1, h = 10$;



4 Continuous development of shoulder surface

By the requirement imposed on the two curves C_0 and C_1, we can introduce functions $R(\phi)$ and $l(\phi)$ such that

$$R(\phi)^2 = (\mathbf{r}_0(\phi) - \mathbf{k})^2 = (\mathbf{r}_1(\phi) - \mathbf{k})^2, \tag{18}$$

$$l(\phi)^2 = \left(\frac{\partial \mathbf{r}_0(\phi)}{\partial \phi}\right)^2 = \left(\frac{\partial \mathbf{r}_1(\phi)}{\partial \phi}\right)^2, \tag{19}$$

where $\mathbf{k} = (k_x, k_y, k_z)$. In this section we will interpolate the two curves continuously, that is, to find a set of curves parametrized by a 'time' parameter t

$$C_t = \{\mathbf{r}(\phi, t) : 0 \leqq \phi \leqq \pi\},$$

such that

$$\mathbf{r}(\phi, 0) = \mathbf{r}_0(\phi),\ \ \mathbf{r}(\phi, 1) = \mathbf{r}_1(\phi), \tag{20}$$

$$(\mathbf{r}(\phi, t) - \mathbf{k})^2 = R(\phi)^2, \tag{21}$$

$$\left(\frac{\partial \mathbf{r}(\phi, t)}{\partial \phi}\right)^2 = l(\phi)^2. \tag{22}$$

Let us write

$$\mathbf{r}(\phi, t) = \mathbf{k} + \begin{pmatrix} R(\phi) \sin\theta(\phi, t) \cos\psi(\phi, t) \\ R(\phi) \sin\theta(\phi, t) \sin\psi(\phi, t) \\ R(\phi) \cos\theta(\phi, t) \end{pmatrix}.$$

Then we have

$$\left(\frac{\partial \mathbf{r}(\phi, t)}{\partial \phi}\right)^2 = R'(\phi)^2 + R(\phi)^2 \left\{ \left(\frac{\partial \theta}{\partial \phi}\right)^2 + \sin^2\theta \left(\frac{\partial \psi}{\partial \phi}\right)^2 \right\}.$$

Hence

$$\frac{\partial \psi}{\partial \phi} = \frac{1}{\sin\theta} \sqrt{\frac{l(\phi)^2 - R'(\phi)^2}{R(\phi)^2} - \left(\frac{\partial \theta}{\partial \phi}\right)^2}. \tag{23}$$

Note that, since $\mathbf{r}_0(\phi), \mathbf{r}_1(\phi)$ are given, $\theta(\phi, 0), \psi(\phi, 0), \theta_1(\phi, 1), \psi_1(\phi, 1)$ are known. If a function $\theta(\phi, t)$ of the variable ϕ is given for each t, then the equation (23) can determine the function $\psi(\phi, t)$ uniquely. Although $\theta(\phi, t)$ can be defined freely to some extent, one simple way is to define it as

$$\cos\theta(\phi, t) = (1 - t) \cos\theta_0(\phi, 0) + t \cos\theta_1(\phi, 1). \tag{24}$$

In the below we give details of how to construct the interpolating curves in the simplest case, i.e., the case that a base curve is a circl of radius $r_0 = 1$.

1. Recall $\mathbf{r}_0(\phi) = (\cos\phi, \sin\phi, z(\phi))$, where

$$z(\phi) = h - \lambda\{-k_x(1-\cos\phi) - k_y(\phi - \sin\phi) + \frac{1}{2}\phi^2\}.$$

Hence we have

$$\begin{aligned} R(\phi)^2 &= (\cos\phi - k_x)^2 + (\sin\phi - k_y)^2 + (z(\phi) - k_z)^2, \\ R(\phi)R'(\phi) &= k_x \sin\phi - k_y \cos\phi + (z(\phi) - k_z)z'(\phi), \end{aligned}$$

where

$$z'(\phi) = \lambda\{k_x \sin\phi + k_y(1-\cos\phi) - \phi\}.$$

Moreover, since

$$\frac{\partial \mathbf{r}_0(\phi)}{\partial\phi} = (-\sin\phi, \cos\phi, z'(\phi)),$$

we see

$$l(\phi)^2 = 1 + z'(\phi)^2.$$

Consequently we can compute the quantity

$$\frac{l(\phi)^2 - R'(\phi)^2}{R(\phi)^2}.$$

2. By the definition

$$\mathbf{r}_0(\phi) - \mathbf{k} = \mathbf{r}(\phi, 0) - \mathbf{k} = (R\sin\theta\cos\psi, R\sin\theta\sin\psi, R\cos\theta),$$

we have

$$\cos\theta(\phi, 0) = \frac{z(\phi) - k_z}{R(\phi)} = \frac{-(h - z(\phi)) - e\sin\alpha}{R(\phi)}.$$

Note that

$$\mathbf{r}_1(\phi) = (r_0 - Y(\phi)\cos\alpha, X(\phi), h + Y(\phi)\sin\alpha).$$

Hence, recalling $Y(\phi) = h - z(\phi)$, we have

$$\cos\theta(\phi, 1) = \frac{h + Y(\phi)\sin\alpha - k_z}{R(\phi)} = \frac{(h - z(\phi) - e)\sin\alpha}{R(\phi)}.$$

Consequently, by (32), we get

$$\cos\theta(\phi, t) = \frac{1}{R(\phi)}\{(t(1+\cos\alpha) - 1)(h - z(\phi)) - e\sin\alpha\}.$$

3. From the above expression of $\cos\theta(\phi, t)$, we can compute

$$\sin\theta(\phi, t), \text{ and } \frac{\partial}{\partial\phi}\cos\theta(\phi, t) = \sin\theta(\phi, t)\frac{\partial\theta(\phi, t)}{\partial\phi}.$$

Accordingly the right hand side of (23) can be computed because it equals

$$\frac{1}{\sin^2\theta}\sqrt{\frac{l(\phi)^2 - R'(\phi)^2}{R(\phi)^2}\sin^2\theta - \left(\frac{\partial}{\partial\phi}\cos\theta\right)^2}.$$

4. Integrating (23), we have

$$\psi(\phi, t) = \psi(0, t) + \int_0^\phi \frac{\partial\psi}{\partial\phi}\, d\phi.$$

Here $\psi(0,t)$ can be obtained because we have assumed that the starting points of the interpolating curves remains to be the starting point of C_0. Explicitly we proceed as

$$\begin{aligned} R(0) &= e^2 \sec^2\beta, \\ \cos\theta(0,t) &= \frac{h-k_z}{R(0)} = -\sin\alpha\cos\beta, \\ \sin\theta(0,t) &= \sqrt{1-\cos^2\theta(0,t)} = \sqrt{\sin^2\beta+\cos^2\alpha\cos^2\beta}, \\ \cos\psi(0,t) &= \frac{r_0-k_x}{R(0)\sin\theta(0,t)} = \frac{\cos\alpha\cos\beta}{\sqrt{\sin^2\beta+\cos^2\alpha\cos^2\beta}}. \end{aligned}$$

5. Since the above procedures give both $\theta(\phi,t)$ and $\psi(\phi,t)$, we have obtained the interpolating curves.

5 Concavity of shoulder surfaces

Let $C=\{\mathbf{r}(\phi): 0 \leqq \phi \leqq \pi\}$ be a curve in space and consider a cone that is made of straight lines combining a fixed point $\mathbf{k}$ and every point on the curve C. A shoulder surface S is defined to be a subset of the cone. We ask when the surface S is concave.

Let $\mathbf{t}(\phi)$ be the tangent vector of unit length at the point $\mathbf{r}(\phi)$ of C. It can be computed as $\mathbf{t}(\phi)=\mathbf{r}'(\phi)/|\mathbf{r}'(\phi)|$. Then the tangent plane that passes through $\mathbf{k}$ and $\mathbf{r}(\phi)$, which we denote by $TP(\phi)$, is spanned by two vectors $\mathbf{r}(\phi)-\mathbf{k}$ and $\mathbf{t}(\phi)$. The surface S is concave if and only if any point $\mathbf{r}(\phi')$ of C that lies in the neighbourhood of the point $\mathbf{r}(\phi)$ is below or on the tangent plane $TP(\phi)$. Thus we have

$$\Delta = (\mathbf{r}(\phi')-\mathbf{k})\cdot((\mathbf{r}(\phi)-\mathbf{k})\times\mathbf{t}(\phi)) \leqq 0.$$

Hence, setting $\phi'=\phi+d\phi$, where $d\phi$ is an infinitesimal, and neglecting infinitesimals of higher order, we can deduce

$$\begin{aligned} \Delta &\approx \{\mathbf{r}(\phi)-\mathbf{k}+d\phi\,\mathbf{r}'(\phi)+\frac{d\phi^2}{2}\mathbf{r}''(\phi)\}\cdot((\mathbf{r}(\phi)-\mathbf{k})\times\mathbf{t}(\phi)) \\ &= \frac{d\phi^2}{2}\mathbf{r}''(\phi)\cdot((\mathbf{r}(\phi)-\mathbf{k})\times\mathbf{t}(\phi)) \\ &= \frac{d\phi^2}{2}\det(\mathbf{r}(\phi)-\mathbf{k},\ \mathbf{t}(\phi),\ \mathbf{r}''(\phi)), \end{aligned}$$

where the symbol $\det(\cdot,\ \cdot,\ \cdot)$ denotes the determinant of the matrix made of three column vectors. Therefore the surface S is concave if and only if

$$\det(\mathbf{r}(\phi)-\mathbf{k},\ \mathbf{r}'(\phi),\ \mathbf{r}''(\phi)) \leqq 0 \qquad (25)$$

for all ϕ.

Now we recall that

$$\begin{aligned} \mathbf{r}''(\phi) &= \frac{d}{d\phi}\mathbf{r}'(\phi) = \frac{d}{d\phi}\left(\frac{ds}{d\phi}\mathbf{t}(\phi)\right) = \frac{d^2s}{d\phi^2}\mathbf{t}+\frac{ds}{d\phi}\frac{d\mathbf{t}}{d\phi} = \frac{d^2s}{d\phi^2}\mathbf{t}+\left(\frac{ds}{d\phi}\right)^2\frac{d\mathbf{t}}{d\phi} \\ &= \frac{d^2s}{d\phi^2}\mathbf{t}+\left(\frac{ds}{d\phi}\right)^2\kappa\mathbf{n}, \end{aligned}$$

where $\mathbf{n}=\mathbf{n}(\phi)$ is the principal normal vector of unit length and $\kappa=\kappa(\phi)$ is the curvature at the point $\mathbf{r}(\phi)$. Accordingly we have

$$\det(\mathbf{r}(\phi)-\mathbf{k},\ \mathbf{t}(\phi),\ \mathbf{r}''(\phi)) = \left(\frac{ds}{d\phi}\right)^2\kappa(\phi)\det(\mathbf{r}-\mathbf{k},\ \mathbf{t},\ \mathbf{n}).$$

Thus, urther recalling that the bi-normal vector $\mathbf{b}$ of unit length is given by $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, we get

$$\det(\mathbf{r}(\phi) - \mathbf{k},\ \mathbf{t}(\phi),\ \mathbf{r}''(\phi)) = (\mathbf{r}(\phi) - \mathbf{k}) \cdot \mathbf{b}(\phi).$$

Therefore we obtain the following lemma.

Lemma 1. *The shoulder surface is concave if and only if*

$$\det(\mathbf{r}(\phi) - \mathbf{k},\ \mathbf{r}'(\phi),\ \mathbf{r}''(\phi)) \leqq 0$$

for all ϕ; in other words,

$$(\mathbf{r}(\phi) - \mathbf{k}) \cdot \mathbf{b}(\phi) \leqq 0$$

for all ϕ.

Let us consider the set

$$D = \{\mathbf{k} : \det(\mathbf{r}(\phi) - \mathbf{k},\ \mathbf{r}'(\phi),\ \mathbf{r}''(\phi)) \leqq 0 \ \text{ for all } \phi\}.$$

Since it can be written as

$$D = \bigcap_{\phi} \{\mathbf{k} : \mathbf{k} \cdot \mathbf{b}(\phi) \geqq \mathbf{r}(\phi) \cdot \mathbf{b}(\phi)\}.$$

it is an intersection of half-spaces and thus a convex set. However note that it may be empty. We shall give a sufficient condition that it becomes non-empty.

Let us express the curve C as

$$\mathbf{r}(\phi) = \begin{pmatrix} x(\phi) \\ y(\phi) \\ z(\phi) \end{pmatrix} = \begin{pmatrix} r(\phi)\cos\phi \\ r(\phi)\sin\phi \\ z(\phi) \end{pmatrix}.$$

Then, since

$$\mathbf{r}'(\phi) \times \mathbf{r}''(\phi) = \begin{pmatrix} y'z'' - z'y'' \\ z'x'' - x'z'' \\ x'y'' - y'x'' \end{pmatrix},$$

we have

$$\begin{aligned} &\det(\mathbf{r}(\phi), \mathbf{r}'(\phi), \mathbf{r}''(\phi)) \\ &= x(y'z'' - z'y'') + y(z'x'' - x'z'') + z(x'y'' - y'x'') \\ &= (xy' - yx')z'' + (yx'' - xy'')z' + (x'y'' - y'x'')z \\ &= r^2 z'' - 2rr'z' + (r^2 + 2r'^2 - rr'')z \end{aligned}$$

Similarly we have

$$\det(\mathbf{k}, \mathbf{r}'(\phi), \mathbf{r}''(\phi)) = k_x(y'z'' - z'y'') + k_y(z'x'' - x'z'') + k_z(r^2 + 2r'^2 - rr'').$$

Thus the condition stated in Lemma 1 reduces to

$$\begin{aligned} &r^2 z'' - 2rr'z' - (r^2 + 2r'^2 - rr'')(h - z) \\ &\leqq \ k_x(y'z'' - z'y'') + k_y(z'x'' - x'z'') + (k_z - h)(r^2 + 2r'^2 - rr''). \end{aligned} \tag{26}$$

Now suppose that

$$z(\phi) = h - \lambda s^n,$$

where $s = s(\phi)$ denotes the length of the curve C from $\phi = 0$ to ϕ, n is a positive exponent, and λ a positive coefficient. For a while we assume that$n \neq 1$. Then the left-hand side of (26) becomes to

$$\lambda s^{n-2}\left[nrs(2r's' - rs'') - (r^2 + 2r'^2 - rr'')s^2 - n(n-1)r^2 s'^2\right].$$

Now note that

$$s' = \sqrt{r'^2 + r^2},\ s'' = \frac{r'(r'' + r)}{\sqrt{r'^2 + r^2}}$$

and

$$\kappa = \frac{r^2 + 2r'^2 - rr''}{(r'^2 + r^2)^{\frac{3}{2}}},$$

where κ denotes the curvature of the plane curve $r = r(\phi)$. Then the left-hand side of (26) further reduces to

$$\lambda s^{n-2}(r'^2 + r^2)\left[\kappa(nrr's - \sqrt{r'^2 + r^2}s^2) - n(n-1)r^2\right]. \tag{27}$$

Note that (27) remains to hold even if $n = 1$.

On the other hand,

$$\begin{aligned} y'z'' - z'y'' &= n\lambda s^{n-2}\left[-y'(ss'' + (n-1)s'^2) + y''ss'\right] \\ &= n\lambda s^{n-2}\left[(y''s' - y's'')s - (n-1)y's'^2\right]. \end{aligned}$$

and

$$\begin{aligned} y''s' - y's'' &= (r''\sin\phi + 2r'\cos\phi - r\sin\phi)\sqrt{r'^2 + r^2} \\ &\qquad -(r'\sin\phi + r\cos\phi)\frac{r'(r'' + r)}{\sqrt{r'^2 + r^2}} \\ &= \frac{1}{\sqrt{r'^2 + r^2}}\left[\cos\phi\left\{2r'(r'^2 + r^2) - rr'(r'' + r)\right\}\right. \\ &\qquad \left.+\sin\phi\left\{(r'' - r)(r'^2 + r^2) - r'^2(r'' + r)\right\}\right] \\ &= \frac{r^2 + 2r'^2 - rr''}{\sqrt{r'^2 + r^2}}\left(r'\cos\phi - r\sin\phi\right) \\ &= (r'^2 + r^2)\,\kappa\, x'. \end{aligned}$$

Hence we have

$$y'z'' - z'y'' = n\lambda s^{n-2}(r'^2 + r^2)\left(\kappa s x' - (n-1)y'\right). \tag{28}$$

Similarly

$$\begin{aligned} x'z'' - x''z' &= n\lambda s^{n-2}\left[-x'(ss'' + (n-1)s'^2) + x''ss'\right] \\ &= n\lambda s^{n-2}\left[(x''s' - x's'')s - (n-1)x's'^2\right]. \end{aligned}$$

and

$$\begin{aligned} x''s' - x's'' &= (r''\cos\phi - 2r'\sin\phi - r\cos\phi)\sqrt{r'^2 + r^2} \\ &\qquad -(r'\cos\phi - r\sin\phi)\frac{r'(r'' + r)}{\sqrt{r'^2 + r^2}} \\ &= \frac{1}{\sqrt{r'^2 + r^2}}\left[\cos\phi\left\{(r'' - r)(r'^2 + r^2) - r'^2(r'' + r)\right\}\right. \\ &\qquad \left.+\sin\phi\{-2r'(r'^2 + r^2) + rr'(r'' + r)\}\right] \\ &= -\frac{r^2 + 2r'^2 - rr''}{\sqrt{r'^2 + r^2}}\left(r'\sin\phi + r\cos\phi\right) \\ &= -(r'^2 + r^2)\,\kappa\, y'. \end{aligned}$$

Hence we have

$$x'z'' - x''z' = -n\lambda s^{n-2}(r'^2 + r^2)\left(\kappa s y' + (n-1)x'\right). \tag{29}$$

Substitute (27), (28), and (29) into (26). Then we see that the surface S is concave if and only if

$$g_0 \leqq k_x g_1 - k_y g_2 + (k_z - h)g_3$$

for all ϕ, where we define

$$\begin{aligned} g_0 &= \kappa(nrr's - \sqrt{r'^2 + r^2}s^2) - n(n-1)r^2, \\ g_1 &= \kappa s x' - (n-1)y', \\ g_2 &= \kappa s y' + (n-1)x', \\ g_3 &= \frac{\kappa\sqrt{r'^2 + r^2}}{n\lambda s^{n-2}}. \end{aligned}$$

Hence we obtain the following proposition.

Proposition 1. *Suppose that* $z(\phi) = h - \lambda s^n$. *If* $g_0 \leqq 0, g_2 \geqq 0$ *for all* ϕ *and* $k_x = 0, k_y \leqq 0, k_z \geqq h$, *then* $\mathbf{k} = (k_x, k_y, k_z) \in D$. *In general, even if* $g_0 > 0$ *for some* ϕ, *it holds that* $\mathbf{k} = (0, 0, k_z) \in D$ *for any sufficiently large* k_z.

6 Method of constructing concave shoulder surfaces

Suppose that the space curve $\mathbf{r}(\phi)$ will be developed to a plane curve $\{(X(\phi), Y(\phi)) : 0 \leqq \phi \leqq \pi\}$ while both the fixed point $\mathbf{k}$ and the start point $\mathbf{r}(0)$ remain to stay at the original positions and the lengths $\mathbf{r}(\phi) - \mathbf{k}$ remain to be constants during development. Suppose the fixed point $\mathbf{k}$ is situated at (k_X, k_Y) on the plane that contains the developed curve. We can formulate the above manner of the development as

$$|\mathbf{r}'(\phi)| = \sqrt{X'(\phi)^2 + Y'(\phi)^2}, \tag{30}$$

$$|\mathbf{r}(\phi) - \mathbf{k}| = \sqrt{(X(\phi) - k_X)^2 + (Y(\phi) - k_Y)^2}. \tag{31}$$

Denote the above quantities, that are preserveed during the development, by $s'(\phi)$ and $l(\phi)$ respectively. By (31) we can write

$$X(\phi) = k_X + l(\phi)\cos\theta(\phi),\ Y(\phi) = k_Y + l(\phi)\sin\theta(\phi).$$

Then, from (30), it follows that

$$s'^2(\phi) = l'(\phi)^2 + l(\phi)^2\theta'(\phi)^2.$$

Consequently we get

$$\theta(\phi) - \theta(0) = \int_0^\phi \frac{\sqrt{s'^2(\phi) - l'(\phi)^2}}{l(\phi)}\, d\phi. \tag{32}$$

Note that the equation (32) is valid, because

$$l(\phi)^2(s'^2(\phi) - l'(\phi)^2) = \mathbf{r}'^2\,(\mathbf{r} - \mathbf{k})^2 - (\mathbf{r}'(\mathbf{r} - \mathbf{k}))^2 \geqq 0.$$

6.1 An example

Suppose that a base curve is a circle of radius r_0. In the below, to simplify computation, we consider a particular case that $n = 2$, although we can construct concave shoulder surfaces for other values of n. Then, since $s = r_0\phi$ and $\kappa = 1/r_0$, we have

$$g_0 = -r_0^2(\phi^2 + 2),\ g_1 = -r_0(\phi\sin\phi + \cos\phi),\ g_2 = r_0(\phi\cos\phi + \sin\phi).$$

It can be easily seen that $g_2 \leqq 0$ always, and $g_0 \leqq k_x g_1$ if $|k_x| \leqq r_0$. Therefore, when $|k_x| \leqq r_0, k_y \leqq 0, k_z \geqq h$, the surfaces S, that are constructed by the method stated in the previous section, are concave.

Now, supposing that $k_x = k_y = 0$, we shall derive an explicit expression for the developed curve. In this case we have

$$l(\phi)^2 = a^2 + 2(k_z - h)\lambda\phi^2 + \lambda^2\phi^4,$$

where $a = \sqrt{r_0^2 + (k_z - h)^2}$. Furthermore, since $z' = -2\lambda\phi$, we see that

$$s'(\phi)^2 = r_0^2 + 4\lambda^2\phi^2.$$

Hence

$$\sqrt{l(\phi)^2 s'(\phi)^2 - (l(\phi)l'(\phi))^2} = r_0\sqrt{a^2 + 2((k_z - h)\lambda + 2\lambda^2)\phi^2 + \lambda^2\phi^4}.$$

To simplyfy computation, we consider the case that the expression inside the square root is a square of a quadratic polynomial of ϕ. Thus it is necessary that

$$((k_z - h)\lambda + 2\lambda^2)^2 - a^2\lambda^2 = 0,$$

which implies that

$$\lambda = \frac{a - (k_z - h)}{2}.$$

Then

$$\sqrt{l(\phi)^2 s'(\phi)^2 - (l(\phi)l'(\phi))^2} = r_0(a + \lambda\phi^2)$$

and moreover,

$$\begin{aligned} l(\phi)^2 &= (a + \lambda\phi^2)^2 - 4\lambda^2\phi^2 \\ &= (a - 2\lambda\phi + \lambda\phi^2)(a + 2\lambda\phi + \lambda\phi^2) \end{aligned}$$

Accordingly we get

$$\theta(\phi) - \theta(0) = r_0 \int_0^\phi \frac{a + \lambda\phi^2}{(a - 2\lambda\phi + \lambda\phi^2)(a + 2\lambda\phi + \lambda\phi^2)}\, d\phi.$$

Hence

$$\begin{aligned} \theta(\phi) - \theta(0) &= \arctan\left(\sqrt{\frac{\lambda}{a-\lambda}}\,(\phi + 1)\right) + \arctan\left(\sqrt{\frac{\lambda}{a-\lambda}}\,(\phi - 1)\right) \\ &= \arctan\left(\frac{r_0\phi}{a - \lambda\phi^2}\right). \end{aligned}$$

Consequently, since $\theta(0) = -\frac{\pi}{2}$, we have

$$\begin{aligned} \cos\theta(\phi) &= \cos\left(-\frac{\pi}{2} + \arctan\left(\frac{r_0\phi}{a - \lambda\phi^2}\right)\right) \\ &= \sin\left(\arctan\left(\frac{r_0\phi}{a - \lambda\phi^2}\right)\right) \\ &= \frac{r_0\phi}{\sqrt{(a - \lambda\phi^2)^2 + (r_0\phi)^2}} \\ &= \frac{r_0\phi}{l(\phi)}. \end{aligned}$$

Similarly wee see

$$\sin\theta(\phi) = \frac{a - \lambda\phi^2}{l(\phi)}.$$

Therefore we obtain

$$X(\phi) = r_0\phi,\ Y(\phi) = \lambda\phi^2.$$

Thus the developed curve is a parabola whose equation is given by

$$Y = \frac{\sqrt{r_0^2 + (k_z - h)^2} - (k_z - h)}{r_0^2}\, X^2.$$

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