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ON THE DONNELLY-TAVARÉ-GRIFFITHS FORMULA ASSOCIATED WITH THE COALESCENT

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ABSTRACT

We evaluate the moments of the Donnelly-Tavaré-Griffiths formula appearing in the n-coalescent with mutation, which characterizes this formula. The formula is also characterized by using Waring distribution and Yule distribution. The asymptotic distributions of the related statistics are obtained as n tends to infinity.

1. INTRODUCTION

Let $\mathcal{C}_n$ denote the set of all ordered partitions of a positive integer n, that is,

$$\mathcal{C}_n = \{(c_1, ..., c_k) : 1 \leq k \leq n, c_i > 0\, (i = 1, ..., k) \text{ and } c_1 + \cdots + c_k = n\}.$$

The Donnelly-Tavaré-Griffiths formula is a probability distribution of random ordered partition $C_n = (C_{n1}, ..., C_{nk})$ on $\mathcal{C}_n$ defined by

$$(1) \qquad P(C_n = (c_1, ..., c_k)) = \frac{\alpha^k}{\alpha^{[n]}} \cdot \frac{n!}{c_k(c_k + c_{k-1}) \cdots (c_k + c_{k-1} + \cdots + c_1)},$$

where α is a positive constant, $1 \leq k \leq n$, $(c_1, ..., c_k) \in \mathcal{C}_n$ and $\alpha^{[n]} = \alpha(\alpha+1)\cdots(\alpha+n-1)$. This distribution was named Donnelly-Tavaré-Griffiths formula by Ewens (1990), based on the paper by Donnelly and Tavaré (1986) and an unpublished note by Griffiths. Joyce and Tavaré (1987) derive this distribution using the linear birth process with immigration. The distribution can be derived as the distribution of frequencies of order statistics from GEM distribution (Donnelly and Tavaré (1991)). Considering the frequencies of the sample associated with the order of appearance of the sample from an infinite random proportions, it has the distribution (1) if and only if the size-biased permutation of the infinite random proportions has the GEM distribution (Donnelly (1986) and Sibuya and Yamato (1995)). The distribution (1) can also be derived by using urn models. One is a Pólya-like urn (Hoppe(1984) or Sibuya and Yamato(1995)). The random clustering process in Sibuya (1993) is equivalent to this model. Another model is an urn with a continuum of colors (Blackwell and MacQueen (1973) or Yamato (1993)). Pitman's Chinese restaurant process gives also the distribution (1) (see, for example, Donnelly and Tavaré (1990)). Many properties of the Donnelly-Tavaré-Griffiths formula are derived, concerning with the distribution given by (1) (see, for example, Hoppe (1987) or Ewens (1990)).

For the n-coalescent with mutation, we denote by D_{n1} the number of individuals of new equivalence class with the youngest allelic type, by D_{nj} the one with the j-th youngest allelic type ($j = 1, 2, ...$) and by D_{nk} the one with the oldest allelic type. Then the random ordered partition $D_n = (D_{n1}, ..., D_{nk})$ on $\mathcal{C}_n$ has the probability distribution given by

$$(2) \qquad P(D_n = (d_1, ..., d_k)) = \frac{\alpha^k}{\alpha^{[n]}} \cdot \frac{n!}{d_1(d_1 + d_2) \cdots (d_1 + d_2 + \cdots + d_k)},$$

where $(d_1, ..., d_k) \in \mathcal{C}_n$ (see Donnelly and Tavaré(1986)). Ethier (1990) derives this distribution using a diffusion model. For $D_n = (D_{n1}, ..., D_{nk})$, its rearrangement $\bar{D}_n = (D_{nk}, ..., D_{n1})$ in a reverse order has the probability given by (1). Distinguishing between the distributions given by (1) and (2), we shall call the distribution given by (2) Donnelly-Tavaré-Griffiths II formula and abbreviate it DTG II(n, α). The purpose of this paper is to show properties of DTG II, which is different from the properties of the Donnelly-Tavaré-Griffiths formula given by (1).

In Section 2 we evaluate the moments, which characterize DTG II. In section 3, we state Waring distribution, Yule distribution and the related distributions, which appear in Section 4. In Section 4, we give the marginal distribution of D_n using a simple pure birth chain instead of the distribution (2) itself. Then we give the conditional distribution of $D_{n,r}$ given $D_{n1}, ..., D_{n,r-1}$ for $r = 1, ..., n-1$. These conditional distributions and the marginal distribution of D_{n1} are described using Waring distribution and Yule distribution, respectively. The asymptotic distributions as $n \to \infty$ of $D_{n1}, ..., D_{nr}$ with r fixed and their sum are also given.

2. MOMENTS

For any random ordered partition D_n of a positive integer n, we have the following.

Proposition 1 *Any random ordered partition D_n of a positive integer n satisfies*

(3)
$$E[\prod_{j=1}^{t}(D_{n1}+D_{n2}+\cdots+D_{nj})(D_{nj}-1)^{(r_j-1)}]$$
$$=\prod_{j=1}^{t}[(r_j-1)!(r_1+\cdots+r_j)]P(D_n=(r_1,...,r_t))$$

for $(r_1,...,r_t)\in\mathcal{C}_n$, where $x^{(r)}=x(x-1)\cdots(x-r+1)$ and $x^{(0)}=1$.

Proof. We have

$$E[\prod_{j=1}^{t}(D_{n1}+D_{n2}+\cdots+D_{nj})(D_{nj}-1)^{(r_j-1)}]$$
$$=\sum{}_1\prod_{j=1}^{t}[(d_1+\cdots+d_j)(d_j-1)^{(r_j-1)}]P(D_n=(d_1,...,d_l)),$$

where the summation $\sum_1$ is taken over all $(d_1,...,d_l)$ belonging to $\mathcal{C}_n$. It must be $d_j\geq r_j$ for $(d_j-1)^{(r_j-1)}\neq 0$. Since $d_1+\cdots+d_l=n=r_1+\cdots+r_t$, we have $l=t$ and $d_j=r_j$, $j=1,...,t$. Thus we have the relation (3). □

By Proposition 1, we get the following characterization of DTG II.

Proposition 2 *A random ordered partition D_n of a positive integer n has* DTG II(n,α) *if and only if the moments of D_n satisfies*

$$E[\prod_{j=1}^{t}(D_{n1}+D_{n2}+\cdots+D_{nj})(D_{nj}-1)^{(r_j-1)}]=\frac{\alpha^t}{\alpha^{[n]}}n!\prod_{j=1}^{t}(r_j-1)!$$

for $(r_1,...,r_t)\in\mathcal{C}_n$.

3. WARING DISTRIBUTIONS

We shall state Waring distribution, Yule distribution and their grouped distributions for the next section. The Waring distribution is the probability distribution of the random variable W taking on the values $0,1,2,...$ such that

$$P(W=x)=(c-a)\frac{a^{[x]}}{c^{[x+1]}},\ \ x=0,1,2,...,$$

where c, a are positive constants such that $c>a$. We shall denote this Waring distribution by Wa(c,a). Its mean and variance are $E(W)=a/(c-a-1)$ if $c-a>1$ and $Var(W)=a(c-a)(c-1)/[(c-a-1)^2(c-a-2)]$ if $c-a>2$, respectively. It holds that $P(W\geq x)=a^{[x]}/c^{[x]},\ \ x=0,1,2,...$ The Waring distribution with $a=1$ is Yule distribution shifted to the support $0, 1, ...$ The Yule distribution with the support $1, 2, ...$ has the probability distribution such that

$$P(Y = y) = \frac{\rho(y-1)!}{(1+\rho)^{[y]}}, \quad \rho > 0 \ \text{ and } \ y = 1, 2, ...,$$

which we shall denote by Yu(ρ). (See, for example, Johnson et al. (1992), 6.10.3 and 6.10.4.)

By grouping the events $\{W = n\}$,$\{W = n+1\}$,$\{W = n+2\}$,... with respect to W having Wa(c, a) for a non-negative integer n, we have the probability distribution given by

$$P(W = x) = (c-a)\frac{a^{[x]}}{c^{[x+1]}}, \quad x = 0, 1, 2, ..., n-1,$$

$$\frac{a^{[n]}}{c^{[n]}}, \quad x = n.$$

We shall call this distribution bounded Waring distribution and denote it by BWa$(n; c, a)$. For $n = 0$, the bounded Waring distribution degenerates to zero. Similarly, bounded Yule distribution BYu$(n; \rho)$ is defined by

$$P(Y = y) = \frac{\rho(y-1)!}{(1+\rho)^{[y]}}, \quad y = 1, 2, ..., n-1,$$

$$\frac{\rho(n-1)!}{(1+\rho)^{[n-1]}}, \quad y = n.$$

4. MARGINAL AND CONDITIONAL DISTRIBUTIONS

We consider the following urn model (Yamato (1990), Example 1.1). There are many red balls of mass one, and a single black ball of mass $\alpha > 0$. An urn contains only the black ball at the beginning. A ball is randomly chosen from the urn in proportion to its mass and replaced along with a red ball. Let Y_1 be 1. Let Y_{j+1} be equal to Y_j or $Y_j + 1$ if the color of the ball chosen at the $(j+1)$-th trial is red or black, respectively, for $j = 1, 2$,... Then we have a pure birth chain $\{Y_j; j = 1, 2, ...\}$ with states $1, 2, ...$ Its initial state is $Y_1 = 1$ and the transition probabilities are

$$P\{Y_{j+1} = y_j \mid Y_1 = y_1, ..., Y_j = y_j\} = \frac{j}{\alpha + j} \tag{4}$$

$$P\{Y_{j+1} = y_j + 1 \mid Y_1 = y_1, ..., Y_j = y_j\} = \frac{\alpha}{\alpha + j}$$

for $j = 1, 2, ...$ and all states $y_1 (= 1), y_2, ..., y_j$. The equivalent model is obtained from a Pólya-like urn (Hoppe (1984)) and sampling from Ferguson's Dirichlet process (Blackwell and MacQueen (1973) or Yamato (1993)). In this model we let $Y_1 = 1$, and Y_{j+1} be Y_j or

Y_j+1 if the $(j+1)$-th observation (or the color of the ball chosen at the $(j+1)$-th trial) is equal to any one of the previous ones or a new one, respectively, for $i = 2, 3, ...$ Pitman's Chinese restaurant process (see, for example, Donnelly and Tavaré (1990)) gives also the equivalent model, in which Y_{j+1} is equal to Y_j+1 or Y_j if the $(j+1)$-th person sits at a new empty table or not, respectively.

For the first n observations $Y_1, ..., Y_n$ of this chain $\{Y_j; j = 1, 2, ...\}$, we put

$$D_{n1} = l \text{ such that } Y_1 = \cdots = Y_l < Y_{l+1}, \quad 1 \le l \le n,$$

$$D_{ni} = l \text{ such that } Y_{D_{n,i-1}+1} = \cdots = Y_{D_{n,i-1}+l} < Y_{D_{n,i-1}+l+1}, \quad D_{n,i-1} + l \le n$$

for $i = 2, ..., n$. That is, D_{n1} is the number of observations equal to Y_1, D_{n2} is the number of observations equal to the first one which exceeds Y_1, and so on.

Proposition 3 *For the pure birth chain given by* (4)*,* $D_n = (D_{n1}, ..., D_{nk})$ *has the* DTG II(n, α)*, where* k *is the number of different observations among the first* n *observations. That is, the probability distribution of* D_n *is given by* (2).

Proof. For $(d_1, ..., d_k) \in \mathcal{C}_n$, we have

$$P(D_{n1} = d_1, D_{n2} = d_2, ..., D_{nk} = d_k)$$

$$= P(Y_1 = \cdots = Y_{d_1} < Y_{d_1+1} = \cdots = Y_{d_1+d_2} < \cdots < Y_{d_1+\cdots+d_{k-1}+1} = \cdots = Y_n).$$

Writing the right-hand side as the products of the conditional probabilities and using the transition probabilities (4), we get that D_n has the DTG II. $\square$

For any ordered partition $(d_1, ..., d_i, d_{i+1}, ..., d_k) \in \mathcal{C}_n$ such that $d_i \ge d_{i+1}$ $(i = 1, ..., k-1)$, we have $P(D_n = (d_1, ..., d_{i-1}, d_i, d_{i+1}, ..., d_k)) \le P(D_n = (d_1, ..., d_{i-1}, d_{i+1}, d_i, ..., d_k))$, because of $d_1 + \cdots + d_{i-1} + d_i \ge d_1 + \cdots + d_{i-1} + d_{i+1}$. Therefore for any permutation $(d^0_{i_1}, d^0_{i_2}, ..., d^0_{i_k})$ of a partition $(d^0_1, d^0_2, ..., d^0_k) \in \mathcal{C}_n$ such that $d^0_1 \ge d^0_2 \ge \cdots \ge d^0_k$,

$$P(D_n = (d^0_1, d^0_2, ..., d^0_k)) \le P(D_n = (d^0_{i_1}, d^0_{i_2}, ..., d^0_{i_k})) \le P(D_n = (d^0_k, d^0_{k-1}, ..., d^0_1)).$$

The marginal distribution of D_n is given by the following.

Proposition 4 (Donnelly and Tavaré (1990), Prop. 1 of Chap. 2)*Suppose that* D_n *have* DTG II(n, α). *Let* r *be a positive integer such that* $1 \le r \le n-1$. *Then,* $D_{n1}, D_{n2}, ..., D_{nr}$ *has the probability given by*

$$(5)\quad P(D_{n1} = d_1, D_{n2} = d_2, ..., D_{nr} = d_r) = \frac{\alpha^r}{(\alpha+1)^{[d(r)]}} \cdot \frac{d(r)!}{d_1(d_1+d_2)\cdots(d_1+\cdots+d_r)}$$

for $d_1, d_2, ..., d_r(= 1, 2, ..., n-1)$ *satisfying* $d(r) = d_1 + \cdots + d_r < n$. *For* $d_1, d_2, ..., d_r(= 1, 2, ..., n-1)$ *satisfying* $d_1 + \cdots + d_r = n$, *the probability* $P(D_{n1} = d_1, D_{n2} = d_2, ..., D_{nr} = d_r)$ *is given by* (2) *with* r *instead of* k.

Remark: For $r = n$, it is only possible that $D_{n1} = D_{n2} = \cdots = D_{nn} = 1$ since $(D_{n1}, D_{n2}, ..., D_{nn}) \in \mathcal{C}_n$.

Proof. In order to derive the marginal distributions of D_n, we use the pure birth chain defined by (4). For $d_1, ..., d_r(= 1, 2, ..., n-1)$ satisfying $d_1 + \cdots + d_r < n$, we have

$$P(D_{n1} = d_1, D_{n2} = d_2, ..., D_{nr} = d_r) = P(Y_1 = \cdots = Y_{d_1} < Y_{d_1+1} = \cdots$$

$$= Y_{d_1+d_2} < \cdots < Y_{d_1+\cdots+d_{r-1}+1} = \cdots = Y_{d_1+\cdots+d_r} < Y_{d_1+\cdots+d_r+1}).$$

Thus we get the relation (5) by the similar method to the proof of Proposition 4. $\square$

Especially for $r = 1$, from Proposition 4 we have for $n \geq 2$

$$P(D_{n1} = y) = \begin{cases} \dfrac{\alpha(y-1)!}{(\alpha+1)^{[y]}}, & y = 1, 2, ..., n-1 \\ \dfrac{(n-1)!}{(\alpha+1)^{[n-1]}}, & y = n. \end{cases}$$

Thus we have the following corollary.

Corollary 1 (Branson(1991), Th. 4.12 and Donnelly and Tavaré (1990), 2.23) *For* $n \geq 2$, D_{n1} *has the bounded Yule distribution* $\mathrm{BYu}(n; \alpha)$.

Proposition 5 *Suppose that* D_n *have* $\mathrm{DTG\ II}(n, \alpha)$. *Then given* $D_{n1} = d_1, ..., D_{n,r-1} = d_{r-1}$, $D_{nr} - 1$ *has the bounded Waring distribution* $\mathrm{BWa}(n - d(r-1) - 1; \alpha + d(r-1) + 1, d(r-1) + 1)$, *where* $r = 2, ..., n-1$, $d_1, ..., d_{r-1} = 1, 2, ..., n-1$ *and* $d(r-1) = d_1 + \cdots + d_{r-1} < n$.

Proof. For $x_r = 0, 1, ..., n - d(r-1) - 2$, by (5) we have

$$P(D_{nr} - 1 = x_r \mid D_{n1} = d_1, ..., D_{n,r-1} = d_{r-1}) = \frac{\alpha(d(r-1)+1)^{[x_r]}}{(\alpha + d(r-1)+1)^{[x_r+1]}}.$$

For $x_r = n - d(r-1) - 1$, by (2) and (5) we have

$$P(D_{nr} - 1 = x_r \mid D_{n1} = d_1, ..., D_{n,r-1} = d_{r-1}) = \frac{(d(r-1)+1)^{[x_r]}}{(\alpha + d(r-1)+1)^{[x_r]}}. \ \square$$

Since this conditional distribution depends on $d_1, ..., d_{r-1}$ only through their sum, we have the following.

Corollary 2 *Given* $D_{n1} + \cdots + D_{n,r-1} = d(r-1)$, $D_{nr} - 1$ *has the bounded Waring distribution* $\mathrm{BWa}(n - d(r-1) - 1; \alpha + d(r-1) + 1, d(r-1) + 1)$, *where* $r = 2, ..., n-1$ *and* $r - 1 \leq d(r-1) < n$.

By the property of Waring distribution stated in Section 2, the conditional distribution $\mathrm{BWa}(n - d_1 - 1; \alpha + d_1 + 1, d_1 + 1)$ of $D_{n2} - 1$ given $D_{n1} = d_1$ gives the following.

Corollary 3 *For* $d_1 = 1, 2, ..., [(n-1)/2]$ *and* $n > 2d_1$,

$$P(D_{n2} > D_{n1} \mid D_{n1} = d_1) = \frac{(d_1 + 1)^{[d_1]}}{(\alpha + d_1 + 1)^{[d_1]}},$$

where $[(n-1)/2]$ *is the greatest integer not greater than* $(n-1)/2$.

For example, for $n > 2$, $P(D_{n2} > D_{n1} \mid D_{n1} = 1) = 2/(\alpha + 2)$, which is greater than $1/2$ for $0 < \alpha < 2$. For $n > 4$, $P(D_{n2} > D_{n1} \mid D_{n1} = 2) = 12/(\alpha+3)(\alpha+4)$, which is greater than $1/2$ for $\alpha < (\sqrt{97} - 7)/2 \simeq 1.42$. Since the marginal distribution of D_{n1} and the conditional distribution of D_{nr} given D_{n1},..., $D_{n,r-1}$, $r = 2, ..., n$, determines the joint distribution of D_n, we have the following.

Proposition 6 *Let* $D_n = (D_{n1}, ..., D_{nk})$ *be a random ordered partition of a positive integer* n *and* α *be a positive constant. Suppose that* D_{n1} *has the bounded Yule distribution* $\mathrm{BYu}(n; \alpha)$ *and given* $D_{n1} = d_1, ..., D_{n,r-1} = d_{r-1}$, $D_{nr} - 1$ *has the bounded Waring distribution* $\mathrm{BWa}(n - d(r-1) - 1; \alpha + d(r-1) + 1, d(r-1) + 1)$, *where* $r = 2, ..., n-1$, $d_1, ..., d_{r-1} = 1, 2, ..., n-1$ *and* $d(r-1) = d_1 + \cdots + d_{r-1} < n$. *Then* D_n *has the* DTG II(n, α).

It is well-known that D_n gives the Ewens sampling formula if we neglect the order of elements of $D_n = (D_{n1}, ..., D_{nk})$ (see, for example, Donnelly and Tavaré (1986) or Sibuya and Yamato (1995)). We consider the joint distribution of $(D_{n1}, ..., D_{nr})$ neglecting their order for a positive integer $r (< n)$ fixed. Given $D_{n1} = d_1, ..., D_{nr} = d_r$, we let

$$S_{nj}^r = \text{no. of } \{i : d_i = j (i = 1, ..., r)\}, \quad j = 1, 2, ..., d(r),$$

where $d(r) = d_1 + \cdots + d_r$. It holds that $1 \cdot S_{n1}^r + 2 \cdot S_{n2}^r + \cdots + d(r) \cdot S_{n,d(r)}^r = d(r)$ and $S_{n1}^r + \cdots + S_{n,d(r)}^r = r$. That is, $(D_{n1}, ..., D_{nr})$ is the ordered partition of the positive integer $d(r)$ and $(S_{n1}^r, ..., S_{n,d(r)}^r)$ is the corresponding unordered partition of $d(r)$. $(S_{n1}^r, ..., S_{n,d(r)}^r)$ has the following joint distribution.

Proposition 7 *Suppose that* D_n *have* DTG II(n, α). *For positive integers* r *and* $d(r)$ *such that* $r \le d(r) < n$, $(s_1, ..., s_{d(r)})$ *denotes an unordered partition of* $d(r)$ *such that* $s_1, ..., s_{d(r)} \ge 0$, $s_1 + \cdots + s_{d(r)} = r$ *and* $1 \cdot s_1 + 2 \cdot s_2 + \cdots + d(r) \cdot s_{d(r)} = d(r)$. *Then we have*

$$P(S^r_{n1} = s_1, S^r_{n2} = s_2, ..., S^r_{n,d(r)} = s_{d(r)}) = \frac{\alpha^r}{\alpha^{[n]}} \cdot \frac{n!}{\prod_{i=1}^n i^{s_i} s_i!}, \quad d(r) = n, \tag{6}$$

$$\frac{\alpha^r}{(\alpha+1)^{[d(r)]}} \cdot \frac{d(r)!}{\prod_{i=1}^r i^{s_i} s_i!}, \ d(r) = r, r+1, ..., n-1.$$

Proof. We have $P(S^r_{n1} = s_1, ..., S^r_{n,d(r)} = s_{d(r)}) = \sum_2 P(D_{n1} = d_1, ..., D_{nr} = d_r)$, where the summation $\sum_2$ is taken over all distinct ordered partitions $(d_1, ...d_r)$ of $d(r)$ which give the unordered partition $(s_1, ..., s_{d(r)})$ of $d(r)$. Using the relation $\sum_2 [1/\prod_{j=1}^r (\sum_{i=1}^j d_i)] = 1/\prod_{i=1}^r i^{s_i} s_i!$(see Donnelly and Tavaré (1986) or Sibuya (1993)), by Proposition 4 we have (6) for $d(r) = r, r+1, ..., n-1$. For $d(r) = n$, we have (6) from (2) by the similar method. $\square$

We put $D(r) = D_{n1} + \cdots + D_{nr}$ for a positive integer r less than or equal to the number k of distinct partitions in D_n.

Proposition 8 *Suppose that* D_n *have* DTG II(n, α). $D(r) = D_{n1} + \cdots + D_{nr}$ *satisfies the relation given by*

$$P(D(r) = j, r < k) = \begin{bmatrix} j \\ r \end{bmatrix} \frac{\alpha^r}{(\alpha+1)^{[j]}}, \ j = r, r+1, ..., n-1,$$

$$P(D(r) = n) = \begin{bmatrix} n \\ r \end{bmatrix} \frac{\alpha^r}{\alpha^{[n]}},$$

where $[\ ]$ *denotes the unsigned Stirling number of the first kind.*

Proof. For $j(= r, r+1, ..., n-1)$, using the notations in Proposition 7 we have $P(D(r) = j, r < k) = \sum_3 P(S^r_{n1} = s_1, ..., S^r_{nj} = s_j)$, where the summation $\sum_3$ is taken over all unordered partitions $(s_1, ...s_j)$ of j such that $s_1 + \cdots + s_j = r$ and $1 \cdot s_1 + 2 \cdot s_2 + \cdots + j \cdot s_j = j$. Using the representation of the unsigned Stirling number of the first kind $\begin{bmatrix} j \\ r \end{bmatrix}$ $= \sum_3 j!/\prod_{i=1}^j i^{s_i} s_i!$ (see, for example, Riordan(1968)), from the second relation of (6) we have $P(D(r) = j, r \le k) = \begin{bmatrix} j \\ r \end{bmatrix} \alpha^r/(\alpha+1)^{[j]}$. By the similar method and from the first relation of (6) we have $P(D(r) = n) = \begin{bmatrix} n \\ r \end{bmatrix} \alpha^r/\alpha^{[n]}$. $\square$

From Propositions 4 and 8, we have the conditional distribution of $D_{n1}, ..., D_{nr}$ given $D(r)$ and $r < k$. In addition, from Proposition 4 and the probability $P(D(r) = n)$ of Proposition 8 we have the conditional distribution of $D_{n1}, ..., D_{nk}$ given k, since $D(r) = n$ means $r = k$.

Corollary 4 *For $j = r, r+1, ..., n-1$ $(r < n)$ and positive integers $d_1, ..., d_r$ satisfying $d_1 + \cdots + d_r = j$, we have*

$$P(D_{n1} = d_1, ..., D_{nr} = d_r \mid D(r) = j, r < k) = \begin{bmatrix} j \\ r \end{bmatrix}^{-1} \frac{j!}{d_1(d_1 + d_2) \cdots (d_1 + \cdots + d_r)}.$$

Furthermore, we have

$$(7)\ P(D_n = (d_1, ..., d_k) \mid k) = \begin{bmatrix} n \\ k \end{bmatrix}^{-1} \frac{n!}{d_1(d_1 + d_2) \cdots (d_1 + \cdots + d_k)}, \quad (d_1, ..., d_k) \in \mathcal{C}_n.$$

The relation (7) may be also obtained by another approach. If we neglect the order of elements of $D_n = (D_{n1}, ..., D_{nk})$, D_n gives the Ewens sampling formula as stated following Proposition 6. Thus the number k of distinct partitions in D_n has the distribution $P(K = k) = \alpha^k \begin{bmatrix} n \\ k \end{bmatrix} / \alpha^{[n]}$, $k = 1, 2, ..., n$ (Ewens (1972)). Dividing the equation (2) by this probability $P(K = k)$, we have also the relation (7).

For the asymptotic distributions as $n \to \infty$, by Propositions 4, 5, 7 and 8 we have the following.

Proposition 9 *Suppose that D_n have* DTG II(n, α). *Let r be a positive integer. Then*
(i) *D_{n1} has the Yule distribution* Yu(α) *asymptotically as $n \to \infty$.*
(ii) *$(D_{n1}, ..., D_{nr})$ has the asymptotic distribution given by*

$$P(D_{n1} = d_1, ..., D_{nr} = d_r)$$
$$= \frac{\alpha^r}{(\alpha+1)^{[d_1 + \cdots + d_r]}} \cdot \frac{(d_1 + \cdots + d_r)!}{d_1(d_1 + d_2) \cdots (d_1 + \cdots + d_r)}, \quad d_1, ..., d_r = 1, 2, ...$$

(iii) *Given $D_{n1} = d_1, ..., D_{n,r-1} = d_{r-1}$, $D_{nr} - 1$ has the Waring distribution* Wa$(\alpha + d(r-1) + 1, d(r-1) + 1)$ *asymptotically, where $d(r-1) = d_1 + \cdots + d_{r-1}$.*
(iv) *For $d(r) = r, r+1, ...,$ $(S^r_{n1}, S^r_{n2}, ..., S^r_{n,d(r)})$ has the asymptotic probability given by*

$$P(S^r_{n1} = s_1, S^r_{n2} = s_2, ..., S^r_{n,d(r)} = s_{d(r)}) = \frac{\alpha^r}{(\alpha+1)^{[d(r)]}} \cdot \frac{d(r)!}{\prod_{i=1}^r i^{s_i} s_i!},$$

where $(s_1, ..., s_{d(r)})$ denotes an unordered partition of $d(r)$ such that $s_1, ..., s_{d(r)} \geq 0$, $s_1 + \cdots + s_{d(r)} = r$ and $1 \cdot s_1 + 2 \cdot s_2 + \cdots + d(r) \cdot s_{d(r)} = d(r)$.
(v) *$D(r) = D_{n1} + \cdots + D_{nr}$ has the asymptotic distribution given by*

$$P(D(r) = j) = \begin{bmatrix} j \\ r \end{bmatrix} \frac{\alpha^r}{(\alpha+1)^{[j]}}, \quad j = r, r+1, ...$$

Though this probability $P(D(r) = j)$ is easily derived from (iv) of Proposition 9, we can also obtain it by applying to Proposition 8 the fact that the number k of distnct partitions in D_n diverges as $n \to \infty$ with probability one (Korwar and Hollander(1973), Cor. 2.2). The asymptotic distribution of $D(r)+1$ is the $Str1\,W(r+1,\alpha)$ distribution by Sibuya(1988).

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BIBLIOGRAPHY

Blackwell, D. and MacQueen, J. (1973). "Ferguson distribution via Pólya urn schemes," *Ann. Statist.*, **1**, 353–355.

Branson, D. (1991). "An urn model and the coalescent in neutral infinite-alleles genetic processes," In *Selected proceedings of the Sheffield symposium on applied probability*, eds. I. V. Basawa and R. L. Taylor, IMS, 174–192.

Donnelly, P. (1986). "Partition structures, Polya urns, the Ewens sampling formula, and the ages of alles," *Theor. Popul. Biol.*, **30**, 271–288.

Donnelly, P. and Joyce, P. (1991). "Consistent ordered sampling distribution: Characterization and convergence," *Adv. Appl. Prob.*, **23**, 229–258.

Donnelly, P. and Tavaré, S. (1986). "The ages of alleles and a coalescent," *Adv. Appl. Prob.*, **18**, 1–19.

Donnelly, P. and Tavaré, S. (1990). *Chinese restaurant process*, Chap. 2 of unpublished lecture notes.

Ethier, S.N. (1990). "The infinitely-many-neutral-alleles-diffusion model with ages," *Adv. Appl.Prob.*, **22**, 1–24

Ewens, W.J. (1972). "The sampling theory of selectively neutral alleles," *Theor. Popul. Biol.*, **3**, 87–112.

Ewens, W.J. (1990). "Population genetics theory – the past and the future," In *Mathematical and Statistical Developments of Evolutionary Theory*, ed. S. Lessard, Kluwer Academic Publishers, Dordrecht, 177– 227.

Hoppe, F.M. (1984). "Pólya-like urns and the Ewens sampling formula,*J. Math. Biol.*, **20**, 91–99.

Hoppe, F.M. (1987). "The sampling theory of neutral alleles and an urn model in population genetics," *J. Math. Biol.* **25**, 123–159.

Johnson, N.L., Kotz, S. and Kemp, A.W. (1992). *Univariate discrete distributions*, Wiley, New York.

Korwar R.M. and Hollander, M. (1973). "Contribution to the theory of Dirichlet processes," Ann. Probab., **1**, 705–711.

Riordan, J. (1968). *Combinatorial identities.* John Wiley & Sons, New York.

Joyce, P. and Tavaré, S. (1987). "Cycles, permutation and the structure of the Yule process with immigration," *Stoch. Processes Appl.*, **25**, 309–314.

Sibuya, M. (1988). "Log-concavity of Stirling numbers and unimodality of Stirling distributions," *Ann. Inst. Statist. Math.*, **40**, 693–714.

Sibuya, M. (1993). "A random clustering process," *Ann. Inst. Statist. Math.*, **45**, 459–465.

Sibuya, M. and Yamato, H.(1995). "Ordered and unordered random partitions of an integer and the GEM distribution," *Statist. Prob. Letters*, **25**, 177-183.

Yamato, H.(1990). "A probabilistic approach to Stirling numbers of the first kind," *Commun. Statist.-Theory Meth*, **19**, 3915–3923.

Yamato, H.(1993). "A Pólya urn model with a continuum of colors," *Ann. Inst. Statist. Math.*,**45**, 453–458.