A Formula in Combinatorics Derived from the Zeta Functions of RLL(m,n) Shift Dynamical Systems

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A Formula in Combinatorics Derived from the Zeta Functions of RLL(m,n) Shift Dynamical Systems

Hiroshi Sainohira and Shoji Tsuboi

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Abstract. In this article we will compute the numbers of periodic points and the zeta functions of $\mathsf{RLL}(m,n)$ shift dynamical systems to obtain a formula in combinatorics. We will generalize this formula and prove it directly. It will turn out that the generalized formula is related to the numbers of periodic points of $\mathsf{RLL}(a_1,\cdots,a_s)$ shift dynamical systems which is a generalization of $\mathsf{RLL}(m,n)$ shift dynamical systems.

Keywords and Phrases. Number of periodic points of RLL(m,n) shift; Zeta function of Symbolic Dynamical System; A formula in Combinatorics

1 Preliminaries

Throughout this paper we will use the notation and terminology in [1]. We begin with recalling a minimum of them. Let $\mathcal{A}$ be a finite set of symbols which are called the **alphabet**. Elements of alphabet are also called **letters**, and they will typically be donoted by $a, b, c, \cdots$, or sometimes by digits like $0, 1, 2, \cdots$.

Definition 1.1 *If $\mathcal{A}$ is a finite alphabet, then the full $\mathcal{A}$-**shift** $\mathcal{A}^{\mathbb{Z}}$ is the collection of all bi-infinite sequences of symbols from $\mathcal{A}$, i.e.,*

$$\mathcal{A}^{\mathbb{Z}} = \{ x = (x_i)_{i \in \mathbb{Z}} \mid x_i \in \mathcal{A} \quad \mathtt{for} \quad \mathtt{all} \quad i \in \mathbb{Z} \}.$$

The **full r-shift** (*or simply* **r-shift**) *is the full shift over the alphabet* $\{0, 1, \cdots, r-1\}$.

When writting a specific sequence in $\mathcal{A}^{\mathbb{Z}}$, we need to specify which is the 0-th coordinate. This is conveniently done with a decimal point to separate the x_i with $i \geq 0$ from those with $i < 0$. For example,

$$x = \cdots 010.1101 \cdots$$

means that $x_{-3} = 0, x_{-2} = 1$, $x_{-1} = 0, x_0 = 1, x_1 = 1$, $x_2 = 0, x_3 = 1$, and so on.

Definition 1.2 *The* **shift map** σ *on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose i-th coordinate is $y_i = x_{i+1}$.*

A **block** (or **word**) over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. It is convenient to include the sequence of no symbols, called the **empty block** (or **empty word**) and denoted ε. The **length** of a block u is the number of symbols it contains, and is denoted by $|u|$. Thus if $u = a_1a_2\cdots a_k$ is a nonempty block, then $|u| = k$, while $|\varepsilon| = 0$. A k-**block** is simply a block of length k. The set of all k-blocks over $\mathcal{A}$ is denoted by $\mathcal{A}^k$. If x is a point in $\mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then the block of coordinates in x from position i to position j is denoted by

$$x_{[i,j]} = x_i x_{i+1} \cdots x_j$$

If $i > j$, define $x_{[i,j]}$ to be ε. If $x \in \mathcal{A}^{\mathbb{Z}}$ and w is a block over $\mathcal{A}$, it is said that w **occurs in** x if there are indices i and j so that $w = x_{[i,j]}$. Note that the empty block ε in every x, since $\varepsilon = x_{[1,0]}$. Let $\mathcal{F}$ be a collection of blocks over $\mathcal{A}$, which is thougut of as being the **forbidden blocks**. For any such $\mathcal{F}$, define $X_{\mathcal{F}}$ to be the set of subsequences in $\mathcal{A}^{\mathbb{Z}}$ which do not contain any block in $\mathcal{F}$.

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Definition 1.3 *A* **shift space** *(or simply* **shift***) over $\mathcal{A}$ is a subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks over $\mathcal{A}$. If $\mathcal{F}$ is a finite set, then we call $X = X_{\mathcal{F}}$ a* **shift of finite type***. When a shift space X is contained in a shift space Y, we say that X is a* **subshift** *of Y.*

Definition 1.4 *Let X be a subset of a full shift, and let $\mathcal{B}_n(X)$ denote the set of all n-blocks that occure in points in X. The* **language** *of X is the collection*

$$\mathcal{B}(X) = \cup_{n=0}^{\infty}\mathcal{B}_n(X)$$

We consider a map ϕ from a shift space X over $\mathcal{A}$ to a shift space Y over another alphabet $\mathfrak{A}$ described as follows. Fix integer m and n with $-m \leq n$. Let $\Phi : \mathcal{B}_{m+n+1}(X) \to \mathfrak{A}$ be a fixed map from $\mathcal{B}_{m+n+1}(X)$, the set of all $(m+n+1)$-blocks to the alphabet $\mathfrak{A}$, which is called an **(m+n+1)-block map** from allowed $(m+n+1)$-blocks in X to symbols in $\mathfrak{A}$.

Definition 1.5 *The map $\phi : X \to \mathfrak{A}^{\mathbb{Z}}$ defined by $y = \phi(x)$ with y_i given by*

$$y_i = \Phi(x_{i-m}x_{i-m+1}\cdots x_{i+n}) = \Phi(x_{[i-m,i+n]})$$

is called the **sliding block code with memory m and anticipation n induced by Φ***. We will denote the formation of ϕ from Φ by $\phi = \Phi_{\infty}^{[-m,n]}$, or simply by $\phi = \Phi_{\infty}$ if the memory and anticipation of ϕ are understood. If not specified, the memory is taken to be 0. If Y is a shift space contained in $\mathfrak{A}^{\mathbb{Z}}$ and $\phi(X) \subset Y$, we write $\phi : X \to Y$.*

Obviously, if $\phi : X \to Y$ is a sliding block code between shift spaces, then $\phi \circ \sigma_X = \sigma_Y \circ \phi$, i.e., the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\sigma_X} & X \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y & \xrightarrow{\sigma_Y} & Y, \end{array}$$

where σ_X and σ_Y are the shfts maps of X and Y, respectively. If a sliding block code $\phi : X \to Y$ has an inverse, i.e., a sliding block code $\psi : Y \to X$ such that $\psi(\phi(x)) = x$ for all $x \in X$ and $\phi(\psi(y)) = y$ for all $y \in Y$, we call ϕ **invertible**. If ϕ is invertible, its inverse ψ is unique, so we can write $\psi = \phi^{-1}$.

Definition 1.6 *A sliding block code $\phi : X \to Y$ is a* **conjugacy from X to Y***, if it is invertible. Two shift spaces X and Y are* **conjugate** *(written $X \cong Y$) if there is a conjugacy from X to Y.*

Definition 1.7 *A* **graph** *G consists of a finite set $\mathcal{V} = \mathcal{V}(G)$ of* **vertices** *(or* **states***) together with a finite set $\mathcal{E} = \mathcal{E}(G)$ of* **edges***. Each edge $e \in \mathcal{E}(G)$* **starts** *at a vertex denoted by $i(e) \in \mathcal{V}(G)$ and* **terminates** *at a vertex $t(e) \in \mathcal{V}(G)$ (which can be the same as $i(e)$). Equivalently, the edge has* **initial state** *$i(e)$ and* **terminale state** *$t(e)$.*

There may be more than one edge between a given initial state and terminal state; a set of such edges is called a set of **multiple edges**.

Definition 1.8 *A graph G is* **irreducible** *if for every ordered pair of vertices I and J there is a path in G starting at I and terminating at J.*

Definition 1.9 *Let G be a graph with vertex set $\mathcal{V}$. For vertices $I, J \in \mathcal{V}$, let A_{IJ} denote the number of edges in G with initial state I and terminal state J. Then the* **adjacency matirix** *of G is $A = [A_{IJ}]$, and its formation from G is denoted by $A = A(G)$ or $A = A_G$.*

Definition 1.10 *Let G be a graph with edge set $\mathcal{E}$ and adjacency matrix A. The edge shift X_G or X_A is the shift space over the alphabet $\mathcal{A} = \mathcal{E}$ specified by*

$$X_G = X_A = \{\, \xi = (\xi_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} \mid t(\xi_i) = i(\xi_{i+1}) \text{ for all } i \in \mathbb{Z}\}.$$

The shift map on X_G or X_A is called the **edge shift map** *and is denoted by σ_G or σ_A.*

According to the definiton, a bi-infinite sequence of edges is in X_G exactly when the terminal state of each edges is the initial state of the next one;, i.e., the sequence describes a **bi-infinite walk** or **bi-infinite trip** on G.

Definition 1.11 *A* **labeled graph** $\mathcal{G}$ *is a pair* $(G, \mathcal{L})$*, where* G *is a graph with edge set* $\mathcal{E}$*, and the* **labeling** $\mathcal{L} : \mathcal{E} \to \mathcal{A}$ *assigns to each edge* e *of* $\mathcal{E}$*, and a label* $\mathcal{L}(e)$ *from the finite alphabet* $\mathcal{A}$*. The* **underlying graph** *of* $\mathcal{G}$ *is* G*. A labeled graph is* **irreducible** *if its underlying graph is irreducible.*

Definition 1.12 *A subset* X *of a full shift is a* **sofic shift** *if* $X = X_{\mathcal{G}}$ *for some labeled graph* $\mathcal{G}$*. A* **presentation** *of a sofic shift* X *is a labeled graph* $\mathcal{G}$ *for which* $X_{\mathcal{G}} = X$*. The shift map on* $X_{\mathcal{G}}$ *is denoted by* $\sigma_{\mathcal{G}}$*.*

Definition 1.13 *A shift space* X *is* **mixing** *if, for every ordered pair* $u, v \in \mathcal{B}(X)$*, there is an* N *such that for each* $n \geq N$ *there is word* $w \in \mathcal{B}_n(X)$ *such that* $uwv \in \mathcal{B}(X)$*.*

For two points x, y of a full shift $\mathcal{A}^{\mathbb{Z}}$ over an alphabet $\mathcal{A}$, we put

$$\rho(x,y) = \begin{cases} 2^{-k} & \text{if } x \neq y \text{ and } k \text{ is maximal so that } x_{[-k,k]} = y_{[-k,k]} \\ 0 & \text{if } x = y. \end{cases} \tag{1.1}$$

This ρ satisfies the axioms of a metric, so it makes $\mathcal{A}^{\mathbb{Z}}$ a metric space. As we can see easily the topology on $\mathcal{A}^{\mathbb{Z}}$ induced by this metric is equivalent to the product topology of discrete topology on $\mathcal{A}^{\mathbb{Z}}$. The discrete topology on a finite set $\mathcal{A}$ makes it compact. Hence, due to Tikhonov's theorem, $\mathcal{A}^{\mathbb{Z}}$ with the metric ρ is also compact. In what follows we always consider this metric on $\mathcal{A}^{\mathbb{Z}}$. The shift map σ is a homeomorphism with respect to the topology indeuced by the metric ρ, so $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ can be considered as an **invertible dynamical system** (cf. [1], Definition 6.2.1). A subset X of $\mathcal{A}^{\mathbb{Z}}$ is a shift space if and only if it is shift-invariant, i.e., $\sigma(X) = X$, and compact (cf. [1], Thorem 6.1.21). From this fact it follows that any shift space can also be considered as an invertible dynamical system. When we consider a shift space with the metric ρ in (1.1) as a dynamical sysytem, we call it a **shift dynamical sysytem**. A homeomorphism ϕ from a shift (X, σ_X) to another one (Y, σ_Y) is said to be a **topological conjugacy** if $\phi \circ \sigma_X = \sigma_Y \circ \phi$. Two shift spaces are said to be **topological conjugate** if there is a topological conjugacy between them. The fact that two shifts spaces are conjugate as a shift space if and only if they are topologically conjugate as a dynamical system follows from the following thorem ([1], Theorem 6.2.9).

Theorem 1.14 (Curtis-Lyndon-Hedlund Theorem) *Suppose that* (X, σ_X) *and* (Y, σ_Y) *are shift dynamical systems, and that* $\theta : X \to Y$ *is a (not necessarily continuous) map. Then* θ *is a sliding block code if and only if it is continuous and commutes with shift maps, i.e.,* $\theta \circ \sigma_X = \sigma_Y \circ \theta$*.*

2 RLL(m,n) shifts and their characteristic polynomials

For each pair (m, n) of positive integers with $m < n$, we define $X(m, n)$ to be the set of all binary sequences for all 1's occur infinitely often in x in both directions, and there are at least m 0's, but no more than n 0's, between two 1's. $X(m, n)$ is called **(m,n) run-length limited shift**. $X(m, n)$ is the **sofic shift** associated to the following labeled graph:

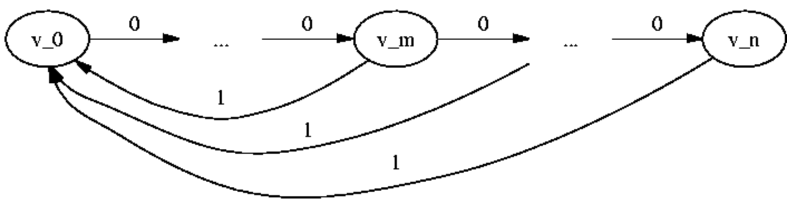

Figure 2.1: X(m,n)

We denote by $G(m, n)$ the underlying graph of the labeled graph in Figure 2.1. We name each vertex and each edge in $G(m, n)$ as follows:

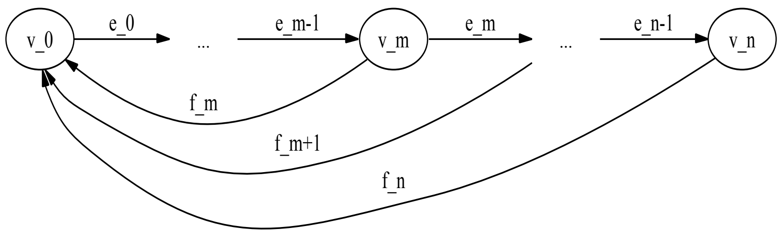

Figure 2.2: G(m,n)

Proposition 2.1 *If we put*

$$\mathcal{F} = \Big\{ 11,\ 101,\ 1001,\ \cdots,\ 1\underbrace{0\cdots0}_{m-1}1,\ \underbrace{0\cdots0}_{n+1} \Big\},$$

then $X(m,n) = X_{\mathcal{F}}$.

Proof: Let x be a point in $\mathcal{A}^{\mathbb{Z}}$ where $\mathcal{A} = \{0,1\}$. Assume $x \notin X_{\mathcal{F}}$. Then x must contain a block in $\mathcal{F}$, and so $x \notin X(m,n)$. Hence $X(m,n) \subset X_{\mathcal{F}}$. To prove the converse inclusion, we assume $x \in X_{\mathcal{F}}$. Then, since the number of consecutive 0's which occurs in x can not exceed $n+1$, 1 occurs infinitely many times in both directions. If we look at a block w in x which has consecutive 0's sandwiched by two 1's like $10\cdots01$, then the number of 0's in the block is more than $m-1$ and less than $n+1$. There exists the unique path π in the graph in Figure 2.2 such that $\mathcal{L}(\pi) = w$ where $\mathcal{L} : \mathcal{E}(G(m,n)) \to \{0,1\}$ is the labeling in Figure 2.1. The path π always ends at the vertex v_0. Hence, there exists the unique infinite path $\pi_\infty(x)$ in both direction in the graph in Figure 2.2 such that $\mathcal{L}(\pi_\infty(x)) = x$, that is, $x \in X(m,n)$. Therefore $X_{\mathcal{F}} \subset X(m,n)$. Consequently, $X_{\mathcal{F}} = X(m,n)$. ∎

Proposition 2.2 $X(m,n)$ *is conjugate to* $X_{G(m,n)}$ *as a shift space.*

Proof: We will construct a sliding block code $\phi : X(m,n) \to X_{G(m,n)}$ which gives a conjugacy between them. As shown in the proof of Proposition 2.1, for any point x of $X(m,n)$ there exists the unique infinite path $\pi_\infty(x)$ in both directions in the graph in Figure 2.2 such that $\mathcal{L}(\pi_\infty(x)) = x$. We define $\phi(x) = \pi_\infty(x)$. Obviously, ϕ commutes with the shift maps. Furthermore, $\phi(x)_0$ is a function of $x_{[-n,n]}$. To see this fact, argue as follows: If $x_0 = 1$, then all of x_{-i} are zeros for $1 \le i \le n$, or there exists k with $m \le k \le n-1$ such that $x_{-1} = \cdots = x_{-k} = 0$ and $x_{-(k+1)} = 1$. In the former case $\phi(x)_0 = f_n$, and in the latter case $\phi(x)_0 = f_k$. If $x_0 = 0$, then there exist k and ℓ with $0 \le k \le n-1$, $0 \le \ell \le n-1$ and $m+1 \le k+\ell+1 \le n$ such that $x_{-k+1} = x_{\ell+1} = 1$, $x_{-1} = \cdots = x_{-k} = 0$ $(1 \le k)$, and $x_1 = \cdots = x_\ell = 0$ $(1 \le \ell)$. In this case $\phi(x)_0 = e_k$. In all cases $\phi(x)_0$ is decided by $x_{[-n,n]}$, which means that $\phi(x)_0$ is a function of $x_{[-n,n]}$. Therefore ϕ is a sliding block code ([1], Proposition 1.5.8). The map $\psi : X_{G(m,n)} \to X(m,n)$ defined by $\psi(\pi_\infty) = \mathcal{L}(\pi_\infty)$ for $\pi_\infty \in X_{G(m,n)}$ gives the inverse of ϕ, so ϕ gives a conjugacy between $X(m,n)$ and $X_{G(m,n)}$. ∎

Proposition 2.3 $X(m,n)$ *is mixing.*

Proof: We consider $X_{G(m,n)}$ instead of $X(m,n)$ due to Proposition 2.2. In $X_{G(m,n)}$ there are periodic points of period $m+1$ and $m+2$. Since $m+1$ and $m+2$ are co-prime, $\mathrm{per}(X_{G(m,n)}) = 1$, where $\mathrm{per}(X_{G(m,n)})$ is the greatest common divisor of the periods of $X_{G(m,n)}$'s periodic points. Obviously, $X_{G(m,n)}$ is irreducible. Therefore we can conclude that $X_{G(m,n)}$ is mixing ([1], Proposition 4.5.10). ∎

In what follows we always consider $X_{G(m,n)}$ instead of $X(m,n)$. The adjacency matrix of the graph $G(m,n)$ denoted by $A(m,n)$ is as follows:

$$
A(m,n) \quad = \quad
\begin{array}{c}
\\ v_0 \\ v_1 \\ \cdot \\ \cdot \\ v_{m-1} \\ v_m \\ \cdot \\ \cdot \\ \cdot \\ v_n
\end{array}
\begin{array}{c}
\begin{array}{ccccc|cccccc}
v_0 & v_1 & \cdot & v_{m-1} & & v_m & \cdot & \cdot & \cdot & & v_n
\end{array} \\
\left(\begin{array}{ccccc|cccccc}
0 & 1 & 0 & \cdot & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
\cdot & 0 & 1 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & & \cdot & & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & & & \cdot & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\
\hline
1 & 0 & \cdot & \cdot & 0 & 0 & 1 & 0 & \cdot & \cdot & 0 \\
1 & 0 & \cdot & \cdot & 0 & 0 & 0 & 1 & 0 & \cdot & \cdot \\
\cdot & & & & \cdot & \cdot & & & \cdot & \cdot & 0 \\
\cdot & & & & \cdot & \cdot & & & \cdot & \cdot & 1 \\
1 & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0
\end{array}\right)
\end{array}
$$

We define the characteristic polynomial of RLL(m,n) shift space is to be that of the adjacency matrix $A(m,n)$ of the graph $G(m,n)$, which is calculated as follows:

$$
\begin{aligned}
\chi_{A(m,n)}(t) &= \left| tE_{n+1} - A(m,n) \right| \\
&= \left|\begin{array}{ccccc|cccccc}
t & -1 & 0 & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & t & -1 & & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & & t & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & t & -1 & 0 & \cdot & \cdot & \cdot & 0 \\
\hline
-1 & 0 & \cdot & \cdot & 0 & t & -1 & 0 & \cdot & \cdot & 0 \\
-1 & 0 & \cdot & \cdot & 0 & 0 & t & -1 & 0 & \cdot & \cdot \\
\cdot & \cdot & & & \cdot & \cdot & & & \cdot & \cdot & 0 \\
\cdot & \cdot & & & \cdot & \cdot & & & \cdot & t & -1 \\
-1 & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & t
\end{array}\right| \\
&= \left|\begin{array}{ccccc|cccccc}
0 & -1 & 0 & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
t^2 & 0 & -1 & & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & & t & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & & t & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & t & -1 & 0 & \cdot & \cdot & \cdot & 0 \\
\hline
-1 & 0 & \cdot & \cdot & 0 & t & -1 & 0 & \cdot & \cdot & 0 \\
-1 & 0 & \cdot & \cdot & 0 & 0 & t & -1 & 0 & \cdot & \cdot \\
\cdot & \cdot & & & \cdot & \cdot & & & \cdot & \cdot & 0 \\
\cdot & \cdot & & & \cdot & \cdot & & & \cdot & t & -1 \\
-1 & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & t
\end{array}\right|
\end{aligned}
$$

$$
= \left|\begin{array}{ccccc|cccccc}
0 & -1 & 0 & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & -1 & & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & \cdot & 0 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & & \cdot & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
t^m & 0 & \cdot & \cdot & 0 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\
\hline
-1 & 0 & \cdot & \cdot & 0 & t & -1 & 0 & \cdot & \cdot & 0 \\
-1 & 0 & \cdot & \cdot & 0 & 0 & t & -1 & 0 & \cdot & \cdot \\
\cdot & \cdot & & & \cdot & \cdot & & & \cdot & \cdot & 0 \\
\cdot & \cdot & & & \cdot & \cdot & & & \cdot & t & -1 \\
-1 & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & t
\end{array}\right|
$$

$$
= \left|\begin{array}{ccccc|cccccc}
0 & -1 & 0 & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & -1 & & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & & \cdot & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & 0 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\
\hline
t^{m+1}-1 & 0 & \cdot & \cdot & 0 & t & -1 & 0 & \cdot & \cdot & 0 \\
-1 & 0 & \cdot & \cdot & 0 & 0 & t & -1 & 0 & \cdot & \cdot \\
\cdot & \cdot & & & \cdot & \cdot & & & \cdot & \cdot & 0 \\
\cdot & \cdot & & & \cdot & \cdot & & & \cdot & t & -1 \\
-1 & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & t
\end{array}\right|
$$

$$
= \left|\begin{array}{ccccc|cccccc}
0 & -1 & 0 & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & -1 & & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & & \cdot & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & 0 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\
\hline
0 & 0 & \cdot & \cdot & 0 & 0 & -1 & 0 & \cdot & \cdot & 0 \\
t(t^{m+1}-1)-1 & 0 & \cdot & \cdot & 0 & 0 & t & -1 & 0 & \cdot & \cdot \\
\cdot & \cdot & & & \cdot & \cdot & & & \cdot & \cdot & 0 \\
\cdot & \cdot & & & \cdot & \cdot & & & \cdot & t & -1 \\
-1 & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & t
\end{array}\right|
$$

$$
= \left|\begin{array}{ccccc|cccccc}
0 & -1 & 0 & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & -1 & & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & & 0 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & 0 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\
\hline
0 & \cdot & \cdot & \cdot & 0 & 0 & -1 & 0 & \cdot & \cdot & 0 \\
0 & \cdot & \cdot & \cdot & 0 & 0 & & -1 & 0 & \cdot & \cdot \\
\cdot & \cdot & & & \cdot & \cdot & & & \cdot & \cdot & 0 \\
0 & \cdot & & & \cdot & \cdot & & & \cdot & \cdot & -1 \\
p(t) & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0
\end{array}\right|
$$

$$
= (-1)^{n+2} \times (-1)^n p(t) = p(t)
$$

where $p(t) = t^{n+1} - \sum_{k=0}^{n-m} t^{n-m-k}$.

3 The numbers of periodic points of RLL(m,n) shifts

First, we give the definitions of a periodic point, a cycle and the zeta function of a topological dynamical sysytem. Let (M, ϕ) be a topological dynamical system. We donote ϕ^k the composition of ϕ with itself $k > 0$ times, and call it **k-th iteration** of ϕ.

Definition 3.1 *A point $x \in M$ is* **periodic for** ϕ *if $\phi^k(x) = x$ for some $k \geq 1$, and we say that x has*

period k *under* ϕ. *If* x *is periodic, the smallest positive integer* k *for which* $\phi^k(x) = x$ *is the* **least period** *of* x. *If* $\phi(x) = x$, *then* x *is called a* **fixed point** *for* ϕ.

Definition 3.2 *For a periodic point* x_0 *for* ϕ *of the least period* $k > 0$, *we call the subset* $C = \{x_0, \phi(x_0), \cdots, \phi^{k-1}(x_0)\}$ *of finite points of* M *a* **cycle** *of period* k *for* ϕ.

Let $p_k(\phi)$ denote the number of periodic points of period k, and $q_k(\phi)$ that of the least period k. For a shift space (X, ϕ_X), we write $p_k(X)$ for $p_k(\sigma_X)$ (resp. $q_k(X)$ for $q_k(\sigma_X)$). From the Möbius inversion formula ([2]), $q_k(\phi)$ and $p_k(\phi)$ are related as follows:

$$q_k(\phi) = \sum_{\ell | k} \mu\Big(\frac{k}{\ell}\Big) p_\ell(\phi)$$

where μ is the Möbius function defined by

$$\mu(k') = \begin{cases} (-1)^r & \text{if } k' \text{ is the product of } r \text{ distinct primes,} \\ 0 & \text{if } k' \text{ contains a square factor,} \\ 1 & \text{if } k' = 1. \end{cases}$$

We are now going to calculate $p_k(X(m,n))$ the number of periodic points of period k in the $\mathrm{RLL}(m,n)$ shift $X(m,n)$ for $k \geq 1$. In what follows we write $p_k(m,n)$ for $p_k(X(m,n))$ for simplicity. We denote by $\mathbb{Z}_{\geq 0}$ the set of all non-negative integers, and by $\mathbb{Z}_{\geq 0}^{n-m+1}$ the product of $n-m+1$ copies of $\mathbb{Z}_{\geq 0}$. For a non-negative integer k, we define the subset $D_k(m,n)$ of $\mathbb{Z}_{\geq 0}^{n-m+1}$ by

$$D_k(m,n) = \Big\{ (p_{m+1}, \cdots, p_{n+1}) \in \mathbb{Z}_{\geq 0}^{n-m+1} \mid \sum_{i=1}^{n-m+1} (m+i) p_{m+i} = k \Big\}.$$

Then, the number $p_{k+1}(m,n)$ for $k \geq 0$ is given as follows:

Proposition 3.3

$$p_{k+1}(m,n) = \begin{cases} 0 & (0 \leq k \leq m-1), \\ \displaystyle\sum_{(p_{m+1},\cdots,p_{n+1}) \in D_{k+1}(m,n)} \frac{(k+1)(p_{m+1} + \cdots + p_{n+1} - 1)!}{p_{m+1}! \cdots p_{n+1}!} & (m \leq k), \end{cases} \tag{3.1}$$

where we understand that if $D_{k+1}(m,n) = \emptyset$, *then the sum is equal to zero.*

Proof: First, note that $p_{k+1}(m,n)/k+1$ is the number of cycles of period $k+1$ in $X(m,n)$, though, if a cycle has period $k'+1$ with $k+1 = \ell(k'+1)$ $(\ell \geq 1)$, then we should count this cycle as $1/\ell$ of a cycle of period $k+1$. There is one-to-one correspondence between closed paths of length $k+1$ in the graph $G(m,n)$ and cycles of peiod $k+1$ in the shift $X(m,n)$ if we do not take account of which vertex and edge closed paths start at. Therefore, it suffices to count the number of closed paths starting at the vertex v_0 of length $k+1$ in the graph $G(m,n)$. In $G(m,n)$ there are $n-m+1$ closed paths $\pi_{m+1}, \cdots, \pi_{n+1}$ starting at v_0 specified by

$$\begin{cases} \pi_{m+1} = e_0 \circ \cdots \circ e_{m-1} \circ f_m, \\ \pi_{m+2} = e_0 \circ \cdots \circ e_{m-1} \circ e_m \circ f_{m+1}, \\ \quad \vdots \\ \pi_{n+1} = e_0 \circ \cdots \circ e_{m-1} \circ e_m \circ \cdots \circ e_{n-1} \circ f_n. \end{cases}$$

We call these closed paths "fundamental paths" since every closed path π starting at v_0 in $G(m,n)$ can be expressed as

$$\pi = \pi_{i_1} \circ \pi_{i_2} \circ \cdots \circ \pi_{i_s}, \tag{3.2}$$

where each π_{i_α} $(1\leq\alpha\leq s)$ in this expression is one of $\pi_{m+1},\cdots,\pi_{n+1}$. If we denote by $p_{m+i}(\pi)$ the number of π_{m+i} which occurs in the expression in (3.2) for $1\leq i\leq n-m+1$, then $s=\sum_{i=1}^{n-m+1} p_{m+i}(\pi)$ and the length of the closed path π is $\sum_{i=1}^{n-m+1}(m+i)p_{m+i}(\pi)$. Note that, in the expression in (3.2), all of $\pi_{i_1}\circ\pi_{i_2}\circ\cdots\circ\pi_{i_s}$, $\pi_{i_2}\circ\pi_{i_3}\circ\cdots\circ\pi_{i_s}\circ\pi_{i_1}$, $\cdots$, $\pi_{i_s}\circ\pi_{i_1}\circ\cdots\circ\pi_{i_{s-1}}$ represent the same closed path in $G(m,n)$. To count the number of closed paths π's which can be expressed as in (3.2) with $p_{m+i}(\pi)=p_{m+i}$ for $1\leq i\leq n-m+1$ for a given $(p_{m+1},\cdots,p_{n+1})\in D_{k+1}(m,n)$, we think of each p_{m+i} as the number of positions where the closed path π_{m+i} occurs in the expression in (3.2). Then the number we seek is

$$\begin{aligned}&\frac{1}{(p_{m+1}+\cdots+p_{n+1})}\binom{p_{m+1}+\cdots+p_{n+1}}{p_{m+2}}\binom{p_{m+1}+\cdots+p_{n+1}}{p_{m+2}}\cdots\binom{p_n+p_{n+1}}{p_n}\\&=\frac{(p_{m+1}+\cdots+p_{n+1}-1)!}{p_{m+1}!\cdots p_{n+1}!}\end{aligned}$$

Thus we have the equality in (3.1). ∎

4 Zeta functions of RLL(m,n) shifts

First, we give the definition of zeta function of a topological dynamical system.

Definition 4.1 *Let* (M,ϕ) *be a topological dynamical system for which* $p_k(\phi)<\infty$ *for all* $k\geq 1$. *The* **zeta function** $\zeta_\phi(t)$ *of* (M,ϕ) *is defined by*

$$\zeta_\phi(t)=\exp\Big(\sum_{k=1}^{\infty}\frac{p_k(\phi)}{k}t^k\Big). \tag{4.1}$$

We know that if X is a shift of finite type, then there is an $r\times r$ non-negative integer matrix A with $X=X_A$ ([1], Theorem 2.3.2) and the zeta function of X_A is given by

$$\zeta_{\sigma_A}(t)=\frac{1}{t^r\chi_A(t^{-1})}=\frac{1}{\det(E_r-tA)}, \tag{4.2}$$

where σ_A denotes the shift map of X_A, and χ_A the characteristic polynomial of A ([1], Theorem 6.4.6). The equality in (4.2) follows from the definition of $\zeta_\phi(t)$ and the fact that

$$p_k(\sigma_A)=\operatorname{tr}A^k=\lambda_1^k+\lambda_2^k+\cdots+\lambda_r^k$$

where $\lambda_1,\cdots,\lambda_r$ the root of $\chi_A(t)$ listed with multiplicity ([1], Proposition 2.2.12).

Now, we calculate $p_{k+1}(m,n)$ for $k\geq 0$ by use of the zeta function of the shift dynamical system $X_{G(m,n)}$. Since the characteristic polynomial $\chi_{A(m,n)}(t)$ of the adjacency matrix $A(m,n)$ of the graph $G(m,n)$ is specified by

$$\chi_{A(m,n)}(t)=t^{n+1}-(t^{n-m}+t^{n-m-1}+\cdots+t+1),$$

the zeta function $\zeta_{\sigma_{A(m,n)}}(t)$ of $X_{A(m,n)}=X_{G(m,n)}$ is given by

$$\zeta_{\sigma_{A(m,n)}}(t)=\frac{1}{t^{n+1}\chi_{A(m,n)}(t^{-1})}=\frac{1}{1-(t^{m+1}+\cdots+t^{n+1})}.$$

Therefore, by (4.1),

$$\sum_{k=0}^{\infty}\frac{p_{k+1}(m,n)}{k+1}t^{k+1}=\log\zeta_{\sigma_{A(m,n)}}(t)=-\log\{1-(t^{m+1}+\cdots+t^{n+1})\}. \tag{4.3}$$

We will calculate the Taylor's expanssion of the R.H.S. of (4.3) at $t=0$. Put

$$\begin{aligned}\varphi(t) &= -\log\{1-(t^{m+1}+\cdots+t^{n+1})\},\\ g(t) &= (m+1)t^m+\cdots+(n+1)t^n, \quad \text{and}\\ f(t) &= 1-(t^{m+1}+\cdots+t^{n+1}).\end{aligned}$$

Then

$$\varphi'(t)=\frac{g(t)}{f(t)}, \quad \text{and}$$

$$\varphi^{(k+1)}(t)=\sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{1}{f}\right)^{(k-\ell)}(t)g^{(\ell)}(t) \quad (k\geq 0),$$

and so

$$\varphi^{(k+1)}(0)=\sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{1}{f}\right)^{(k-\ell)}(0)g^{(\ell)}(0) \quad (k\geq 0).$$

Since

$$g^{(\ell)}(0)=\begin{cases}(\ell+1)! & m\leq\ell\leq n,\\ 0 & \text{otherwise},\end{cases}$$

we have

$$\varphi^{(k+1)}(0)=\begin{cases}0 & (0\leq k\leq m-1),\\ \displaystyle\sum_{\ell=m}^{k}\binom{k}{\ell}\left(\frac{1}{f}\right)^{(k-\ell)}(0)(\ell+1)! & (m\leq k\leq n-1)\\ \displaystyle\sum_{\ell=m}^{n}\binom{k}{\ell}\left(\frac{1}{f}\right)^{(k-\ell)}(0)(\ell+1)! & (n\leq k).\end{cases} \tag{4.4}$$

To know $(1/f)^{(k-\ell)}(0)$, we calculate the Tayler expansion of $(1/f)(t)$ at $t=0$, which is given by

$$\begin{aligned}\left(\frac{1}{f}\right)(t) &= \sum_{\alpha_0=0}^{\infty}(t^{n+1}+\cdots+t^{m+1})^{\alpha_0}\\ &= \sum_{\alpha_0=0}^{\infty}\sum_{\alpha_1=0}^{\alpha_0}\sum_{\alpha_2=0}^{\alpha_0-\alpha_1}\cdots\sum_{\alpha_{n-m}=0}^{\alpha_0-(\alpha_1+\cdots+\alpha_{n-m-1})}\binom{\alpha_0}{\alpha_1}\binom{\alpha_0-\alpha_1}{\alpha_2}\cdots\binom{\alpha_0-(\alpha_1+\cdots+\alpha_{n-m-1})}{\alpha_{n-m}}\\ &\qquad\times t^{\sum_{i=1}^{n-m}(m+i)\alpha_i+(n+1)(\alpha_0-\sum_{i=1}^{n-m}\alpha_i)}.\\ &= \sum_{\alpha_0=0}^{\infty}\left\{\sum_{\alpha_1=0}^{\alpha_0}\sum_{\alpha_2=0}^{\alpha_0-\alpha_1}\cdots\sum_{\alpha_{n-m}=0}^{\alpha_0-(\alpha_1+\cdots+\alpha_{n-m})}\frac{\alpha_0!}{\alpha_1!\alpha_2!\cdots\alpha_{n-m}!(\alpha_0-\alpha_1-\cdots-\alpha_{n-m})!}\right.\\ &\qquad\left.\times t^{\sum_{i=1}^{n-m}(m+i)\alpha_i+(n+1)(\alpha_0-\sum_{i=1}^{n-m}\alpha_i)}\right\}.\end{aligned} \tag{4.5}$$

Hence, if we put

$$D_k(m,n)=\left\{(\alpha_1,\cdots,\alpha_{n-m},\alpha_{n-m+1})\in\mathbb{Z}_{\geq 0}^{n-m+1} \mid \alpha_1\geq 0,\cdots,\alpha_{n-m+1}\geq 0, \sum_{i=1}^{n-m+1}(m+i)\alpha_i=k\right\},$$

then (4.5) can be written as

$$\left(\frac{1}{f}\right)(t) = \sum_{k=0}^{\infty}\Big\{\sum_{(\alpha_1,\cdots,\alpha_{n-m},\alpha_{n-m+1})\in D_k(m,n)} \frac{(\alpha_1+\cdots+\alpha_{n-m+1})!}{\alpha_1!\cdots\alpha_{n-m+1}!}\Big\}t^k,$$

and so

$$\left(\frac{1}{f}\right)^{(k-\ell)}(0) = (k-\ell)! \sum_{(\alpha_1,\cdots,\alpha_{n-m},\alpha_{n-m+1}\in D_{k-\ell}(m,n)} \frac{(\alpha_1+\cdots+\alpha_{n-m+1})!}{\alpha_1!\cdots\alpha_{n-m+1}!}.$$

Hence, by (4.4),
(4.6)

$$\frac{\varphi^{(k+1)}(0)}{(k+1)!} = \begin{cases} 0 & (0\leq k\leq m-1) \\ \dfrac{1}{k+1}\displaystyle\sum_{\ell=m}^{n}\sum_{(\alpha_1,\cdots,\alpha_{n-m}\alpha_{n-m+1})\in D_{k-\ell}(m,n)} \frac{(\ell+1)(\alpha_1+\cdots+\alpha_{n-m+1})!}{\alpha_1!\cdots\alpha_{n-m+1}!} & (m\leq k). \end{cases},$$

By the definition of $\varphi(t)$,

$$\frac{\varphi^{(k+1)}(0)}{(k+1)!} = \frac{p_{k+1}(m,n)}{k+1}.$$

Therefor, by (4.6) and (3.1) we have the following equality:

(4.7)

$$\begin{aligned} &\sum_{(p_{m+1},\cdots,p_{n+1})\in D_{k+1}(m,n)} \frac{(k+1)(p_{m+1}+\cdots+p_{n+1}-1)!}{p_{m+1}!\cdots p_{n+1}!} \\ = &\sum_{\ell=m}^{n}\sum_{(p_{m+1},\cdots,p_{n+1})\in D_{k-\ell}(m,n)} \frac{(\ell+1)(p_{m+1}+\cdots+p_{n+1})!}{p_{m+1}!\cdots p_{n+1}!} \qquad (k\geq m) \end{aligned}$$

5 A formula in combinatorics

We will generalize the formula in (4.7). Let $a_1,\cdots,a_s$ $(s\geq 1)$ be positive integers with $a_1<a_2<\cdots<a_s$, and k a non-negative integer. We define $D_k(a_1,\cdots,a_s)$ by

$$D_k(a_1,\cdots,a_s) = \Big\{(p_1,\cdots,p_s)\in\mathbb{Z}^s_{\geq 0} \mid \sum_{i=1}^{s}(a_i+1)p_i = k\Big\}.$$

When $s=n-m+1$ and $a_1=m, a_2=m+1,\cdots,a_{n-m+1}=n$, $D_k(a_1,\cdots,a_s)$ is nothing but $D_k(m,n)$ defined before. With this notation, we have:

Theorem 5.1

(5.1)

$$\begin{aligned} &\sum_{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)} \frac{(k+1)(p_1+\cdots+p_s-1)!}{p_1!\cdots p_s!} \\ = &\sum_{i=1}^{s}\sum_{(p_1,\cdots,p_s)\in D_{k-a_i}(a_1,\cdots,a_s)} \frac{(a_i+1)(p+\cdots+p_s)!}{p_1!\cdots p_s!} \qquad (k\geq a_1) \end{aligned}$$

Proof: We use the induction on s.

(I) In the case $s=1$: $D_{k+1}(a_1)$ $(k \geq a_1)$ is not empty if and only if $k+1$ is a multiple of a_1+1. If this is the case, we have

$$\frac{k+1}{p_1} = a_1 + 1$$

for $p_1 \in D_{k+1}(a_1)$ and the map which assigns p_1 to $p_1 - 1$ gives a one-to-one correspondence from $D_{k+1}(a_1)$ to $D_{k-a_1}(a_1)$. Hence the equality in (5.1) holds.

(II) In the case $s \geq 2$: We assume that the equality in (5.1) holds for $s-1$. We put

$$S = \sum_{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)} \frac{(k+1)(p_1+\cdots+p_s-1)!}{p_1!\cdots p_s!}$$

For i with $1 \leq i \leq s$,

$$\begin{aligned} S = & \sum_{\substack{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)\\ p_i\geq 1}} \frac{(k+1)(p_1+\cdots+p_s-1)!}{p_1!\cdots p_s!} \\ & + \sum_{\substack{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)\\ p_i=0}} \frac{(k+1)(p_1+\cdots+\check{p}_i+\cdots+p_s-1)!}{p_1!\cdots \check{p}_i!\cdots p_s!} \end{aligned}$$

where $\check{p}_i$ and $\check{p}_i!$ denotes deleting the symbols p_i and $p_i!$. Hence, we have

$$\begin{aligned} S = \frac{1}{s}\Big\{ & \sum_{i=1}^{s} \sum_{\substack{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)\\ p_i\geq 1}} \frac{(k+1)(p_1+\cdots+p_s-1)!}{p_1!\cdots p_s!} \\ & + \sum_{i=1}^{s} \sum_{\substack{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)\\ p_i=0}} \frac{(k+1)(p_1+\cdots+\check{p}_i+\cdots+p_s-1)!}{p_1!\cdots \check{p}_i!\cdots p_s!}\Big\} \end{aligned} \tag{5.2}$$

Since each $(p_1,\cdots,p_s) \in D_{k+1}(a_1,\cdots,a_s)$ satisfies

$$\sum_{j=1}^{s}(a_j+1)p_j = k+1 \quad \text{and so,} \quad \frac{k+1}{p_i} = (a_i+1) + \sum_{\substack{j=1\\ j\neq i}}^{s}(a_j+1)\frac{p_j}{p_i}$$

for every i with $1 \leq i \leq s$, the first term in the braces in (5.2) is transformed as follows:

(5.3)
$$\begin{aligned}
&\sum_{i=1}^{s} \sum_{\substack{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)\\ p_i\geq 1}} \frac{(k+1)(p_1+\cdots+p_s-1)!}{p_1!\cdots p_s!}\\
&= \sum_{i=1}^{s} \sum_{\substack{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)\\ p_i\geq 1}} \frac{k+1}{p_i}\frac{\{(p_1+\cdots+(p_i-1)+\cdots+p_s\}!}{p_1!\cdots(p_i-1)!\cdots p_s!}\\
&= \sum_{i=1}^{s} \sum_{\substack{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)\\ p_i\geq 1}} (a_i+1)\frac{\{(p_1+\cdots+(p_i-1)+\cdots+p_s\}!}{p_1!\cdots(p_i-1)!\cdots p_s!}\\
&\quad + \sum_{i=1}^{s} \sum_{\substack{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)\\ p_i\geq 1}} \sum_{\substack{j=1\\ j\neq i}}^{s}(a_j+1)\frac{p_j}{p_i}\frac{\{(p_1+\cdots+(p_i-1)+\cdots+p_s\}!}{p_1!\cdots(p_i-1)!\cdots p_s!}\\
&= \sum_{i=1}^{s} \sum_{(p_1,\cdots,p_s)\in D_{k-a_i}(a_1,\cdots,a_s)} (a_i+1)\frac{(p_1+\cdots+p_s)!}{p_1!\cdots p_s!}\\
&\quad + \sum_{i=1}^{s} \sum_{\substack{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)\\ p_i\geq 1}} \sum_{\substack{j=1\\ j\neq i}}^{s}(a_j+1)\frac{p_j}{p_i}\frac{\{(p_1+\cdots+(p_i-1)+\cdots+p_s\}!}{p_1!\cdots(p_i-1)!\cdots p_s!}
\end{aligned}$$

On the other hand, by the induction hypothesis the second term in the braces in (5.2) is transformed as follows:

(5.4)
$$\begin{aligned}
&\sum_{i=1}^{s} \sum_{\substack{(p_1,\cdots,p_i,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)\\ p_i=0}} (k+1)\frac{(p_1+\cdots+\check{p}_i+\cdots+p_s-1)!}{p_1!\cdots\check{p}_i!\cdots p_s!}\\
&= \sum_{i=1}^{s} \sum_{\substack{j=1\\ j\neq i}}^{s} \sum_{\substack{(p_1,\cdots,p_s)\in D_{k-a_j}(a_1,\cdots,a_s)\\ p_i=0}} (a_j+1)\frac{(p_1+\cdots+\check{p}_i+\cdots+p_s)!}{p_1!\cdots\check{p}_i!\cdots p_s!}.
\end{aligned}$$

We claim that

(5.5)
$$\begin{aligned}
&\sum_{i=1}^{s} \sum_{\substack{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)\\ p_i\geq 1}} \sum_{\substack{j=1\\ j\neq i}}^{s}(a_j+1)\frac{p_j}{p_i}\frac{\{(p_1+\cdots+(p_i-1)+\cdots+p_s\}!}{p_1!\cdots(p_i-1)!\cdots p_s!}\\
&\quad + \sum_{i=1}^{s} \sum_{\substack{j=1\\ j\neq i}}^{s} \sum_{\substack{(p_1,\cdots,p_s)\in D_{k-a_j}(a_1,\cdots,a_s)\\ p_i=0}} (a_j+1)\frac{(p_1+\cdots+\check{p}_i+\cdots+p_s)!}{p_1!\cdots\check{p}_i!\cdots p_s!}\\
&= (s-1)\sum_{i=1}^{s} \sum_{(p_1,\cdots,p_s)\in D_{k-a_i}(a_1,\cdots,a_s)} (a_i+1)\frac{(p_1+\cdots+p_s)!}{p_1!\cdots p_s!}.
\end{aligned}$$

Indeed, this can be proved as follows:

$$
\begin{aligned}
&\text{The first term on the L.H.S. in (5.5)}\\
&=\sum_{i=1}^{s}\sum_{\substack{j=1\\ j\neq i}}^{s}\sum_{\substack{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)\\ p_i\geq 1}}(a_j+1)\frac{p_j}{p_i}\frac{\{(p_1+\cdots+(p_i-1)+\cdots+p_s\}!}{p_1!\cdots(p_i-1)!\cdots p_s!}\\
&=\sum_{i=1}^{s}\sum_{\substack{j=1\\ j\neq i}}^{s}\sum_{\substack{(p_1,\cdots,p_s)\in D_{k+1}(a_1,\cdots,a_s)\\ p_i\geq 1,p_j\geq 1}}(a_j+1)\frac{\{(p_1+\cdots+(p_j-1)+\cdots+p_s\}!}{p_1!\cdots(p_j-1)!\cdots p_s!}\\
&=\sum_{i=1}^{s}\sum_{\substack{j=1\\ j\neq i}}^{s}\sum_{\substack{(p_1,\cdots,p_s)\in D_{k-a_j}(a_1,\cdots,a_s)\\ p_i\geq 1}}(a_j+1)\frac{(p_1+\cdots+p_s)!}{p_1!\cdots p_s!}
\end{aligned}
$$

Hence:

$$
\begin{aligned}
&\text{The L.H.S in (5.5)}\\
&=\sum_{i=1}^{s}\sum_{\substack{j=1\\ j\neq i}}^{s}\sum_{\substack{(p_1,\cdots,p_s)\in D_{k-a_j}(a_1,\cdots,a_s)\\ p_i\geq 1}}(a_j+1)\frac{(p_1+\cdots+p_s)!}{p_1!\cdots p_s!}\\
&\quad+\sum_{i=1}^{s}\sum_{\substack{j=1\\ j\neq i}}^{s}\sum_{\substack{(p_1,\cdots,p_s)\in D_{k-a_j}(a_1,\cdots,a_s)\\ p_i=0}}(a_j+1)\frac{(p_1+\cdots+\check{p}_i+\cdots+p_s)!}{p_1!\cdots\check{p}_i!\cdots p_s!}\\
&=\sum_{i=1}^{s}\sum_{\substack{j=1\\ j\neq i}}^{s}\sum_{(p_1,\cdots,p_s)\in D_{k-a_j}(a_1,\cdots,a_s)}(a_j+1)\frac{(p_1+\cdots+p_s)!}{p_1!\cdots p_s!}\\
&=\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{(p_1,\cdots,p_s)\in D_{k-a_j}(a_1,\cdots,a_s)}(a_j+1)\frac{(p_1+\cdots+p_s)!}{p_1!\cdots p_s!}\\
&\quad-\sum_{i=1}^{s}\sum_{(p_1,\cdots,p_s)\in D_{k-a_i}(a_1,\cdots,a_s)}(a_i+1)\frac{(p_1+\cdots+p_s)!}{p_1!\cdots p_s!}\\
&=(s-1)\sum_{i=1}^{s}\sum_{(p_1,\cdots,p_s)\in D_{k-a_i}(a_1,\cdots,a_s)}(a_i+1)\frac{(p_1+\cdots+p_s)!}{p_1!\cdots p_s!}.
\end{aligned}
$$

Thus, the equality in (5.5) holds. Consequently, by (5.2), (5.3), (5.4) and (5.5), we obtain the equality in (5.1). ∎

6 $\mathsf{RLL}(a_1,\cdots,a_s)$ shifts

The number on the L.H.S. in (5.1) relates to the number of $(k+1)$ cycles of a certain shift dynamical system. For positive integers $a_1,\cdots,a_s$ $(s\geq 1)$ with $a_1<a_2<\cdots<a_s$, we define the $(a_1,\cdots,a_s)$ **run-length limited shift**, denoted by $\mathsf{RLL}(a_1,\cdots,a_s)$, or $X(a_1,\cdots,a_s)$, to be the shift space associated to the labeled graph in Figure 6.1. We denote by $G(a_1,\cdots,a_s)$ the underlying graph of the labeled one in Figure 6.1, by $A(a_1,\cdots,a_s)$ the adjacency matrix of the graph $G(a_1,\cdots,a_s)$ and by $X_{G(a_1,\cdots,a_s)}$, or $X_{A(a_1,\cdots,a_s)}$, the edge shift associated to $G(a_1,\cdots,a_s)$. As in the case of $\mathsf{RLL}(m,n)$ shift, $X(a_1,\cdots,a_s)$ is conjugate to $X_{G(a_1,\cdots,a_s)}$ as a shift space. The number on the L.H.S. in (5.1) is equal to $p_{k+1}(X(a_1,\cdots,a_s))$, the number of periodic points of period $k+1$ in $X(a_1,\cdots,a_s)$. The adjacency matrix $A(a_1,\cdots,a_s)$ is given by

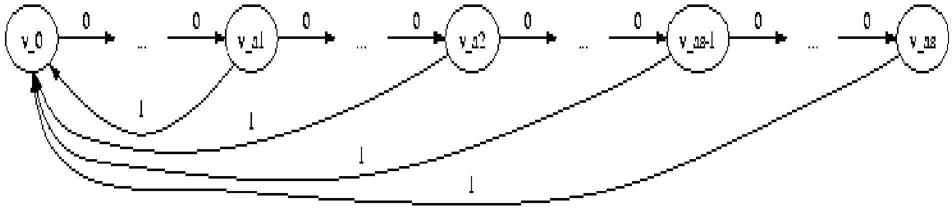

Figure 6.1: $X(a_1, \cdots, a_s)$

$$
A(a_1,\cdots,a_s) \quad = \quad
\begin{array}{c}
\\ v_0 \\ v_1 \\ \cdot \\ \cdot \\ v_{a_1-1} \\ v_{a_1} \\ v_{a_1+1} \\ \cdot \\ \cdot \\ v_{a_2-1} \\ v_{a_2} \\ v_{a_2+1} \\ \cdot \\ \cdot \\ v_{a_s-1} \\ v_{a_s}
\end{array}
\begin{array}{c}
v_0 \;\; v_1 \;\; \cdot \;\; \cdot \;\; \cdot \;\; v_{a_1-1} \;\; v_{a_1} \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; v_{a_s} \\
\left(\begin{array}{ccccccc|ccccccccccc}
0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & \cdot & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & & \cdot & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & & & \cdot & & & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\hline
1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & & & & & & \cdot & \cdot & \cdot & \cdot & & & & & & & \cdot & \cdot \\
\cdot & & & & & & \cdot & \cdot & & & & & & & & \cdot & \cdot & \cdot \\
0 & 0 & & & & & \cdot & \cdot & & & & & & & & \cdot & \cdot & 0 \\
1 & 0 & & & & & \cdot & \cdot & & & & & & & & \cdot & \cdot & 0 \\
0 & 0 & & & & & \cdot & \cdot & & & & & & & & \cdot & \cdot & 0 \\
\cdot & & & & & & \cdot & \cdot & & & & & & & & \cdot & \cdot & \cdot \\
\cdot & & & & & & \cdot & \cdot & & & & & & & & \cdot & \cdot & \cdot \\
0 & & & & & & \cdot & \cdot & & & & & & & & \cdot & \cdot & 1 \\
1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0
\end{array}\right)
\end{array}
$$

and so the characteristic polynomial $\chi_{A(a_1,\cdots,a_s)}(t)$ of $A(a_1, \cdots, a_s)$ is given by

$$\chi_{A(a_1,\cdots,a_s)}(t) = \left| tE_{a_s+1} - A(a_1, \cdots, a_s) \right| = t^{a_s+1} - (t^{a_s-a_1} + t^{a_s-a_2} + \cdots + t^{a_s-a_{s-1}} + 1).$$

Hence the zeta function $\zeta_{\sigma_{A(a_1,\cdots,a_s)}}(t)$ of $X_{A(a_1,\cdots,a_s)} = X_{G(a_1,\cdots,a_s)}$ is given by

$$\zeta_{A(a_1,\cdots,a_s)}(t) = \frac{1}{t^{a_s+1}\chi_{A(a_1,\cdots,a_s)}(t^{-1})} = \frac{1}{1-(t^{a_1+1}+\cdots+t^{a_s+1})}.$$

As in the case of $RLL(m,n)$ shift, we can prove the equality in (5.1) by use of the identity

$$\text{(6.1)} \qquad \sum_{k=0}^{\infty} \frac{p_{k+1}(X(a_1, \cdots, a_s))}{k+1} t^{k+1} = \log \zeta_{\sigma_{A(a_1,\cdots,a_s)}}(t) = -\log\{1-(t^{a_1+1}+\cdots+t^{a_s+1})\}.$$

Notice that if we want to prove the equality in (4.7) directly, not using the zeta function of $X_{G(m,n)}$, we need to generalize it to that of the form in (5.1), since we cannot apply the induction argument to the equality in (4.7) to prove it.

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Hiroshi Sainohira
Kagoshima National College of Technology
(*Part-Time Lecturer*)

Shoji Tsuboi
Department of Mathematics and Computer Science
Faculty of Science
Kagoshima University
(*E-mail* : *tsuboif@msb.biglobe.ne.jp*)