Design of RLS Wiener Fixed-Lag Smoother in Linear Discrete-Time Stochastic Systems

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(Received 28 October, 2014)

Abstract

This paper newly presents the recursive least-squares (RLS) fixed-lag smoother using the covariance information and then the RLS Wiener fixed-lag smoother in linear discrete-time wide-sense stationary stochastic systems. Here, the additional disturbance in the measurement of the signal is white noise. The signal is uncorrelated with observed noise. It is assumed that the signal process is fitted to the autoregressive (AR) model of order N. For this AR model of order N, in the proposed fixed-lag smoother, the fixed-lag smoothing estimate for the fixed lag L, $1 \leq L \leq N-1$, can be calculated. The RLS Wiener fixed-lag smoother requires information of the system matrix, the autovariance function of the state vector, the observation vector, the variance of the observation noise and the coefficients for $K(k-L,s)$ in (19). It is advantageous that the proposed RLS Wiener fixed-lag smoother shows stable and feasible estimation characteristics in comparison with the RLS Wiener fixed-lag smoother [9].

Keyword : Discrete-time stochastic systems, RLS Wiener fixed-lag smoother, covariance information, Wiener-Hopf equation

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1. Introduction

In control and communication systems, within acceptable delay, the smoothing estimate with improved estimation accuracy is preferable to the filtering estimate [1]. Also, it is pointed out that some fixed-lag smoothing algorithms in the literature, for example [2], [3], [4], are computationally unstable and therefore impractical. In the fixed-lag smoother the estimate $\hat{z}(k-L,k)$, at time $k-L$, of the signal $z(k-L)$ uses measurements until k. The fixed-lag smoothing algorithm developed in [1] utilizes the augmented state equation. In this method, the Kalman filter [5] is applicable to the augmented state equation for recursive calculation of the linear least-squares fixed-lag smoothing estimates. Moore [1] proposes various stable fixed-lag smothers for the finite-dimensional state-space model.

From the previous works in the literature, the fixed-point smoother with the following viewpoints might be useful. (1) In contrast to the filter, the fixed-lag smoother with improved estimation accuracy is advantageous. (2) Contrary to the unstable smoothers, computationally stable fixed-lag smoother is indispensable.

In [6], Nakamori et al., using the covariance information, propose the recursive least-squares (RLS) fixed-lag smoother. However, from the restriction that the covariance function of the signal is expressed in the degenerate kernel form, the smoother is not suitable for estimating the general stochastic signal processes with the autocovariance function in the semi-degenerate kernel form.

Previously, in linear discrete-time stochastic systems, the RLS Wiener fixed-point smoother [7], the RLS Wiener fixed-lag smoother [8], [9] and the RLS fixed-lag smoother [9], [10] using the covariance information are proposed. Also, the RLS fixed-lag smoother [11] using the covariance information is presented in linear continuous-time stochastic systems.

To avoid the undesirable instability of the RLS Wiener fixed-lag smoother, the aim of this paper is to design the computationally stable RLS Wiener fixed-lag smoother for the signal observed with additive white noise. The signal is uncorrelated with the observation noise. It is assumed that the signal process is fitted to the autoregressive (AR) model of order N. The key idea adopted in this paper is to express the autocovariance function $K(k-L,s)$ of the signal, which appears in (8), by (19) for the fixed lag L, $1 \le L \le N-1$. Since $K(k-L,s)$ is expressed as a linear combination of $K(k,s)$, $K(k+1,s)$, ⋯, $K(k+L,s)$, $1 \le s \le k$ (see (19)), the invariant imbedding method used in the derivation of the RLS Wiener estimators [7] can be applied to the derivations of the current fixed-lag smoothing algorithms. As a step for obtaining the RLS Wiener fixed-lag smoothing algorithm in Theorem 2, the fixed-lag smoothing algorithm using the covariance information is proposed in Theorem 1. Here, it should be noted that the instability of the fixed-lag smoother in Theorem 1 might be caused by Φ^{-k}, included in $B^T(k)$, for large values of k under the condition where more than one eigenvalue is outside of the unit circle. The RLS

Wiener fixed-lag smoother requires the information of the system matrix Φ, the autovariance function $K_x(k,k)$ of the state vector $x(k)$, the observation vector H and the coefficients $\bar{a}_{i,N}$, $i=1, \quad 2, \quad \cdots, \quad L+1,$ in (19).

In section 5, by introducing the fixed-lag smoothing error variance function, it is shown that the current RLS Wiener fixed-lag smoother is stable.

In section 6, two numerical simulation examples are demonstrated to show the stable and feasible estimation characteristics of the current RLS Wiener fixed-lag smoother. From the viewpoints of the estimation accuracy and the stability, the proposed RLS Wiener fixed-lag smoother is superior to the RLS Wiener fixed-lag smoother [9]. In the numerical simulation examples, the signal processes are fitted to the AR model of order $N=10$.

2. Linear least-squares fixed-lag smoothing problem

Let a scalar observation equation and a state-space model be given by

$$\begin{aligned} &y(k)=z(k)+v(k), \quad z(k)=Hx(k), \\ &x(k+1)=\Phi x(k)+\Gamma w(k), \end{aligned} \tag{1}$$

in linear discrete-time wide-sense stationary stochastic systems. Here, $z(k)$ is signal, H is the $1\times N$ observation vector, $x(k)$ is the state vector, $v(k)$ is white observation noise, Φ is the system matrix and $w(k)$ is the white noise input, which is uncorrelated with the observation noise. It is also assumed that the signal and the observation noise are zero mean and mutually independent. Let the autocovariance function of $v(k)$ be given by

$$E[v(k)v^T(s)]=R\delta_K(k-s), \quad R>0. \tag{2}$$

Here, $\delta_K(\cdot)$ denotes the Kronecker δ function.

Let $K(k,s)$ represent the autocovariance function of $z(k)$ and let $K(k,s)$ be expressed in the semi-degenerate kernel form of

$$K(k,s)=\begin{cases} A(k)B^T(s), & 0\le s\le k, \\ B(k)A^T(s), & 0\le k\le s. \end{cases} \tag{3}$$

Hypothesis of (3) is motivated by the fact that in many applications the covariance function of the signal to be estimated admits a semi-degenerate kernel form. Note that when the system matrix Φ in the state-space model, the observation vector H in the observation equation and the variance $K_x(k,k)$ of the state vector are available, the signal autocovariance function can be expressed as $K(k,s)=H\Phi^{k-s}K_x(s,s)H^T$, $s\le k$, and, consequently, hypothesis (3) is clearly satisfied, taking for example $A(k)=H\Phi^k$ and $B^T(s)=$

$\Phi^{-s}K_x(s,s)H^T$[9] and clearly, this factorization is not unique. Actually, processes with finite-dimensional state-space models, have covariance functions expressed in the semi-degenerate kernel form (3). Consequently, since this semi-degenerate kernel form is suitable for expressing autocovariance functions of stochastic signals in general, the fixed-lag smoothing algorithm proposed in the paper have a wide applicability.

Let a fixed-lag smoothing estimate $\hat{z}(k-L,k)$ of $z(k-L)$ be given by

$$\hat{z}(k-L,k) = \sum_{i=1}^{k} h(k,i)y(i) \tag{4}$$

as a linear transformation of the observed values $\{y(i),\quad 1 \le i \le k\}$. Here, $h(k,i)$ and L are called the impulse response function and the fixed lag.

The impulse response function, which minimizes the mean-square value of the fixed-lag smoothing error $z(k-L) - \hat{z}(k-L,k)$,

$$J = E[\|z(k-L) - \hat{z}(k-L,k)\|^2], \tag{5}$$

satisfies that the estimation error $z(k-L) - \hat{z}(k-L,k)$ is orthogonal to the observations [2]; that is

$$E[(z(k-L) - \hat{z}(k-L,k))y^T(s)] = 0,\quad 1 \le s \le k. \tag{6}$$

From (4) and (6), the Wiener-Hopf equation, which the impulse response satisfies, is given by

$$K(k-L,s) = \sum_{i=1}^{k} h(k,i)E[y(i)y^T(s)]. \tag{7}$$

Substituting (1) into (7) and using (2), we obtain

$$h(k,s)R = K(k-L,s) - \sum_{i=1}^{k} h(k,i)K(i,s). \tag{8}$$

(8) is the basic equations, which the impulse response function $h(k,s)$ satisfies, for deriving the RLS fixed-lag smoothing algorithm using the covariance information and the RLS Wiener fixed-lag smoothing algorithm of the signal in linear discrete-time stochastic systems.

3. AR model for the signal process

Let us assume that the signal process is fitted to the AR model of order N.

$$z(k) = -a_1 z(k-1) - a_2 z(k-2) - \cdots - a_N z(k-N) + w(k). \tag{9}$$

It is seen that the $1 \times N$ observation vector H in (1) and the state equation for the state vector $x(k)$ are expressed as follows:

$$H = [1 \quad 0 \quad 0 \quad \cdots \quad 0], \tag{10}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{N-1}(k+1) \\ x_N(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{N-1}(k) \\ x_N(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} w(k), \tag{11}$$

$$E[w(k)w(s)] = Q\delta_K(k-s). \tag{12}$$

In (8), the autocovariance function $K(k-L,s)$, $1 \le k-L, s \le k$, $L = 1, 2, \cdots, N-2, N-1$, can not be expressed in the form (3) of the semi-degenerate kernel.

In (9), by proceeding the time k by $N-1$, the following equation is obtained.

$$z(k+N-1) = -a_1 z(k+N-2) - a_2 z(k+N-3) - \cdots - a_N z(k-1) + w(k+N-1). \tag{13}$$

By postmultiplying $z(s)$ to (13) and taking into considerations of the relationship $K(k,s) = E[z(k)z(s)]$,

$$\begin{aligned} &K(k+N-1,s) \\ &= -a_1 K(k+N-2,s) - a_2 K(k+N-3,s) - \cdots - a_N K(k-1,s) \end{aligned} \tag{14}$$

is valid. From (14), we see that

$$\begin{aligned} &K(k-1,s) = (K(k+N-1,s) + a_1 K(k+N-2,s) + a_2 K(k+N-3,s) \\ &+ \cdots + a_{N-1} K(k,s))/(-a_N). \end{aligned} \tag{15}$$

Similarly, the equations (16)-(18) are obtained.

$$\begin{aligned} &K(k-2,s) = (K(k+N-2,s) + a_1 K(k+N-3,s) + a_2 K(k+N-4,s) \\ &+ \cdots + a_{N-1} K(k-1,s))/(-a_N) \end{aligned} \tag{16}$$

$$\begin{aligned} &K(k-3,s) = (K(k+N-3,s) + a_1 K(k+N-4,s) + a_2 K(k+N-5,s) \\ &+ \cdots + a_{N-1} K(k-2,s))/(-a_N) \end{aligned} \tag{17}$$

....

$$K(k-(N-1),s)=(K(k+1,s)+a_1K(k,s)+a_2K(k-1,s) \\ +\cdots+a_{N-1}K(k-(N-2),s))/(-a_N) \quad (18)$$

$K(k-1,s)$ is given by (15) in terms of $K(k+i-1,s)$, $1\le i\le N$. By substituting (15) into (16), $K(k-2,s)$ is expressed in terms of $K(k+i-1,s)$, $1\le i\le N$. By successive substitutions, $K(k-(N-1),s)$ is also expressed in terms of $K(k+i-1,s)$, $1\le i\le N$.
From this viewpoint, by introducing the coefficients $\bar{a}_{1,L}$, $\bar{a}_{2,L}$, $\bar{a}_{3,L}$, $\cdots$, $\bar{a}_{N,L}$, let us represent $K(k-L,s)$, $1\le L\le N-1$, as follows.

$$K(k-L,s)=\bar{a}_{1,L}K(k,s)+\bar{a}_{2,L}K(k+1,s)+\bar{a}_{3,L}K(k+2,s)+\cdots+\bar{a}_{N,L}K(k+L,s) \quad (19)$$

In the derivations of the RLS fixed-lag smoother and the RLS Wiener fixed-lag smoother, the relationship (19), with regard to the autocovariance functions, is applied to (8). Here, it should be noted, in the proposed approach, that the fixed-lag smoothing estimates $\hat{z}(k-L,k)$, $1\le L\le N-1$, are calculated for the signal process fitted to the AR model of order N.

4. RL Wiener fixed-lag smoothing algorithm

Based on the estimation problems introduced in sections 2 and 3, in Theorem 1, using the covariance information, the discrete-time RLS fixed-lag smoothing algorithm is presented in linear wide-sense stationary stochastic systems.

Theorem 1 *Let the observation equation be given by (1). Let the autocovariance function of the signal* $z(k)$ *be given by (3) in the semi-degenerate kernel form. Let the variance of white observation noise* $v(k)$ *be* R. *Let the signal process be fitted to the AR model of order* N. *Let the autocovariance function* $K(k-L,s)$ *be expressed by (19). Then the algorithm for the RLS fixed-lag smoothing estimate* $\hat{z}(k-L,k)$ *of* $z(k-L)$, $1\le L\le N-1$, *based on the observed values* $y(i)$, $1\le i\le k$, *consists of (20)-(24) in the linear discrete-time wide-sense stationary stochastic systems.*
Fixed-lag smoothing estimate of $z(k-L)$*:* $\hat{z}(k-L,k)$

$$\hat{z}(k-L,k)=(\bar{a}_{1,L}A(k)+\bar{a}_{2,L}A(k+1)+\cdots+\bar{a}_{N,L}A(k+L))e(k) \quad (20)$$

Filtering estimate of $z(k)$*:* $\hat{z}(k,k)$

$$\hat{z}(k,k)=A(k)e(k) \quad (21)$$

$$e(k) = e(k-1) + J(k,k)(y(k) - A(k)e(k-1)), \quad e(0) = 0 \tag{22}$$

$$r(k) = r(k-1) + J(k,k)(B(k) - A(k)r(k-1)), \quad r(0) = 0 \tag{23}$$

$$J(k,k) = (B^T(k) - r(k-1)A^T(k))(R + K(k,k) - A(k)r(k-1)A^T(k))^{-1} \tag{24}$$

Proof of Theorem 1 is deferred to the Appendix 1.

Similarly, based on the RLS fixed-lag smoothing algorithm in Theorem 1, using the covariance information, the RLS Wiener fixed-lag smoothing algorithm is proposed in Theorem 2.

Theorem 2 *Let the observation equation be given by (1). Let the state equation for the state vector be given by (1). Let the variance of white observation noise* $v(k)$ *be* R. *Let the signal process be fitted to the AR model of order* N. *Let the autocovariance function* $K(k-L,s)$ *be expressed by (19). Then the algorithm for the RLS Wiener fixed-lag smoothing estimate* $\hat{z}(k-L,k)$ *of* $z(k-L)$, $1 \le L \le N-1$, *based on the observed values* $y(i)$, $1 \le i \le k$, *consists of (25)-(28) in the linear discrete-time wide-sense stationary stochastic systems.*

Fixed-lag smoothing estimate of $z(k-L)$: $\hat{z}(k-L,k)$

$$\hat{z}(k-L,k) = (\bar{a}_{1,L}H + \bar{a}_{2,L}H\Phi + \cdots + \bar{a}_{N,L}H\Phi^{L)})\hat{x}(k,k) \tag{25}$$

Filtering estimate $\hat{x}(k,k)$ *of* $x(k)$

$$\hat{x}(k,k) = \Phi\hat{x}(k-1,k-1) + G(k)(y(k) - H\Phi\hat{x}(k-1,k-1)), \quad \hat{x}(0,0) = 0 \tag{26}$$

Filter gain: $G(k)$

$$G(k) = (K_x(k,k)H^T - \Phi S(k-1)\Phi^T)(R + HK_x(k,k)H^T - \Phi S(k-1)\Phi^T)^{-1} \tag{27}$$

$$S(k) = \Phi S(k-1)\Phi^T + G(k)(HK_x(k,k) - H\Phi S(k-1)\Phi^T)^{-1}, \quad S(0) = 0 \tag{28}$$

Proof of Theorem 2 is deferred to the Appendix 2.

In section 5, the algorithm for the fixed-lag smoothing error variance function is derived from the viewpoints of the estimation accuracy and the stability of the proposed RLS Wiener fixed-lag smoothing algorithm.

5. Fixed-lag smoothing error variance function

The variance function of the fixed-lag smoothing error $z(k-L) - \hat{z}(k-L,k)$ is formulated as

$$P(k-L,k)=E[(z(k-L)-\hat{z}(k-L,k))(z(k-L)-\hat{z}(k-L,k))^T]. \tag{29}$$

The variance of the filtering estimate $\hat{x}(k,k)$ equals $S(k)$. Hence, from (25), (29) might be written as

$$\begin{aligned} &P(k-L,k)=K(k-L,k-L) \\ &-(\bar{a}_{1,L}H+\bar{a}_{2,L}H\Phi+\cdots+\bar{a}_{N,L}H\Phi^{L-1)})S(k)(\bar{a}_{1,L}H+\bar{a}_{2,L}H\Phi+\cdots+\bar{a}_{N,L}H\Phi^{L-1)})^T. \end{aligned} \tag{30}$$

Since $P(k-L,k)\geq 0$ and

$$(\bar{a}_{1,L}H+\bar{a}_{2,L}H\Phi+\cdots+\bar{a}_{N,L}H\Phi^{L-1)})S(k)(\bar{a}_{1,L}H+\bar{a}_{2,L}H\Phi+\cdots+\bar{a}_{N,L}H\Phi^{L-1)})^T\geq 0,$$

it is found that

$$0\leq P(k-L,k)\leq K(k-L,k-L) \tag{31}$$

is valid. (31) shows that the variance of the fixed-lag smoothing error is upper bounded by the variance of the signal and lower bounded by the zero matrix. Hence, it is seen that the proposed RLS Wiener fixed-lag smoothing algorithm in Theorem 2 is stable. As for the estimation accuracy of the proposed RLS Wiener fixed-lag smoother with relation to the fixed lag L is examined in section 6 from the numerical aspects.

6. Numerical simulation examples

6.1. Example 1

Let the scalar observation equation be given by (1), where the observation noise $v(k)$ is the zero-mean white noise sequence.

Let us consider to estimate a vowel signal spoken by the author. Its phonetic symbol is expressed as "/i:/". The sampling frequency of the voice signal is 11.025 [kHz]. The autocovariance data of the signal is calculated in terms of 5,000 sampled signal data.

Let the signal process is fitted to the AR model of order $N=10$ in (9). The 1×10 observation vector is expressed as (10). The state equation for the state vector is given by (11). Here, the system matrix Φ is given by

$$\Phi=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_{10} & -a_9 & -a_8 & \cdots & -a_2 & -a_1 \end{bmatrix},$$

$$\begin{aligned} &a_1=-0.6135,\quad a_2=0.1635,\quad a_3=-1.2912,\quad a_4=0.4335,\quad a_5=-0.6697, \\ &a_6=0.7693,\quad a_7=0.0800,\quad a_8=0.6141,\quad a_9=-0.1770,\quad a_{10}=-0.3007. \end{aligned}$$

Also, the autovariance function $K_x(k,k)$ of the state vector $x(k)$ is calculated as

$$K_x(k,k) = \begin{bmatrix} 1.0889 & 1.0758 & 1.0719 & 1.0745 & 1.0597 & 1.0521 & 1.0421 & 1.0251 & 1.0105 & 0.9948 \\ 1.0758 & 1.0889 & 1.0758 & 1.0719 & 1.0745 & 1.0597 & 1.0521 & 1.0421 & 1.0251 & 1.0105 \\ 1.0719 & 1.0758 & 1.0889 & 1.0758 & 1.0719 & 1.0745 & 1.0597 & 1.0521 & 1.0421 & 1.0251 \\ 1.0745 & 1.0719 & 1.0758 & 1.0889 & 1.0758 & 1.0719 & 1.0745 & 1.0597 & 1.0521 & 1.0421 \\ 1.0597 & 1.0745 & 1.0719 & 1.0758 & 1.0889 & 1.0758 & 1.0719 & 1.0745 & 1.0597 & 1.0521 \\ 1.0521 & 1.0597 & 1.0745 & 1.0719 & 1.0758 & 1.0889 & 1.0758 & 1.0719 & 1.0745 & 1.0597 \\ 1.0421 & 1.0521 & 1.0597 & 1.0745 & 1.0719 & 1.0758 & 1.0889 & 1.0758 & 1.0719 & 1.0745 \\ 1.0251 & 1.0421 & 1.0521 & 1.0597 & 1.0745 & 1.0719 & 1.0758 & 1.0889 & 1.0758 & 1.0719 \\ 1.0105 & 1.0251 & 1.0421 & 1.0521 & 1.0597 & 1.0745 & 1.0719 & 1.0758 & 1.0889 & 1.0758 \\ 0.9948 & 1.0105 & 1.0251 & 1.0421 & 1.0521 & 1.0597 & 1.0745 & 1.0719 & 1.0758 & 1.0889 \end{bmatrix}.$$

By substituting the system matrix Φ, the autovariance function $K_x(k,k)$ of the state vector, the observation vector H and the parameters $\bar{a}_{i,N}$, $i = 1, \ 2, \ \cdots, \ L+1$, in (19) into the fixed-lag smoothing algorithm in Theorem 2, the RLS Wiener fixed-lag smoothing estimates $\hat{z}(k-L,k)$ of the signal $z(k-L)$ are calculated recursively.

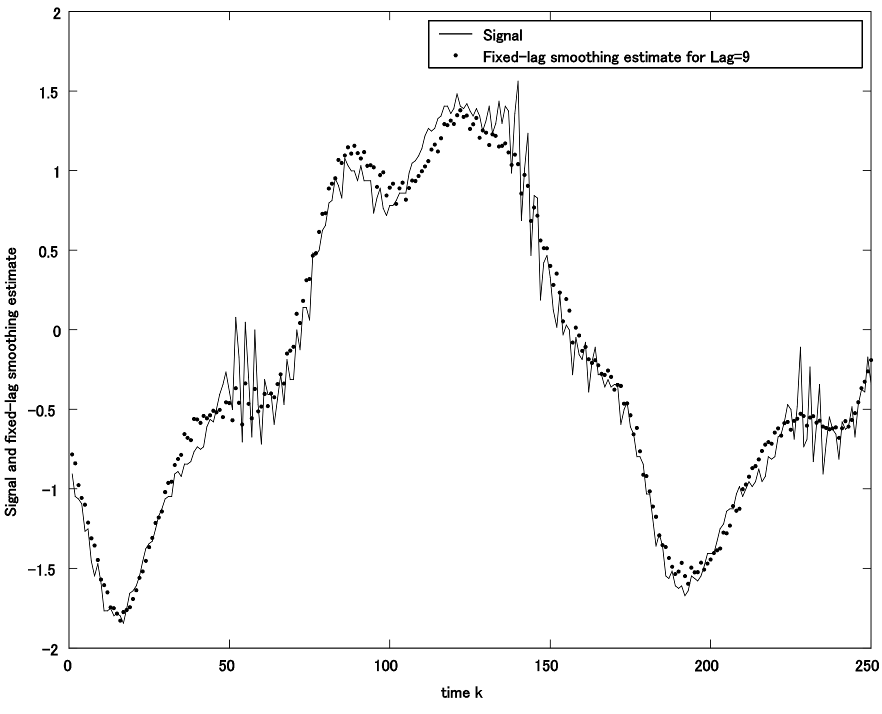

Fig. 1 Signal $z(k-9)$ and the fixed-lag smoothing estimate $\hat{z}(k-9,k)$ by the RLS Wiener fixed-lag smoother in Theorem 2 vs. k, $10 \le k \le 259$, for $SNR = 10[dB]$.

Fig.1 illustrates the signal $z(k-9)$ and the fixed-lag smoothing estimate $\hat{z}(k-9,k)$ by the RLS Wiener fixed-lag smoother in Theorem 2 vs. k, $10 \le k \le 259$, for the signal-to-noise ratio $SNR = 10$ $[dB]$. Fig.2 illustrates the mean-square values (MSVs) of the filtering and fixed-lag smoothing errors by the RLS Wiener fixed-lag smoother in Theorem 2 for $SNR = 0,\ 5,\ 10,\ 20[dB]$. For $L = 0$, the MSVs of the filtering errors are plotted. In Fig.2, particularly for $SNR = 0[dB]$ and $SNR = 5[dB]$, as the fixed lag L increases, the MSVs of the fixed-lag smoothing errors decrease. For $SNR = 10[dB]$, as the fixed lag increases, the MSV of the fixed-lag smoothing errors decreases gradually. The larger the value of the SNR becomes, the smaller the MSVs of the filtering and smoothing errors become. Under the same stochastic assumptions for the signal and the observation noise, for each value of the SNR, the fixed-lag smoothing estimates by the fixed-lag smoother [9] diverge.

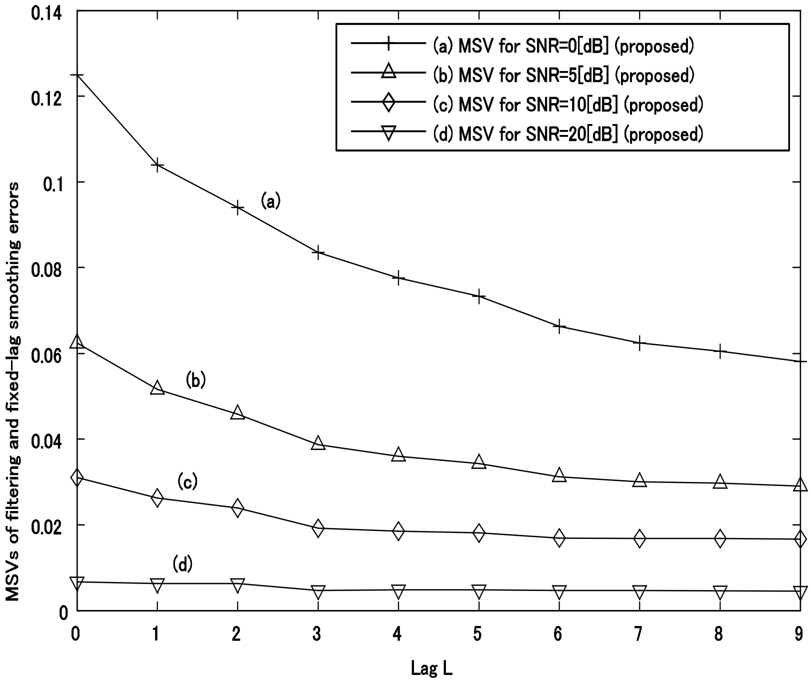

Fig. 2 Mean-square values of the filtering and fixed-lag smoothing errors by the RLS Wiener fixed-lag smoother in Theorem 2 for $SNR = 0,\ 5,\ 10,\ 20[dB]$.

Here, the MSVs of the fixed-lag smoothing and filtering errors are evaluated by $\sum_{k=L+1}^{1000+L}(z(k-L)-\hat{z}(k-L,k))^2/1000$ and $\sum_{k=1}^{1000}(z(k)-\hat{z}(k,k))^2/1000$ respectively.

6.2. Example 2

As the second example, let us adopt the sound signal ”Laughter” which is usable in the MATLAB program. The sampling frequency of the voice signal is 8.192 [kHz]. The autocovariance data of the signal is calculated in terms of 5,000 sampled signal data.

Suppose that the signal process is modeled in terms of the AR model of order 10 in (9). The 1×10 observation vector is expressed as (10). The state equation is given by (11). The parameters in the system matrix Φ are as follows:

$$a_1 = -0.9372, \quad a_2 = 0.9500, \quad a_3 = -0.1625, \quad a_4 = 0.4429, \quad a_5 = -0.1555,$$
$$a_6 = 0.3668, \quad a_7 = 0.0207, \quad a_8 = 0.3125, \quad a_9 = -0.2216, \quad a_{10} = 0.3069.$$

Also, the autovariance function $K_x(k,k)$ of the state $x(k)$ is calculated as

$$K_x(k,k)$$
$$= \begin{bmatrix} 0.1418 & 0.0800 & -0.0256 & -0.0809 & -0.0653 & -0.0152 & 0.0148 & 0.0114 & -0.0005 & -0.0002 \\ 0.0800 & 0.1418 & 0.0800 & -0.0256 & -0.0809 & -0.0653 & -0.0152 & 0.0148 & 0.0114 & -0.0005 \\ -0.0256 & 0.0800 & 0.1418 & 0.0800 & -0.0256 & -0.0809 & -0.0653 & -0.0152 & 0.0148 & 0.0114 \\ -0.0809 & -0.0256 & 0.0800 & 0.1418 & 0.0800 & -0.0256 & -0.0809 & -0.0653 & -0.0152 & 0.0148 \\ -0.0653 & -0.0809 & -0.0256 & 0.0800 & 0.1418 & 0.0800 & -0.0256 & -0.0809 & -0.0653 & -0.0152 \\ -0.0152 & -0.0653 & -0.0809 & -0.0256 & 0.0800 & 0.1418 & 0.0800 & -0.0256 & -0.0809 & -0.0653 \\ 0.0148 & -0.0152 & -0.0653 & -0.0809 & -0.0256 & 0.0800 & 0.1418 & 0.0800 & -0.0256 & -0.0809 \\ 0.0114 & 0.0148 & -0.0152 & -0.0653 & -0.0809 & -0.0256 & 0.0800 & 0.1418 & 0.0800 & -0.0256 \\ -0.0005 & 0.0114 & 0.0148 & -0.0152 & -0.0653 & -0.0809 & -0.0256 & 0.0800 & 0.1418 & 0.0800 \\ -0.0002 & -0.0005 & 0.0114 & 0.0148 & -0.0152 & -0.0653 & -0.0809 & -0.0256 & 0.0800 & 0.1418 \end{bmatrix}.$$

By substituting the system matrix Φ, the autovariance function $K_x(k,k)$ of the state vector, the bservation vector H and the coefficients $\bar{a}_{i,N}$, $i = 1, \ 2, \ \cdots, \ L+1$, in (19) into the fixed-lag smoothing algorithm in Theorem 2, the RLS Wiener fixed-lag smoothing estimates $\hat{z}(k-L,k)$ of the signal $z(k-L)$ are calculated recursively.

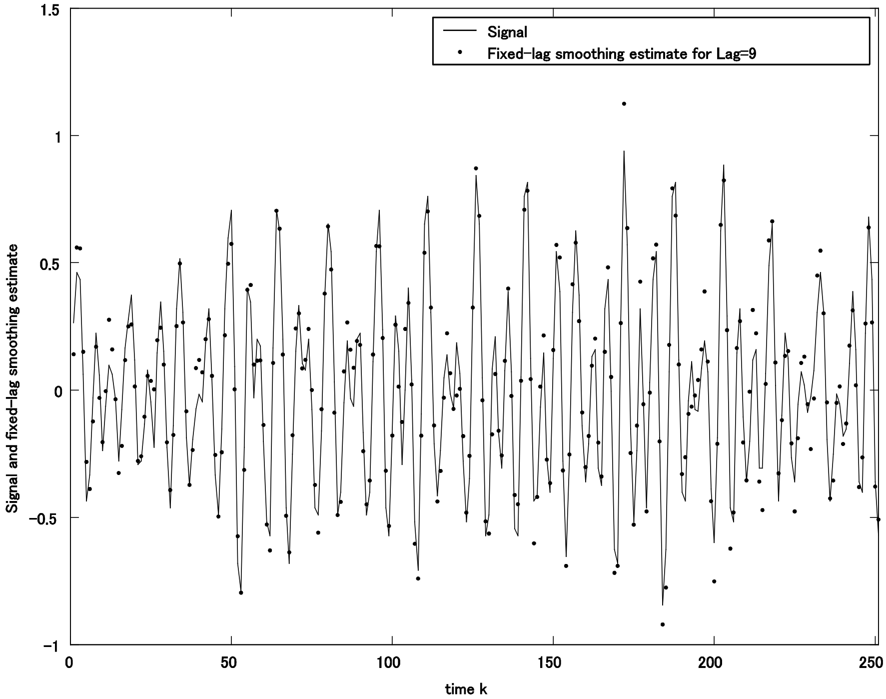

Fig. 3 Signal $z(k-9)$ and the fixed-lag smoothing estimate $\hat{z}(k-9,k)$ by the RLS Wiener fixed-lag smoother in Theorem 2 vs. k, $10 \le k \le 259$, for $SNR = 10[dB]$.

Fig.3 illustrates the signal $z(k-9)$ and the fixed-lag smoothing estimate $\hat{z}(k-9,k)$ by the RLS Wiener fixed-lag smoother in Theorem 2 vs. k, $10 \le k \le 259$, for $SNR = 10 \quad [dB]$. Fig.4 illustrates the MSVs of the filtering and fixed-lag smoothing errors by the RLS Wiener fixed-lag smoother in Theorem 2 for $SNR = 0,\ 5,\ 10,\ 20[dB]$ and the RLS Wiener fixed-lag smoother [9] for $SNR = 0,$ $5[dB]$. For $L = 0$, the MSVs of the filtering errors are plotted. In Fig.4, it is shown that the proposed RLS Wiener fixed-lag smoother shows better estimation accuracy than the fixed-lag smoother [9]. For $SNR = 10[dB]$, the MSVs of the fixed-lag smoothing errors by the fixed-lag smoother [9] are fairly larger than those by the proposed RLS Wiener fixed-lag smoother. Actually, these MSVs are 0.2704, 2.4815, 2.9695, 2.0858, 5.3556, 3.1665, 2.6676, 3.1021 and 2.7873 for $L = 0, 1, 2, \cdots, 9$.respectively. Also, for $SNR = 20[dB]$, the fixed-lag smoothing estimates by the fixed-lag smoother [9] diverge.

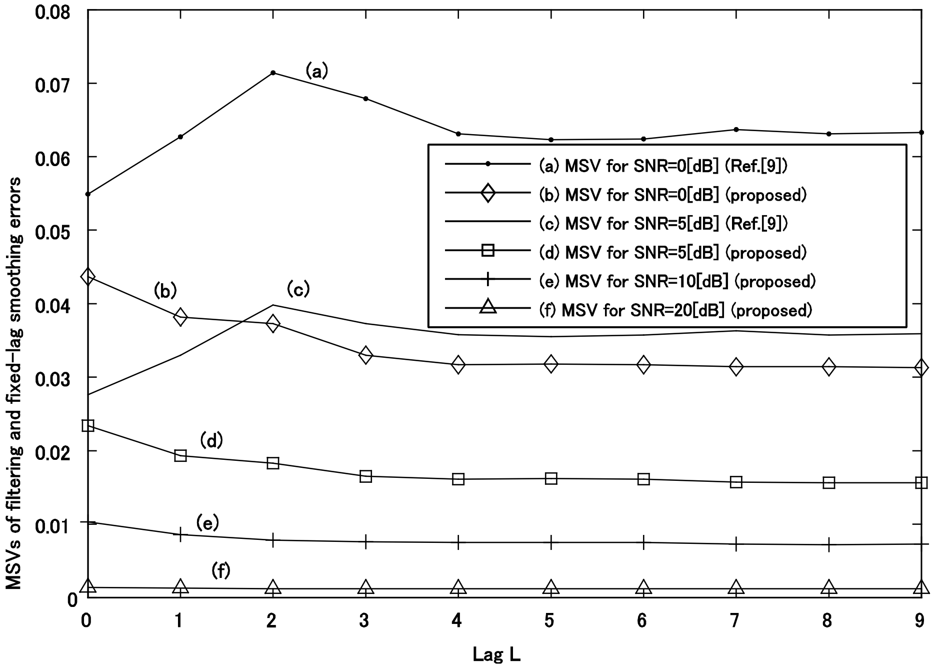

Fig. 4 Mean-square values of the filtering and fixed-lag smoothing errors by the RLS Wiener fixed-lag smoother in Theorem 2 for $SNR = 0,\ 5,\ 10,\ 20[dB]$ and the RLS Wiener fixed-lag smoother [9] for $SNR = 0,\ 5\ [dB]$.

Here, the MSVs of the fixed-lag smoothing and filtering errors are evaluated by $\sum_{k=L+1}^{N_i+L}(z(k-L)-\hat{z}(k-L,k))^2/N_i$ and $\sum_{k=1}^{N_i}(z(k)-\hat{z}(k,k))^2/N_i$, $i = 1,\ 2$, respectively, where $N_1 = 1000$ for the current fixed-lag smoother and $N_2 = 250$ for the previous fixed-lag smoother [9].

7. Conclusions

In this paper, the RLS fixed-lag smoother using the covariance information and the RLS Wiener fixed-lag smoother have been newly devised in linear discrete-time wide-sense stationary stochastic systems.

In the proposed RLS Wiener fixed-lag smoother, for the signal process fitted to the AR model of order N, the fixed-lag smoothing estimate $\hat{z}(k-L,k)$, $1 \le L \le N-1$, can be calculated. The key idea suggested in the current approach is that the autocovariance function $K(k-L,s)$ in the Wiener-Hopf equation is expressed by (19) as the linear combination of $K(k,s)$, $K(k+1,s)$, $\cdots$, $K(k+L-1,s)$ and $K(k+L,s)$.

In the numerical simulation examples, for the two kinds of stochastic signal processes, fitted to the AR model of the order $N = 10$, it has been shown that the proposed RLS Wiener fixed-lag smoother has the stable and superior estimation characteristics in comparison with the fixed-lag smoother [9].

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Appendix 1 (Proof of Theorem 1)

Let us introduce the function $J(k,s)$, which satisfies

$$J(k,s)R = B^T(s) - \sum_{i=1}^{k} J(k,i)K(i,s). \tag{A-1}$$

From (3), (8) and (19), it is seen that

$$h(k,s) = (\bar{a}_{1,L}A(k) + \bar{a}_{2,L}A(k+1) + \bar{a}_{3,L}A(k+2) + \cdots + \bar{a}_{N,L}A(k+L))J(k,s). \tag{A-2}$$

Subtracting the equation by putting $k \to k-1$ in (A.1) from (A-1), we have

$$(J(k,s)-J(k-1,s))R \\ = -J(k,k)K(k,s)-\sum_{i=1}^{k-1}(J(k,i)-J(k-1,s))K(i,s). \tag{A-3}$$

From (A-1) and (A-3), it follows that

$$J(k,s)-J(k-1,s) = -J(k,k)A(k)J(k-1,s). \tag{A-4}$$

Substituting (A-2) into (4), we have

$$\hat{z}(k-L,k) \\ = (\bar{a}_{1,L}A(k)+\bar{a}_{2,L}A(k+1)+\cdots+\bar{a}_{N,L}A(k+L))\sum_{i=1}^{k}J\,(k,i)y(i). \tag{A-5}$$

Introducing

$$e(k)=\sum_{i=1}^{k}J\,(k,i)y(i), \tag{A-6}$$

we obtain (20).

Subtracting the equation obtained by putting $k \to k-1$ in (A-6) from (A-6), we have

$$e(k)-e(k-1)=J(k,k)y(k)+\sum_{i=1}^{k-1}(J(k,i)-J(k-1,i))y(i) \\ = J(k,k)y(k)-J(k,k)A(k)\sum_{i=1}^{k-1}J\,(k-1,i)y(i). \tag{A-7}$$

Here, the initial condition on the difference equation for $e(k)$ at $k=0$ is given by $e(0)=0$ from (A-6). From (A-6) and (A-7), (22) is obtained.

Putting $s=k$ in (A-1), we have

$$J(k,k)R = B^T(k)-\sum_{i=1}^{k}J\,(k,i)K(i,k) \\ = B^T(k)-\sum_{i=1}^{k}J\,(k,i)B(i)A^T(k). \tag{A-8}$$

Introducing the function

$$r(k)=\sum_{i=1}^{k}J\,(k,i)B(i), \tag{A-9}$$

(A-8) is written as

$$J(k,k)R = B^T(k)-r(k)A^T(k). \tag{A-10}$$

Subtracting the equation obtained by putting $k \to k-1$ in (A-9) from (A-9), and using (A-4), we have

$$r(k) - r(k-1) = J(k,k)B(k) + \sum_{i=1}^{k-1}(J(k,i) - J(k-1,i))B(i)$$
$$= J(k,k)B(k) - J(k,k)A(k)r(k-1). \qquad \text{(A-11)}$$

From (A-10) and (A-11), (24) is obtained. Here, the initial condition on the difference equation for $r(k)$ at $k = 0$ is given by $r(0) = 0$ from (A-9).

(Q.E.D.)

Appendix 2 (Proof of Theorem 2)

From (20) and (21), with the relation $A(k) = H\Phi^k$, it is seen that

$$\hat{z}(k-L,k) = \bar{a}_{1,L}H\Phi^k e(k) + \bar{a}_{2,L}H\Phi\Phi^k e(k) + \cdots + \bar{a}_{N,L}H\Phi\Phi^{k+L}e(k)$$
$$= \bar{a}_{1,L}H\hat{x}(k,k) + \bar{a}_{2,L}H\Phi\hat{x}(k,k) + \cdots + \bar{a}_{N,L}H\Phi^L\hat{x}(k,k).$$

Here, the filtering estimate $\hat{x}(k,k)$ of the state vector $x(k)$ is given by $\hat{x}(k,k) = \Phi^k e(k)$. Substitution of (22) into $\hat{x}(k,k) = \Phi^k e(k)$ yields

$$\hat{x}(k,k) = \Phi\hat{x}(k-1,k-1) + \Phi^k J(k,k)\big(y(k) - H\Phi\hat{x}(k-1,k-1)\big),$$
$$\hat{x}(0,0) = 0. \qquad \text{(A-12)}$$

Here, the initial condition on the difference equation for the filtering estimate $\hat{x}(k,k)$ at $k = 0$ is given by $\hat{x}(k,k) = 0$ from $\hat{x}(k,k) = \Phi^k e(k)$ with (A-6). Putting the filter gain in (A-12) as $G(k) = \Phi^k J(k,k)$, we obtain (26). From (24), $J(k,k)$ is given by

$$J(k,k) = (B^T(k) - r(k-1)A^T(k))(R + HK_x(k,k)H^T - A(k)r(k-1)A^T(k))^{-1}.$$

Putting $S(k) = A(k)r(k)A^T(k) = \Phi^k r(k)(\Phi^T)^k$, the filter gain is expressed as

$$G(k) = (A(k)B^T(k) - A(k)r(k-1)A^T(k))(R + HK_x(k,k)H^T - A(k)r(k-1)A^T(k))^{-1}$$
$$= (K(k,k) - \Phi S(k-1)\Phi^T)(R + HK_x(k,k)H^T - \Phi S(k-1)\Phi^T)^{-1}.$$

Finally, substituting (23) into $S(k) = A(k)r(k)A^T(k)$, we obtain

$$S(k) = A(k)(r(k-1) + J(k,k)(B(k) - A(k)r(k-1)))A^T(k)$$
$$= \Phi S(k-1)\Phi^T + G(k)(HK_x(k,k) - H\Phi S(k-1)\Phi^T).$$

Here, the initial condition on the difference equation for $S(k)$ at $k = 0$ is given by $S(0) = 0$ from $S(k) = A(k)r(k)A^T(k)$ with (A-9). (Q.E.D.)