Rep. Fac. Sci., Kagoshima Univ.,
No. 38, pp. 1~81 (2005)

INFINITESIMAL MIXED TORELLI PROBLEM FOR ALGEBRAIC SURFACES WITH ORDINARY SINGULARITIES, II*

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Dedicated to the memory of N. Sasakura

ABSTRACT. In Part I of this paper ([14]), we have formulated the *infinitesimal mixed Torelli problem* for an algebraic surface with ordinary singularities S. In this Part II, we formulate the *cohomological infinitesimal mixed Torelli problem* for such S, which enable us to deal with the problem more easily. We give some cohomological sufficient conditions under which the infinitesimal mixed Torelli problem is affirmatively solved. We also give some examples.

Key words: Surface, Ordinary singularity, Cubic hyper-resolution, Mixed Hodge structure, Infinitesimal mixed Torelli

In this Part II, we inherit terminology and notation from Part I.

§4 Cohomological infinitesimal mixed Torelli problem

In this section we consider the following problem for a given algebraic surface S with ordinary singularities:

Is the homomorphism

$$(4.1)\quad \tau : H^1(S, \Theta_S(b_\bullet)) \to \oplus_{\ell=1}^2 \{\oplus_{p=1}^{\ell} Hom_{\mathbb{C}}(\mathbb{H}^{\ell-p}(\Omega^p_{X_\bullet}[1]), \mathbb{H}^{\ell-p+1}(\Omega^{p-1}_{X_\bullet}[1])\}$$

defined by taking the *contraction* $\Theta_S(b_\bullet) \otimes \Omega^p_{X_\bullet}[1] \to \Omega^{p-1}_{X_\bullet}[1]$ and *cup-product* in Theorem 3.17 is injective ?

We call this problem *cohomological infinitesimal mixed Torelli problem.* Note that if the *cohomological infinitesimal mixed Torelli problem* is affirmatively solved, then the *infinitesimal mixed Torelli problem* formulated in §3 is also affirmatively solved under the condition $H^1(S, \Theta_S(b_\bullet)) \simeq H^1(S, \Theta_S)$. To consider the cohomological infinitesimal mixed Torelli problem we shall first prove the injectivity of the natural map $H^1(S, \Theta_S) \to H^1(X, \Theta_X(-log\, D_X))$ derived from the short exact sequence in Theorem 3.19 in Part I.

2000 *Mathematics Subject Classification.* Primary 32G; Secondary 14D07, 32G13

* This work is supported by the Grant-in-Aid for Scientific Research (C) (No. 15540085), The Ministry of Education, Culture, Sports, Science and Technology of Japan

4.1 Proposition. *The map*

$$H^1(D_S^*, \Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*)) \longrightarrow H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

is injective.

The proof of this proposition will be completed after a few lemmas. First, we will prove general facts about a double covering $\pi : C_1 \longrightarrow C$ between compact Riemann surfaces, or connected, compact complex manifolds of dimension 1. We denote by $\Sigma\mathfrak{c}$ the branch locus of the double covering $\pi : C_1 \longrightarrow C$, and by $[\Sigma\mathfrak{c}]$ the line bundle over C determined by the divisor $\Sigma\mathfrak{c}$. Due to Wavrik's result ([18]), there exists a complex line bundle F over C such that;

(i) $F^{\otimes 2} = [\Sigma\mathfrak{c}]$, and
(ii) C_1 is a submanifold of F and the bundle map $F \longrightarrow C$ realizes the double covering $\pi : C_1 \longrightarrow C$.

The transition functions of the line bundle F are given as follows: We choose a covering $\{U_j, U_\lambda\}$ of C by polycylinders having the following properties;

(i) $U_j \cap \Sigma\mathfrak{c} = \emptyset$, and $U_\lambda \cap \Sigma\mathfrak{c} \neq \emptyset$,
(ii) on U_λ, $\Sigma\mathfrak{c}$ has the equation $u_\lambda = 0$ where u_λ is a local coordinate on U_λ,
(iii) $\pi^{-1}(U_j) = U_j^{(0)} \cup U_j^{(1)}$, $U_j^{(0)} \cup U_j^{(1)} = \emptyset$,
(iv) on $U_j^{(\nu)}$, $\nu = 0, 1$, the map π is given by $u_j = v_j^{(\nu)}$ where u_j and $v_j^{(\nu)}$ are local coordinates on U_j and $U_j^{(\nu)}$, respectively, and
(v) on $U_\lambda^\# := \pi^{-1}(U_\lambda)$, the map π is given by $u_\lambda = v_\lambda^2$ where v_λ is a local coordinate on $U_\lambda^\#$.

We define

$$f_{ij} := \begin{cases} 1 & \text{if } \; U_i^{(0)} \cap U_j^{(0)} \neq \emptyset \\ -1 & \text{if } \; U_i^{(0)} \cap U_j^{(1)} \neq \emptyset, \end{cases}$$

$$f_{\lambda j} := g_{\lambda j}^{(0)},$$

where $g_{\lambda j}^{(0)}$ denotes the coordinate-transformation function, i.e., $v_\lambda = g_{\lambda j}^{(0)}(v_j^{(0)})$. Then $\{f_{ij}, f_{\lambda j}\}$ are the transition functions of the line bundle F over C. We may think that C_1 is a submanifold of F defined by $\xi_i^2 = 1$ on $\pi^{-1}(U_i)$, and by $\xi_\lambda^2 = u_\lambda$ on $\pi^{-1}(U_\lambda)$ where ξ_i and ξ_λ are fiber coordinates of F over U_i and U_λ, respectively.

4.2 Lemma. *There exists an exact sequence of $\mathcal{O}_C$-modules*

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \pi_*\mathcal{O}_{C_1} \longrightarrow \mathcal{O}_C(F^{-1}) \longrightarrow 0. \tag{4.2}$$

Proof. We use the same notation as before. The homomorphism $\pi_*\mathcal{O}_{C_1} \longrightarrow \mathcal{O}_C(F^{-1})$ of $\mathcal{O}_C$-modules is defined as follows: for a local cross-section

$(U_i^{(0)}, \phi_i^{(0)})$, $(U_i^{(1)}, \phi_i^{(1)})$ of $\pi_*\mathcal{O}_{C_1}$ over U_i, we put

$$\psi_i(u_i) := \phi_i^{(0)}(u_i) - \phi_i^{(1)}(u_i).$$

For a local cross-section $(U_\lambda^{\#}, \phi_\lambda)$ of $\pi_*\mathcal{O}_{C_1}$ over U_λ, we put

$$\psi_\lambda(u_\lambda) := \frac{\phi_\lambda(v_\lambda) - \phi_\lambda(-v_\lambda)}{v_\lambda}.$$

We note that the right-hand-side of this is invariant by the transformation $v_\lambda \longrightarrow -v_\lambda$, and so it defines a holomorphic function on U_λ. We can see that the collection $\{\psi_i, \psi_\lambda\}$ defines a local cross-section of $\mathcal{O}_C(F^{-1})$. Indeed, if $U_i^{(0)} \cap U_j^{(0)} \neq \emptyset$, we have $\phi_i^{(0)} = \phi_j^{(0)}$ on $U_i^{(0)} \cap U_j^{(0)}$, $\phi_i^{(1)} = \phi_j^{(1)}$ on $U_i^{(1)} \cap U_j^{(1)}$ and $f_{ij} = 1$. Hence

$$\psi_i = \phi_i^{(0)} - \phi_i^{(1)} = \phi_j^{(0)} - \phi_j^{(1)} = f_{ij}^{-1}\psi_j \text{ on } U_i \cap U_j.$$

If $U_i^{(0)} \cap U_j^{(1)} \neq \emptyset$, we have $\phi_i^{(0)} = \phi_j^{(1)}$ on $U_i^{(0)} \cap U_j^{(1)}$, $\phi_i^{(1)} = \phi_j^{(0)}$ on $U_i^{(1)} \cap U_j^{(0)}$ and $f_{ij} = -1$. Hence

$$\psi_i = \phi_i^{(0)} - \phi_i^{(1)} = -(\phi_j^{(0)} - \phi_j^{(1)}) = f_{ij}^{-1}\psi_j \text{ on } U_i \cap U_j.$$

If $U_\lambda \cap U_i \neq \emptyset$, we have

$$\begin{cases} \phi_\lambda(v_\lambda) = \phi_i^{(0)}(v_i^{(0)}) & \text{on } U_\lambda^{\#} \cap U_i^{(0)}, \textit{and} \\ \phi_\lambda(v_\lambda) = \phi_i^{(1)}(v_i^{(1)}) & \text{on } U_\lambda^{\#} \cap U_i^{(1)}. \end{cases}$$

Hence

$$\phi_\lambda(-v_\lambda) = \phi_i^{(1)}(v_i^{(0)}) \text{ on } U_\lambda^{\#} \cap U_i^{(0)}$$

and

$$\psi_\lambda(u_\lambda) = \frac{\phi_\lambda(v_\lambda) - \phi_\lambda(-v_\lambda)}{v_\lambda} = g_{\lambda i}^{(0)}(v_i^{(0)})^{-1}\{\phi_i^{(0)}(v_i^{(0)}) - \phi_i^{(1)}(v_i^{(0)})\}$$

$$= f_{\lambda i}^{-1}(u_i)\psi_i(u_i) \text{ on } U_\lambda \cap U_i.$$

Thus the collection $\{\psi_i, \psi_\lambda\}$ certainly defines a local cross-section of $\mathcal{O}_C(F^{-1})$. We define the homomorphism $\pi_*\mathcal{O}_{C_1} \longrightarrow \mathcal{O}_C(F^{-1})$ in (4.2) by the correspondence

$$\begin{cases} (\phi_i^{(0)}, \phi_i^{(1)}) \longmapsto \psi_i & \text{over } U_i, \textit{ and} \\ \phi_\lambda \longmapsto \psi_\lambda & \text{over } U_\lambda. \end{cases}$$

The fact that the kernel of the homomorphism $\pi_*\mathcal{O}_{C_1} \longrightarrow \mathcal{O}_C(F^{-1})$ is $\mathcal{O}_C$ is obvious. The surjectivity of the homomorphism $\pi_*\mathcal{O}_{C_1} \longrightarrow \mathcal{O}_C(F^{-1})$ at a point

$p \notin \Sigma\mathfrak{c}_\lambda$ is also obvious. We will show the surjectivity of this homomorphism at a point $p \in \Sigma\mathfrak{c}_\lambda$. Let ψ be a local cross-section of $\mathcal{O}_C(F^{-1})$ at the point p. We may think of it as a holomorphic function defined around p. Let

$$\psi(u_\lambda) = \Sigma_{k=0}^{\infty} a_k u_\lambda^k$$

be the power series expansion of ψ with center p. We put

$$\phi(v_\lambda) = \Sigma_{k=0}^{\infty} \frac{1}{2} a_k v_\lambda^{2k+1}$$

Then, since $u_\lambda = v_\lambda^2$, we have

$$\psi_\lambda(u_\lambda) = \frac{\phi_\lambda(v_\lambda) - \phi_\lambda(-v_\lambda)}{v_\lambda}.$$

Thus the homomorphism $\pi_*\mathcal{O}_{C_1} \longrightarrow \mathcal{O}_C(F^{-1})$ is surjective at the point $p \in \Sigma\mathfrak{c}$.

Q.E.D.

Let $\pi : C_1 \longrightarrow C$ and $\Sigma\mathfrak{c}$ be the same as before, and let Σt be a set of finite distinct points of C with $\Sigma\mathfrak{c} \cap \Sigma t = \emptyset$. We put $\Sigma t_1 := \pi^{-1}(\Sigma t)$.

4.3 Lemma. *With the notation similar to that in* Part I, (3.31), *we have an exact sequence of $\mathcal{O}_C$-modules*

$$(4.3) \qquad 0 \longrightarrow \Theta_C(-\Sigma\mathfrak{c} - \Sigma t) \longrightarrow \pi_*\Theta_{C_1}(-\Sigma t_1) \longrightarrow \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1}) \longrightarrow 0.$$

Proof. Since $\pi_*(\pi^*\Theta_C(-\Sigma t)) \simeq \Theta_C(-\Sigma t) \otimes \pi_*\mathcal{O}_{C_1}$, tensoring the sheaf $\Theta_C(-\Sigma t)$ to the exact sequence in (4.2), we have the exact sequence of $\mathcal{O}_C$-modules

$$(4.4) \qquad 0 \longrightarrow \Theta_C(-\Sigma t) \longrightarrow \pi_*(\pi^*\Theta_C(-\Sigma t)) \longrightarrow \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1}) \longrightarrow 0.$$

We also have the following commutative diagram of exact sequences of $\mathcal{O}_C$-modules:

$$\begin{array}{ccc}
0 \longrightarrow \Theta_C(-\Sigma t) & \xrightarrow{\widehat{\omega\pi}} & \pi_*(\pi^*\Theta_C(-\Sigma t)) \\
\uparrow & & \uparrow \\
0 \longrightarrow \Theta_C(-\Sigma\mathfrak{c} - \Sigma t) & \longrightarrow & \pi_*\Theta_{C_1}(-\Sigma t_1) \\
\uparrow & & \uparrow \\
0 & & 0.
\end{array}$$

We will show that this diagram gives an isomorphism

$$(4.5)\qquad \pi_*\Theta_{C_1}(-\Sigma t_1)/\Theta_C(-\Sigma\mathfrak{c}-\Sigma t) \simeq \pi_*(\pi^*\Theta_C(-\Sigma t))/\Theta_C(-\Sigma t).$$

To prove the surjectivity of the homomorphism in (4.5), we will first show that

$$(4.6)\qquad \hat{t\pi}(\Theta_{C_1}(-\Sigma t_1)_{\pi^{-1}(p)}) + \hat{\omega\pi}(\Theta_C(-\Sigma t)_p) = \pi^*\Theta_C(-\Sigma t)_{\pi^{-1}(p)}$$

for any point $p \in C$. If $p \notin \Sigma\mathfrak{c}$, (4.6) obviously holds. Assume $p \in \Sigma\mathfrak{c}$. We put $q := \pi^{-1}(p)$, and let u and v be local coordinates around p and q with center p and q, respectively. We may assume that the map $\pi : C_1 \longrightarrow C$ is given by $v \longrightarrow u = v^2$ at q. For a local cross-section $a(v)\pi^*(\partial/\partial u)$ of $\pi^*\Theta_C(-\Sigma t)$ around q where $a(v)$ is a holomorphic function of v, we express $a(v)$ as

$$a(v) = a(0) + va_1(v)$$

where $a_1(v)$ is a holomorphic function of v. Then we have

$$\begin{aligned} &\hat{t\pi}(\frac{1}{2}a_1(v)(\frac{\partial}{\partial v})) + \hat{\omega\pi}(a(0)(\frac{\partial}{\partial u})) \\ &= (va_1(v) + a(0))\pi^*(\frac{\partial}{\partial u}) = a(v)\pi^*(\frac{\partial}{\partial u}), \end{aligned}$$

which shows (4.6) holds for the point $p \in \Sigma\mathfrak{c}$. Thus (4.6) holds for any point $p \in C$. To prove the injectivity of the homomorphism in (4.5), it suffices to show that, for any point $p \in C$ and a local holomorphic cross-section θ_1 of $\pi_*\Theta_{C_1}(-\Sigma t_1)$, if $\hat{t\pi}(\theta_{1,p})$ belongs to $\hat{\omega\pi}(\Theta_C(-\Sigma t)_p)$, then $\theta_{1,p}$ belongs to the image $\Theta_C(-\Sigma\mathfrak{c} - \Sigma t)_p$ in $\pi_*\Theta_{C_1}(-\Sigma t_1)_p$. Since this is obvious if $p \notin \Sigma\mathfrak{c}$, we assume $p \in \Sigma\mathfrak{c}$. We take the same local coordinates u and v around p and $q := \pi^{-1}(p)$, respectively, as before. For a local cross-section $\theta_1 = a_1(v)(\partial/\partial v)$ of $\Theta_{C_1}(-\Sigma t_1)$ at q, we assume that there exists a local cross-section $\theta = a(u)(\partial/\partial u)$ of $\Theta_C(-\Sigma t)$ at p such that $\hat{t\pi}(\theta_1) = \hat{\omega\pi}(\theta)$. Then

$$2a_1(v)v\pi^*(\frac{\partial}{\partial v}) = a(v^2)\pi^*(\frac{\partial}{\partial v})$$

Hence $a(0) = 0$, that is, θ belongs to $\Theta_C(-\Sigma t - \Sigma\mathfrak{c})$. This means θ_1 belongs to the image of $\Theta_C(-\Sigma\mathfrak{c} - \Sigma t)$ in $\pi_*\Theta_{C_1}(-\Sigma t_1)$ at p. Now the exact sequence in (4.3) follows from (4.4) and (4.5).

Q.E.D.

4.4 Remark. In the proof of Lemma 4.3, the equality in (4.6) is essential. This equality telles that the double branched covering map $\pi : C_1 \longrightarrow C$ is locally stable in the sense of J. N. Mather.

Proof of Proposition 4.1 We may assume that D_S^* is irreducible, and so it suffices to show that the homomorphism

$$H^1(C, \Theta_C(-\Sigma\mathfrak{c} - \Sigma t)) \longrightarrow H^1(C_1, \Theta_{C_1}(-\Sigma t_1)) \tag{4.7}$$

derived from the exact sequence in (4.3) is injective. For this purpose, we count the degree of the line bundle $\Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})$. We denote by $\mathfrak{K}_C$ and $g(C)$ the canonical line bundle and the genus of the curve C, respectively. Then, since $F^{\otimes 2} = \mathcal{O}_C([\Sigma\mathfrak{c}])$, we have

$$\begin{aligned} \deg(\Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})) &= -\deg \mathfrak{K}_C - \deg F - \#\Sigma t \\ &= -2(g(C) - 1) - \frac{1}{2}\#\Sigma\mathfrak{c} - \#\Sigma t. \end{aligned}$$

Now we have

$$-2(g(C) - 1) - \frac{1}{2}\#\Sigma\mathfrak{c} - \#\Sigma t < 0$$

with the exception of the following cases:

(i) $g(C) = 1$, $\Sigma\mathfrak{c} = \emptyset$, and $\Sigma t = \emptyset$,
(ii) $g(C) = 0$, $\Sigma\mathfrak{c} = \emptyset$, and $0 \leq \#\Sigma t \leq 2$,
(iii) $g(C) = 0$, $\#\Sigma\mathfrak{c} = 2$, and $0 \leq \#\Sigma t \leq 1$,
(iv) $g(C) = 0$, $\#\Sigma\mathfrak{c} = 4$, and $\Sigma t = \emptyset$.

Hence, excluding the exceptional cases listed above, we have

$$H^0(C, \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})) = 0, \tag{4.8}$$

and so the the homomorphism in (4.7) is injective as required. Now we check the exceptional cases listed above, case by case. In the case (i), we have $\Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1}) \simeq \mathcal{O}_C(F^{-1})$. Assume $\mathrm{H}^0(C, \mathcal{O}_C(F^{-1})) \neq 0$. Then the line bundle $\mathcal{O}_C(F^{-1})$ has a global cross-section. Since $\deg F^{-1} = -(1/2)\#\Sigma\mathfrak{c} = 0$, this global section vanishes nowhere. This fact implies $F = 1$, contradicting to $F \neq 1$. Thus we conclude (4.8) holds, and so the homomorphism in (4.8) is also injective for this case. The case (ii) could not happen. Indeed, if $g(C) = 0$ and $\Sigma\mathfrak{c} = \emptyset$, it follows from the Hurwitz formula

$$2g(C_1) - 2 = 2(2g(C) - 2) + \#\Sigma\mathfrak{c}. \tag{4.9}$$

that $g(C_1) = -1$, which is a contradiction.
Next we will show that the homomrohism

$$H^0(C_1, \Theta_{C_1}(-\Sigma t_1)) \longrightarrow H^0(C, \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})) \tag{4.10}$$

is surjective in the cases (iii) and (iv). Note that in these two cases, $C \cong \mathbb{P}^1(\mathbb{C})$. By the duality,

$$H^1(C,\, \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})) \simeq H^0(C,\, \mathfrak{K}_C^{\otimes 2}(\Sigma t) \otimes \mathcal{O}_C(F)).$$

Since

$$\deg[\mathfrak{K}_C^{\otimes 2}(\Sigma t) \otimes \mathcal{O}_C(F)] = -4 + \#\Sigma t + \frac{1}{2}\#\Sigma \mathfrak{c} < 0,$$

we conclude

$$H^1(C, \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})) = 0.$$

Hence by the Riemann-Roch theorem,

$$\begin{aligned}\dim H^0(C, \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})) &= \deg[\Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})] + 1 - g(C) \\ &= 3 - \frac{1}{2}\#\Sigma \mathfrak{c} - \#\Sigma t \qquad (4.11) \\ &= \begin{cases} 2 & \text{if } \#\Sigma \mathfrak{c} = 2,\ \#\Sigma t = 0, \\ 1 & \text{if } \#\Sigma \mathfrak{c} = 2,\ \#\Sigma t = 1, \text{ or } \#\Sigma \mathfrak{c} = 4,\ \#\Sigma t = 0. \end{cases}\end{aligned}$$

By the Hurwitz formula, if $\#\Sigma \mathfrak{c} = 2$, then $g(C_1) = 0$; and if $\#\Sigma \mathfrak{c} = 4$, then $g(C_1) = 1$. Therefore we have

$$\begin{aligned}&\dim H^0(C_1, \Theta_{C_1}(-\Sigma t_1)) \\ (4.12) \qquad &= \begin{cases} 3 & \text{if } \#\Sigma \mathfrak{c} = 2,\ \#\Sigma \mathfrak{t} = 0, \\ 1 & \text{if } \#\Sigma \mathfrak{c} = 2,\ \#\Sigma t = 1, \text{ or } \#\Sigma \mathfrak{c} = 4, \#\Sigma t = 0. \end{cases}\end{aligned}$$

On the other hand, since $C = \mathbb{P}^1(\mathbb{C})$, we have

$$\begin{aligned}&\dim H^0(C, \Theta_C(-\Sigma \mathfrak{c} - \Sigma t)) \\ (4.13) \qquad &= \begin{cases} 1 & \text{if } \#\Sigma \mathfrak{c} = 2,\ \#\Sigma t = 0, \\ 0 & \text{if } \#\Sigma \mathfrak{c} = 2,\ \#\Sigma t = 1, \text{ or } \#\Sigma \mathfrak{c} = 4, \#\Sigma t = 0. \end{cases}\end{aligned}$$

By (4.11), (4.12) and (4.13), we have

$$\begin{aligned}&\dim H^0(C, \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})) \\ &= \dim H^0(C_1, \Theta_{C_1}(-\Sigma t_1)) - \dim H^0(C, \Theta_C(-\Sigma \mathfrak{c} - \Sigma t)),\end{aligned}$$

which means the homomorphism

$$H^0(C_1, \Theta_{C_1}(-\Sigma t_1)) \longrightarrow H^0(C, \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1}))$$

is surjective. Therefore the homomorphism

$$H^1(C, \Theta_C(-\Sigma \mathfrak{c} - \Sigma t)) \longrightarrow H^1(C_1, \Theta_{C_1}(-\Sigma t_1))$$

in the lemma is also injective for these cases.

Q.E.D.

4.5 Corollary. *If the map*

$$H^0(X,\Theta_X(-logD_X))\oplus H^0(D_S^*,\Theta_{D_S^*}(-\Sigma c_S^*-\Sigma t_S^*))\longrightarrow H^0(D_X^*,\Theta_{D_X^*}(-\Sigma t_X^*))$$

is surjective, then the natural map

$$H^1(S,\Theta_S)\longrightarrow H^1(X,\Theta_X(-logD_X))$$

is injective.

Proof. Let $([\theta_X],[\theta_{D_S^*}])$ be the element of

$$Ker\{H^1(X,\Theta_X(-logD_X))\oplus H^1(D_S^*,\Theta_{D_S^*}(-\Sigma c_S^*-\Sigma t_S^*)) \longrightarrow H^1(D_X^*,\Theta_{D_X^*}(-\Sigma t_X^*))\}$$

corresponding to an element $[\theta]\in H^1(S,\Theta_S)$ by the isomorphism in Theorem 3.26, (ii). The homomorphism $H^1(S,\Theta_S)\longrightarrow H^1(X,\Theta_X(-\log D_X))$ is the map which assignes $[\theta_X]$ to $[\theta]$. We assume $[\theta_X]=0$, then $\widehat{\omega g}([\theta_{D_S^*}])=\widehat{\omega\nu_X}([\theta_X])=0$ (cf. Part I, (3.31)). Then, by Theorem 4.1, we have $[\theta_{D_S^*}]=0$, and so by Proposition 3.26, (ii), we have $[\theta]=0$.

Q.E.D.

In order to consider the cohomological Torelli problem we need to analyze the hyper-cohomology groups $\mathbb{H}^0(\Omega^2_{X_\bullet}[1])$, $\mathbb{H}^1(\Omega^1_{X_\bullet}[1])$, $\mathbb{H}^0(\Omega^1_{X_\bullet}[1])$, and $\mathbb{H}^1(\mathcal{O}_{X_\bullet}[1])$.

4.6 Proposition.

(i) $\mathbb{H}^0(\Omega^2_{X_\bullet}[1])\simeq H^0(X,\Omega^2_X)$,

(ii) there exists the following exact sequence

$$0\longrightarrow H^0(D_X^*,\Omega^1_{D_X^*})/Im\{H^0(X,\Omega^1_X)\oplus H^0(D_{S^*},\Omega^1_{D_S^*})\} \longrightarrow \mathbb{H}^1(\Omega^1_{X_\bullet}[1])\longrightarrow Ker\{H^1(X,\Omega^1_X)\oplus H^1(D_{S^*},Omega^1_{D_S^*}) \longrightarrow H^1(D_X^*,\Omega^1_{D_X^*})\}\longrightarrow 0,$$

where the homomorphism $H^0(X,\Omega^1_X)\oplus H^0(D_{S^*},\Omega^1_{D_S^*})\longrightarrow H^0(\Omega^1_{D_X^*})$ is defined by $(\omega_X,\omega_{D_S^*})\longrightarrow \nu_X^*\omega_X-g^*\omega_{D_S^*}$ ($\nu_X: D_X^*\longrightarrow X$, $g: D_X^*\longrightarrow D_S^*$ cf. Part I, (1.4)) for $(\omega_X,\omega_S^*)\in H^0(X,\Omega^1_X)\oplus H^0(D_{S^*},\Omega^1_{D_S^*})$, and the homomorphism $H^1(X,\Omega^1_X)\oplus H^1(D_{S^*},\Omega^1_{D_S^*})\longrightarrow H^1(D_X^*,\Omega^1_{D_X^*})$ is defined by $(\omega_X^{(1)},\omega_{D_S^*}^{(1)})\longrightarrow \nu_X^*\omega_X^{(1)}-g^*\omega_{D_S^*}^{(1)}$ for $(\omega_X^{(1)},\omega_{D_S^*}^{(1)})\in H^1(X,\Omega^1_X)\oplus H^1(D_{S^*}\Omega^1_{D_{S^*}})$,

(iii) $\mathbb{H}^0(\Omega^1_{X_\bullet}[1])\simeq Ker\{H^0(X,\Omega^1_X)\oplus H^0(D_{S^*},\Omega^1_{D_S^*})\longrightarrow H^0(D_X^*,\Omega^1_{D_X^*})\}$

where the homomorphism $H^0(X,\Omega^1_X)\oplus H^0(D_{S^*},\Omega^1_{D^*_S}) \to H^0(D^*_X,\Omega^1_{D^*_X})$ is the same one as in (ii) above,

(iv) there exists the following exact sequence

$$0 \to \frac{Ker\{H^0(D^*_X,\mathcal{O}_{D^*_X})\oplus H^0(\Sigma t_X,\mathbb{C}_{\Sigma t_X})\oplus H^0(\Sigma t^*_S,\mathbb{C}_{\Sigma t^*_S}) \to H^0(\Sigma t^*_X,\mathbb{C}_{\Sigma t^*_X})\}}{Im\{H^0(X,\mathcal{O}_X)\oplus H^0(D^*_S,\mathcal{O}_{D^*_S})\oplus H^0(\Sigma t_S,\mathbb{C}_{\Sigma t_S})\}}$$
$$\to \mathbb{H}^1(\mathcal{O}_{X_\bullet}[1]) \to Ker\{H^1(X,\mathcal{O}_X)\oplus H^1(D^*_S,\mathcal{O}_{D^*_S})$$
$$\to H^1(D^*_X,\mathcal{O}_{D^*_X})\} \to 0,$$

where the homomorphism $H^0(D^*_X,\mathcal{O}_{D^*_X})\oplus H^0(\Sigma t_X,\mathbb{C}_{\Sigma t_X})\oplus H^0(\Sigma t^*_S,\mathbb{C}_{\Sigma t^*_S}) \to H^0(\Sigma t^*_X,\mathbb{C}_{\Sigma t^*_X})$ is defined by

$$(\phi_{D^*_X}, c_{\Sigma t_X}, c_{\Sigma t^*_S}) \to \phi_{D^*_X|\Sigma t^*_X} - \nu^*_{X|\Sigma t^*_X} c_{\Sigma t_X} + g^*_{|\Sigma t^*_X} c_{\Sigma t^*_S}$$
$$(\nu_{X|\Sigma t^*_X} : \Sigma t^*_X \to \Sigma t_X,\ \ g_{|\Sigma t^*_X} : \Sigma t^*_X \to \Sigma t^*_S)$$

for $(\phi_{D^*_X}, c_{\Sigma t_X}, c_{\Sigma t^*_S}) \in H^0(D^*_X,\mathcal{O}_{D^*_X})\oplus H^0(\Sigma t_X,\mathbb{C}_{\Sigma t_X})\oplus H^0(\Sigma t^*_S,\mathbb{C}_{\Sigma t^*_S})$, the homomorphism

$$H^0(X,\mathcal{O}_X)\oplus H^0(D^*_S,\mathcal{O}_{D^*_S})\oplus H^0(\Sigma t_S,\mathbb{C}_{\Sigma t_S})$$
$$\to Ker\{H^0(D^*_X,\mathcal{O}_{D^*_X})\oplus H^0(\Sigma t_X,\mathbb{C}_{\Sigma t_X})\oplus H^0(\Sigma t^*_S,\mathbb{C}_{\Sigma t^*_S})$$
$$\to H^0(\Sigma t^*_X,\mathbb{C}_{\Sigma t^*_X})\}$$

is defined by

$$(\phi_X, \phi_{D^*_S}, c_{\Sigma t_S}) \to (\nu^*_X\phi_X - g^*\phi_{D^*_S}, \phi_{X|\Sigma t_X} - f^*_{|\Sigma t_X} c_{\Sigma t_S}, \phi_{D^*_S|\Sigma t^*_S} - \nu_{S|\Sigma t^*_S} c_{\Sigma t_S})$$

for $(\phi_X, \phi_{D^*_S}, c_{\Sigma t_S}) \in H^0(X,\mathcal{O}_X)\oplus H^0(D^*_S,\mathcal{O}_{D^*_S})\oplus H^0(\Sigma t_S,\mathbb{C}_{\Sigma t_S})$, and the homomrphism $H^1(X,\mathcal{O}_X)\oplus H^1(D^*_S,\mathcal{O}_{D^*_S}) \to H^1(D^*_X,\mathcal{O}_{D^*_X})$ is defined by

$$(\phi^{(1)}_X, \phi^{(1)}_{D^*_S}) \to \nu^*_X\phi^{(1)}_X - g^*\phi^{(1)}_{D^*_S}$$

for $(\phi^{(1)}_X, \phi^{(1)}_{D^*_S}) \in H^1(X,\mathcal{O}_X)\oplus H^1(D^*_S,\mathcal{O}_{D^*_S})$.

Proof. First, we recall the definition of the hyper-cohomology $\mathbb{H}^\bullet(\Omega^p_{X_\bullet}[1])$. It is defined as the cohomology of the complex

$$K^\bullet(\Omega^p_{X_\bullet}[1]) = K^{\bullet+1}(\Omega^p_{X_\bullet}) = s(\bigoplus_{\alpha\in \mathrm{Ob}(\square_2)} C^\bullet(\mathcal{U}_\alpha,\Omega^p_{X_\alpha})),$$

where $X_\bullet \to S$ is the cubic hyper-resolution of S in (1.4) in Part I, and $C^\bullet(\mathcal{U}_\alpha)$, $\alpha \in \mathrm{Ob}(\square_2)$, are the Cěch complex with values in $\Omega^p_{X_\alpha}$ with respect to the Stein covering $\mathcal{U}_\alpha$ of X_α. The system of open coverings $\{\mathcal{U}_\alpha\}_{\alpha\in\mathrm{Ob}(\square^+_2)}$ are such one that satisfies the conditions (i) through (iv) in (2.7) and the condition (v) in (3.16) in Part I. The cobundary map

$$\begin{aligned} D^{(q)}[1] := D^{(q+1)} : K^q(\Omega^p_{X_\bullet}[1]) &= K^{q+1}(\Omega^p_{X_\bullet}) \\ &= s(\textstyle\bigoplus_{\alpha\in \mathrm{Ob}(\Box_2)} C^\bullet(\mathcal{U}_\alpha, \Omega^p_{X_\alpha}))^{q+1} \\ &\quad \to K^{q+1}(\Omega^p_{X_\bullet}[1]) = K^{q+2}(\Omega^p_{X_\bullet}) \\ &\quad = s(\textstyle\bigoplus_{\alpha\in \mathrm{Ob}(\Box_2)} C^\bullet(\mathcal{U}_\alpha, \Omega^p_{X_\alpha}))^{q+2} \end{aligned}$$

of the complex $K^\bullet(\Omega^p_{X_\bullet}[1])$ is defined as

$$D^{(q)}[1] = D^{(q+1)} = \bigoplus_{|\alpha|+r=q+1} \{\Sigma_{1\le j\le 3}(-1)^{\varepsilon_j} d^{(p,r)*}_{\alpha,j} + (-1)^{|\alpha|}\delta^{(p,r)}_\alpha\}$$

where

$$d^{(p,r)*}_{\alpha,j} : C^r(\mathcal{U}_\alpha, \Omega^p_{X_\alpha}) \to C^r(\mathcal{U}_\alpha, \Omega^p_{X_{\alpha+e_j}}) \quad (e_j = (0\cdots\overset{j}{1}\cdots 0) \in \mathbb{Z}^3),$$

$\delta^{(p,r)}_\alpha : C^r(\mathcal{U}_\alpha, \Omega^p_{X_\alpha}) \to C^{r+1}(\mathcal{U}_{D_\alpha}, \Omega^p_{X_\alpha}))$
(the Cěch coboundary map),

$\varepsilon_j = \alpha_0 + \alpha_1 + \cdots + \alpha_{j-2}$
for $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathrm{Ob}(\Box_2))$ and j with $1 \le j \le 3$, and

$|\alpha| = \alpha_0 + \alpha_1 + \alpha_2$

(cf. (2.14) in Part I). Here we understand $\varepsilon_1 = 0$.

(i) By the definition of $\mathbb{H}^0(\Omega^2_{X_\bullet}[1])$, it is $Ker D^{(0)}[1]$ where

$$D^{(0)}[1] : K^0(\Omega^2_{X_\bullet}[1]) = C^0(\mathcal{U}_X, \Omega^2_X) \to K^1(\Omega^2_{X_\bullet}[1]) = C^1(\mathcal{U}_X, \Omega^2_X)$$

Here we denote $C^r(\mathcal{U}_{(001)}, \Omega^2_{X_{(001)}})$, $r = 0, 1$, simply by $C^r(\mathcal{U}_X, \Omega^2_X)$ since $X = X_{(001)}$, $(001) \in \mathrm{Ob}(\Box_2)$. In what follows we will use the similar notation for other $C^r(\mathcal{U}_\alpha, \Omega^p_{X_\alpha})$, $\alpha \in \mathrm{Ob}(\Box_2)$. Since the coboundary map $D^{(0)}[1]$ is simply $-\delta^{(2,0)}_X$, that is, $(-1) \times$ {the Cěch coboudary map}, we have $\mathbb{H}^0(\Omega^2_{X_\bullet}[1]) \simeq H^0(X, \Omega^2_X)$.

(ii) The hyper-cohomology $\mathbb{H}^1(\Omega^1_{X_\bullet}[1])$ is defined to be

$$Ker\, D^{(1)}[1]/Im\, D^{(0)}[1],$$

using the following part of the complex $K^\bullet(\Omega^1_{X_\bullet}[1])$:

$$K^0(\Omega^1_{X_\bullet}[1]) \xrightarrow{D^{(0)}[1]} K^1(\Omega^1_{X_\bullet}[1]) \xrightarrow{D^{(1)}[1]} K^2(\Omega^1_{X_\bullet}[1])$$

where

$$\begin{aligned} &K^0(\Omega^1_{X_\bullet}[1]) := C^0(\mathcal{U}_X, \Omega^1_X) \oplus C^0(\mathcal{U}_{D^*_S}, \Omega^1_{D^*_S}), \\ (4.14) \qquad &K^1(\Omega^1_{X_\bullet}[1]) := C^1(\mathcal{U}_X, \Omega^1_X) \oplus C^1(\mathcal{U}_{D^*_S}, \Omega^1_{D^*_S}) \oplus C^0(\mathcal{U}_{D^*_X}, \Omega^1_{D^*_X}), \text{ and} \\ &K^2(\Omega^1_{X_\bullet}[1]) := C^2(\mathcal{U}_X, \Omega^1_X) \oplus C^2(\mathcal{U}_{D^*_S}, \Omega^1_{D^*_S}) \oplus C^1(\mathcal{U}_{D^*_X}, \Omega^1_{D^*_X}). \end{aligned}$$

Since $X_{(001)} = X$, $X_{(010)} = D^*_S$, $X_{(011)} = D^*_X$, we have

$$\begin{aligned} &D^{(0)}[1] = d^{(1,0)*}_{(001),2} - d^{(1,0)*}_{(010),3} - \delta^{(1,0)}_{(001)} - \delta^{(1,0)}_{(010)}, \text{ and} \\ (4.15) \qquad & \\ &D^{(1)}[1] = d^{(1,1)*}_{(001),2} - d^{(1,1)*}_{(010),3} - \delta^{(1,1)}_{(001)} - \delta^{(1,1)}_{(010)} + \delta^{(1,0)}_{(011)}, \end{aligned}$$

where

$$d^{(1,0)*}_{(001),2} = \nu^*_X : C^0(\mathcal{U}_X, \Omega^1_X) \to C^0(\mathcal{U}_{D^*_X}, \Omega^1_{D^*_X}),$$

$$d^{(1,0)*}_{(010),3} = g^* : C^0(\mathcal{U}_{D^*_S}, \Omega^1_{D^*_S}) \to C^0(\mathcal{U}_{D^*_X}, \Omega^1_{D^*_X})),$$

are the maps defined to be pulling back cochains, and

$$\delta^{(1,0)}_{(001)} : C^0(\mathcal{U}_X, \Omega^1_X) \to C^1(\mathcal{U}_X, \Omega^1_X),$$

$$\delta^{(1,0)}_{(010)} : C^0(\mathcal{U}_{D^*_S}, \Omega^1_{D^*_S}) \to C^1(\mathcal{U}_{D^*_S}, \Omega^1_{D^*_S}),$$

are the Cěch coboundary maps, etc.. The quotient space

$$Ker\, D^{(1)}[1] \cap C^0(\mathcal{U}_{D^*_X}, \Omega^1_{D^*_X})/Im\, D^{(0)}[1] \cap C^0(\mathcal{U}_{D^*_X}, \Omega^1_{D^*_X})$$

is a subspace of $Ker\, D^{(1)}[1]/Im\, D^{(0)}[1]$, which is isomorphic to

$$H^0(D^*_X, \Omega^1_{D^*_X})/Im\{H^0(X, \Omega^1_X) \oplus H^0(D^*_S, \Omega^1_{D^*_S})\}.$$

There exists a homomorphism from $Ker\, D^{(1)}[1]/Im\, D^{(0)}[1]$ to

$$Ker\{H^1(X, \Omega^1_X) \oplus H^1(D^*_S, \Omega^1_{D^*_S}) \to H^1(D^*_X, \Omega^1_{D^*_X})\}$$

which is defined by assigning

$$[\theta_X, \theta_{D_S^*}, \theta_{D_X^*}] \in Ker\, D^{(1)}[1]/Im\, D^{(0)}[1]$$

to $([\theta_X], [\theta_{D_S^*}]) \in H^1(X, \Omega_X^1) \oplus H^1(D_S^*, \Omega_{D_S^*}^1)$, where

$$(\theta_X, \theta_{D_S^*}, \theta_{D_X^*}) \in Ker\, D^{(1)}[1],$$

and "[ ]" denotes cohomology classes. Note that

$$Ker\, D^{(1)}[1] \subset C^1(\mathcal{U}_X, \Omega_X^1) \oplus C^1(\mathcal{U}_{D_S^*}, \Omega_{D_S^*}^1) \oplus C^0(\mathcal{U}_{D_X^*}, \Omega_{D_X^*}^1).$$

This homomorphism is obviously surjective. Assume that the image of $[\theta_X, \theta_{D_S^*}, \theta_{D_X^*}]$ in $H^1(X, \Omega_X^1) \oplus H^1(D_S^*, \Omega_{D_S^*}^1)$ is zero. Then there exists an element

$$(\varphi_X, \varphi_{D_S^*}) \in C^0(\mathcal{U}_X, \Omega_X^1) \oplus C^0(\mathcal{U}_{D_S^*}, \Omega_{D_S^*}^1)$$

such that $-\delta_{(001)}^{(1,0)} \varphi_X = \theta_X$ and $-\delta_{(010)}^{(1,0)} \varphi_{D_S^*} = \theta_{D_S^*}$. Now, we have

$$(\theta_X, \theta_{D_S^*}, \theta_{D_X^*}) - D^{(0)}[1]((\varphi_X, \varphi_{D_S^*})) = (0, 0, \theta_{D_X^*} - \nu_X^* \varphi_X + g^* \varphi_{D_S^*}), \text{ and}$$

$$\begin{aligned} &D^{(1)}[1]((0, 0, \theta_{D_X^*} - \nu_X^* \varphi_X + g^* \varphi_{D_S^*})) \\ &= D^{(1)}[1]((\theta_X, \theta_{D_S^*}, \theta_{D_X^*})) - D^{(1)}[1] D^{(0)}[1]((\varphi_X, \varphi_{D_S^*})) = 0 \end{aligned}$$

That is, $(\theta_X, \theta_{D_S^*}, \theta_{D_X^*})$ is cohomologus to an element of

$$Ker\, D^{(1)}[1] \cap C^0(\mathcal{U}_{D_X^*}, \Omega_{D_X^*}^1) \text{ in } Ker\, D^{(1)}[1]/Im\, D^{(0)}[1] \simeq \mathbb{H}^1(\Omega_{X_\bullet}^1[1]).$$

Consequently, the sequence in (ii) in the proposition is exact.

(iii) By the definition of $\mathbb{H}^0(\Omega_{X_\bullet}^1)$, it is the kernel of

$$D^{(0)}[1] : K^0(\Omega_{X_\bullet}^1[1]) \to K^1(\Omega_{X_\bullet}^1[1])$$

where $K^0(\Omega_{X_\bullet}^1[1])$, $K^1(\Omega_{X_\bullet}^1[1])$ and $D^{(0)}[1]$ are the same ones as in (4.14) and (4.15), respectively. By the definition of $D^{(0)}[1]$, $Ker\, D^{(0)}[1]$ is isomorphic to

$$Ker\{H^0(X, \Omega_X^1) \oplus H^0(D_S^*, \Omega_{D_S^*}^1) \to H^0(D_X^*, \Omega_{D_X^*}^1)\}.$$

(iv) The hyper-cohomology $\mathbb{H}^1(\mathcal{O}_{X_\bullet})$ is defined to be

$$Ker\, D^{(1)}[1]/Im\, D^{(0)}[1],$$

using the following part of the complex $K^\bullet(\mathcal{O}_{X_\bullet}[1])$:

$$K^0(\mathcal{O}_{X_\bullet}[1]) \xrightarrow{D^{(0)}[1]} K^1(\mathcal{O}_{X_\bullet}[1]) \xrightarrow{D^{(1)}[1]} K^2(\mathcal{O}_{X_\bullet}[1])$$

where

$$\begin{aligned}
&K^0(\mathcal{O}_{X_\bullet}[1]) := C^0(\mathcal{U}_X, \mathcal{O}_X) \oplus C^0(\mathcal{U}_{D_S^*}, \mathcal{O}_{D_S^*}) \oplus C^0(\Sigma t_S, \mathbb{C}_{\Sigma t_S}),\\
&K^1(\mathcal{O}_{X_\bullet}[1]) := C^1(\mathcal{U}_X, \mathcal{O}_X) \oplus C^1(\mathcal{U}_{D_S^*}, \mathcal{O}_{D_S^*}) \oplus C^0(\mathcal{U}_{D_X^*}, \mathcal{O}_{D_X^*})\\
(4.14)' \qquad &\qquad\qquad \oplus C^0(\Sigma t_X, \mathbb{C}_{\Sigma t_X}) \oplus C^0(\Sigma t_S^*, \mathbb{C}_{\Sigma t_S^*}), \quad \text{and}\\
&K^2(\mathcal{O}_{X_\bullet}[1]) := C^2(\mathcal{U}_X, \mathcal{O}_X) \oplus C^2(\mathcal{U}_{D_S^*}, \mathcal{O}_{D_S^*}) \oplus C^1(\mathcal{U}_{D_X^*}, \mathcal{O}_{D_X^*})\\
&\qquad\qquad \oplus C^0(\Sigma t_X^*, \mathbb{C}_{\Sigma t_X^*}).
\end{aligned}$$

Since

$$\begin{aligned}
&X_{001} = X, \quad X_{010} = D_S^*, \quad X_{100} = \Sigma t_S,\\
&X_{011} = D_X^*, \quad X_{101} = \Sigma t_X, \quad X_{110} = \Sigma t_S^* \ \text{ and } X_{111} = \Sigma t_X^*,
\end{aligned}$$

we have

$$\begin{aligned}
D^{(0)}[1] &= \{d^{(0,0)*}_{(001),1} + d^{(0,0)*}_{(001),2} - \delta^{(0,0)}_{(001)}\}\\
&\qquad + \{d^{(0,0)*}_{(010),1} - d^{(0,0)*}_{(010),3} - \delta^{(0,0)}_{(010)}\} + \{-d^{(0,0)*}_{(100),3} + d^{(0,0)*}_{(100),2}\}\\
(4.15)' \qquad &= \{d^{(0,0)*}_{(001),1} - d^{(0,0)*}_{(100),3}\} + \{d^{(0,0)*}_{(001),2} - d^{(0,0)*}_{(010),3}\}\\
&\qquad + \{d^{(0,0)*}_{(010),1} - d^{(0,0)*}_{(100),2}\} - \delta^{(0,0)}_{(001)} - \delta^{(0,0)}_{(010)}
\end{aligned}$$

and

$$\begin{aligned}
D^{(1)}[1] &= \{d^{(0,1)*}_{(001),2} - \delta^{(0,1)}_{(001)}\} + \{-d^{(0,1)*}_{(010),3} - \delta^{(0,1)}_{(010)}\}\\
&\qquad + \{d^{(0,0)*}_{(011),1} + \delta^{(0,0)}_{(011)}\} - d^{(0,0)*}_{(101),2} + d^{(0,0)*}_{(110),3}\\
(4.15)'' \qquad &= \{d^{(0,1)*}_{(001),2} - d^{(0,1)*}_{(010),3}\} + \{d^{(0,0)*}_{(011),1} - d^{(0,0)*}_{(101),2} + d^{(0,0)*}_{(110),3}\}\\
&\qquad - \delta^{(0,0)}_{(001)} - \delta^{(0,0)}_{(010)} + \delta^{(0,0)}_{(011)},
\end{aligned}$$

where

$$\begin{aligned}
&d^{(0,0)*}_{(001),1} - d^{(0,0)*}_{(100),3} : C^0(\mathcal{U}_X, \mathcal{O}_X) \oplus C^0(\Sigma t_S, \mathbb{C}_{\Sigma t_S}) \to C^0(\Sigma t_X, \mathbb{C}_{\Sigma t_X})\\
&d^{(0,0)*}_{(001),2} - d^{(0,0)*}_{(010),3} : C^0(\mathcal{U}_X, \mathcal{O}_X) \oplus C^0(D_S^*, \mathcal{O}_{D_S^*}) \to C^0(D_X^*, \mathcal{O}_{D_X^*})\\
&d^{(0,0)*}_{(010),1} - d^{(0,0)*}_{(100),2} : C^0(\mathcal{U}_{D_S^*}, \mathcal{O}_{D_S^*}) \oplus C^0(\Sigma t_S, \mathbb{C}_{\Sigma t_S}) \to C^0(\Sigma t_S^*, \mathbb{C}_{\Sigma t_S^*})
\end{aligned}$$

$$d^{(0,1)*}_{(001),2} - d^{(0,1)*}_{(010),3} : C^1(\mathcal{U}_X, \mathcal{O}_X) \oplus C^1(\mathcal{U}_{D^*_S}, \mathcal{O}_{D^*_S}) \to C^1(\mathcal{U}_{D^*_X}, \mathcal{O}_{D^*_X})$$

$$d^{(0,0)*}_{(011),1} - d^{(0,0)*}_{(101),2} + d^{(0,0)*}_{(110),3} : C^0(\mathcal{U}_{D^*_X}, \mathcal{O}_{D^*_X}) \oplus C^0(\Sigma t_X, \mathbb{C}_{\Sigma t_X}) \oplus C^0(\Sigma t^*_S, \mathbb{C}_{\Sigma t^*_S}) \to C^0(\Sigma t^*_X, \mathbb{C}_{\Sigma t^*_X})$$

are the alternative sums of the maps defined to be pulling back cochians, and

$$\delta^{(0,0)}_{(001)} : C^0(\mathcal{U}_X, \mathcal{O}_X) \to C^1(\mathcal{U}_X, \mathcal{O}_X),$$

$$\delta^{(0,0)}_{(010)} : C^0(\mathcal{U}_{D^*_S}, \mathcal{O}_{D^*_S}) \to C^1(\mathcal{U}_{D^*_S}, \mathcal{O}_{D^*_S})$$

$$\delta^{(0,1)}_{(001)} : C^1(\mathcal{U}_X, \mathcal{O}_X) \to C^2(\mathcal{U}_X, \mathcal{O}_X),$$

$$\delta^{(0,1)}_{(010)} : C^1(\mathcal{U}_{D^*_S}, \mathcal{O}_{D^*_S}) \to C^2(\mathcal{U}_{D^*_S}, \mathcal{O}_{D^*_S}), \text{ and}$$

$$\delta^{(0,0)}_{(011)} : C^0(\mathcal{U}_{D^*_X}, \mathcal{O}_{D^*_X}) \to C^1(\mathcal{U}_{D^*_X}, \mathcal{O}_{D^*_X})$$

are the Cěch coboundary maps. The quotient space of

$$Ker\, D^{(1)}[1] \cap \{C^0(\mathcal{U}_{D^*_X}, \mathcal{O}_{D^*_X}) \oplus C^0(\Sigma t_X, \mathbb{C}_{\Sigma t_X}) \oplus C^0(\Sigma t^*_S, \mathbb{C}_{\Sigma t^*_S})\}$$

devided by

$$Im\, D^{(0)}[1] \cap \{C^0(\mathcal{U}_{D^*_X}, \mathcal{O}_{D^*_X}) \oplus C^0(\Sigma t_X, \mathbb{C}_{\Sigma t_X}) \oplus C^0(\Sigma t^*_S, \mathbb{C}_{\Sigma t^*_S})\}$$

is a subspace of $Ker\, D^{(1)}/Im\, D^{(0)}$ which is isomorphic to

$$\frac{Ker\{H^0(D^*_X, \mathcal{O}_{D^*_X}) \oplus H^0(\Sigma t_X, \mathbb{C}_{\Sigma t_X}) \oplus H^0(\Sigma t^*_S, \mathbb{C}_{\Sigma t^*_S}) \to H^0(\Sigma t^*_X, \mathbb{C}_{\Sigma t^*_X}\}}{Im\{H^0(X, \mathcal{O}_X) \oplus H^0(D^*_S, \mathcal{O}_{D^*_S}) \oplus H^0(\Sigma t_S, \mathbb{C}_{\Sigma t_S})\}}$$

There exists a homomorphism from $Ker\, D^{(1)}/Im\, D^{(0)}$ to

$$Ker\, \{H^1(X, \mathcal{O}_X) \oplus H^1(D^*_S, \mathcal{O}_{D^*_S}) \to H^1(D^*_X, \mathcal{O}_{D^*_X})\}$$

which is defined by assigning

$$[\varphi^{(1)}_X, \varphi^{(1)}_{D^*_S}, \varphi^{(0}_{D^*_X}, c_{\Sigma t_X}, c_{\Sigma t^*_S}] \in Ker\, D^{(1)}[1]/Im\, D^{(0)}[1]$$

to $([\varphi^{(1)}_X], [\varphi^{(1)}_{D^*_S}]) \in H^1(X, \mathcal{O}_X) \oplus H^1(D^*_S, \mathcal{O}_{D^*_S})$, where

$$(\varphi^{(1)}_X, \varphi^{(1)}_{D^*_S}, \varphi^{(0}_{D^*_X}, c_{\Sigma t_X}, c_{\Sigma t^*_S}) \in Ker\, D^{(1)}[1].$$

Note that $Ker\, D^{(1)}[1]$ is a subspace of

$$C^1(\mathcal{U}_X, \mathcal{O}_X) \oplus C^1(\mathcal{U}_{D^*_S}, \mathcal{O}_{D^*_S}) \oplus C^0(\mathcal{U}_{D^*_X}, \mathcal{O}_{D^*_X}) \oplus C^0(\Sigma t_X, \mathbb{C}_{\Sigma t_X}) \oplus C^0(\Sigma t^*_S, \mathbb{C}_{\Sigma t^*_S})$$

Assume that the image of

$$[\varphi^{(1)}_X, \varphi^{(1)}_{D^*_S}, \varphi^{(0}_{D^*_X}, c_{\Sigma t_X}, c_{\Sigma t^*_S}] \text{ in } H^1(X, \mathcal{O}_X) \oplus H^1(D^*_S, \mathcal{O}_{D^*_S})$$

is zero. Then there exists an element

$$(\psi^{(0)}_X, \psi^{(0)}_{D^*_S}) \in C^0(\mathcal{U}_X, \mathcal{O}_X) \oplus C^0(\mathcal{U}_{D^*_S}, \mathcal{O}_{D^*_S})$$

such that $\delta\psi^{(0)}_X = \varphi^{(1)}_X$ and $\delta\psi^{(0)}_{D^*_S} = \varphi^{(1)}_{D^*_S}$. Now, we have

$$\begin{aligned}&(\varphi^{(1)}_X, \varphi^{(1)}_{D^*_S}, \varphi^{(0)}_{D^*_X}, c_{\Sigma t_X}, c_{\Sigma t^*_S}) + D^{(0)}[1]((\psi^{(0)}_X, \psi^{(0)}_{D^*_S}))\\ &= (0, 0, \varphi^{(0)}_{D^*_X} + \nu^*_X\psi^{(0)}_X - g^*\psi_{D^*_S},\ c_{\Sigma t_X} + \psi^{(0)}_{X|\Sigma t_X},\ c_{\Sigma t_S} + \psi^{(0)}_{D^*_S|\Sigma t^*_S}),\end{aligned}$$

and

$$\begin{aligned}&D^{(1)}[1]((0, 0, \varphi^{(0)}_{D^*_X} + \nu^*_X\psi^{(0)}_X - g^*\psi_{D^*_S},\ c_{\Sigma t_X} + \psi^{(0)}_{X|\Sigma t_X},\ c_{\Sigma t_S} + \psi^{(0)}_{D^*_S|\Sigma t_S})\\ &= D^{(1)}[1]((\varphi^{(1)}_X, \varphi^{(1)}_{D^*_S}, \varphi^{(0)}_{D^*_X}, c_{\Sigma t_X}, c_{\Sigma t^*_S})) + D^{(1)}[1]D^{(0)}[1]((\psi^{(0)}_X, \psi^{(0)}_{(D^*_S})) = 0\end{aligned}$$

That is, $(\varphi^{(1)}_X, \varphi^{(1)}_{D^*_S}, \varphi^{(0)}_{D^*_X}, c_{\Sigma t_X}, c_{\Sigma t^*_S})$ is cohomologus to an element of

$$Ker\, D^{(1)}[1] \cap \{C^0(\mathcal{U}_{D^*_X}, \mathcal{O}_{D^*_X}) \oplus C^0(\Sigma t_X, \mathbb{C}_{\Sigma t_X}) \oplus C^0(\Sigma t^*_S, \mathbb{C}_{\Sigma t^*_S})\}$$

in $Ker\, D^{(1)}[1]/Im\, D^{(0)}[1] \simeq \mathbb{H}^1(\mathcal{O}_{X_\bullet}[1])$. Consequently, we have the exact sequence in (ii) in the proposition.

Q.E.D.

4.7 Remark In the exact sequence in (ii) in the proposition above, by use of the Cěch cohomology, the image of $\mathbb{H}^1(\mathcal{O}_{X_\bullet}[1])$ in $Ker\{H^1(X, \mathcal{O}_X) \oplus H^1(D^*_S, \mathcal{O}_{D^*_S}) \to H^1(D^*_X, \mathcal{O}_{D^*_X})\}$ is described as a subspace consisting of every element

$$(\varphi^{(1)}_X, \varphi^{(1)}_{D^*_S}) \in \check{\mathrm{H}}^1(\mathcal{U}_X, \mathcal{O}_X) \oplus \check{\mathrm{H}}^1(\mathcal{U}_{D^*_S}, \mathcal{O}_{D^*_S})$$

such that there exists an element

$$(\varphi^{(0)}_{D^*_X}, c_{\Sigma t_X}, c_{\Sigma t^*_S}) \in C^0(\mathcal{U}_{D^*_X}, \mathcal{O}_{D^*_X}) \oplus C^0(\Sigma t_X, \mathbb{C}_{\Sigma t_X}) \oplus C^0(\Sigma t^*_S, \mathbb{C}_{\Sigma t^*_S})$$

which satisfies the conditions

$$(i)\ \delta\varphi^{(0)}_{D^*_X} = \nu^*_X\varphi^{(1)}_X - g^*\varphi^{(1)}_{D^*_S},$$

$$(ii)\ \varphi^{(0)}_{D^*_X|\Sigma t_X} - \nu^*_X c_{\Sigma t^*_X} + g^* c_{\Sigma t^*_S} = 0.$$

Assume that the map

$$H^0(X, \Theta_X(-log D_X)) \oplus H^0(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)) \to H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

in Theorem 3.26, (ii) in Part I is surjective. Then, by the same theorem, any cohomology class of $H^1(S, \Theta(b_\bullet)) \simeq H^1(S, \Theta_S)$ is represented as a pair of cohomology classes

$$([\theta_X], [\theta_{D_S^*}]) \in H^1(X, \Theta_X(-log\, D_X)) \oplus H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*))$$

which satisfies the condition that

$$\widehat{\omega\nu_X^*}([\theta_X]) = \widehat{\omega g^*}([\theta_{(D_S^*}]) \text{ in } H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*)).$$

We denote by $[\theta'_X]$ (resp. $[\theta'_{D_S^*}]$) the image of

$$[\theta_X] \text{ (resp. } [\theta_{D_S^*}]) \text{ in } H^1(X, \Theta_X) \quad \text{(resp. in } H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^*))).$$

We denote by $\tau^{(2)}$ (resp. $\tau^{(1)}$) the composite of the homomorphism τ in (4.1) and the projection to the factor

$$Hom_{\mathbb{C}}(\mathbb{H}^0(\Omega^2_{X_\bullet}[1]), \mathbb{H}^1(\Omega^1_{X_\bullet}[1])) \quad \text{(resp. } Hom_{\mathbb{C}}(\mathbb{H}^0(\Omega^1_{X_\bullet}[1]), \mathbb{H}^1(\mathcal{O}_{X_\bullet}[1])).$$

4.8 Lemma.

(i) *The condition* $\tau^{(2)}(([\theta_X], [\theta_{D_S^*}])) = 0$ *is equivalent to the combination of the following two conditions:*

(a) *The homomorphism*

$$[\theta'_X] \lfloor : H^0(X, \Omega^2_X) \to Ker\{H^1(X, \Omega^1_X) \to H^1(D_X^*, \Omega^1_{D_X^*})\}$$

defined by taking cup-product of each element of $H^0(X, \Omega^2_X)$ *with* $[\theta'_X]$ *through contraction "*$\lfloor$*" is zero map.*

(b) *The homomorphism*

$$\begin{aligned}\tau^{(2)}(([\theta_X], [\theta_{D_S^*}])' &: H^0(X, \Omega^2_X) \\ &\to H^0(D_X^*, \Omega^1_{D_X^*})/Im\{H^0(X, \Omega^1_X) \oplus H^0(D_S^*, \Omega^1_{D_S^*})\}\end{aligned}$$

induced by $\tau^{(2)}(([\theta_X], [\theta_{D_S^*}])$ *when the condition* (a) *above is satisfied (cf. Proposition 4.6,* (i), (ii)), *is zero map.*

(ii) *The condition* $\tau^{(1)}(([\theta_X], [\theta_{D_S^*}])) = 0$ *is equivalent to the combination of the following two conditions:*

(c) *The homomorphism*

$$([\theta'_X],[\theta'_{D_S^*}])\lfloor : Ker\{H^0(\Omega^1_X)\oplus H^0(\Omega^1_{D_S^*})\to H^0(\Omega^1_{D_X^*})\}$$
$$\to Ker\{H^1(\mathcal{O}_X)\oplus H^1(\mathcal{O}_{D_S^*})\to H^1(\mathcal{O}_{D_X^*})\}$$

defined by taking cup-product of a cohomology class of $Ker\{H^0(\Omega^1_X)\oplus H^0(\Omega^1_{D_S^*}) \to H^0(\Omega^1_{D_X^*})\}$ *with the pair of cohomology classes* $([\theta'_X],[\theta'_{D_S^*}])$ *through contraction "*$\lfloor$*" is zero map.*

(d) *The homomorphism*

$$\tau^{(1)}(([\theta_X],[\theta_{D_S^*}])' : Ker\{H^0(\Omega^1_X)\oplus H^0(\Omega^1_{D_S^*})\to H^0(\Omega^1_{D_X^*})\}$$
$$\to Ker\{H^0(\mathcal{O}_{D_X^*})\oplus H^0(\mathbb{C}_{\Sigma t_X})\oplus H^0(\mathbb{C}_{\Sigma t_S^*})\to H^0(\mathbb{C}_{\Sigma t_X^*}\}$$
$$/Im\{H^0(\mathcal{O}_X)\oplus H^0(\mathcal{O}_{D_S^*})\oplus H^0(\mathbb{C}_{\Sigma t_S^*})\}$$

induced by $\tau^{(1)}(([\theta_X],[\theta_{D_S^*}]))$ *when the condition* (c) *above is satisfied* (*cf. Proposition 4.6,* (iii), (iv)), *is zero map.*

Proof. The essential part of the proof of the lemma is to prove that $[\theta'_X]\lfloor\Omega_X$ really belongs to $Ker\{H^1(X,\Omega^1_X)\to H^1(D_X^*,\Omega^1_{D_X^*})\}$ for any $\Omega_X\in H^0(X,\Omega^2_X)$, a holomorphic 2 forms on X, and that $([\theta'_X],[\theta'_{D_S^*}])\lfloor(\omega_X,\omega_{D_S^*})$ really belongs to

$$Ker\{H^1(\mathcal{O}_X)\oplus H^1(\mathcal{O}_{D_S^*})\to H^1(\mathcal{O}_{D_X^*})\}. \tag{4.16}$$

for any $(\omega_X,\omega_{D_S^*})$, a pair of holomorphic 1-forms on X and D_S^* satisfying the condition $\nu_X*\omega_X = g^*\omega_{D_S^*}$ If the above facts are proved, the lemma follows from Proposition 4.6 and Theorem 3.17 in Part I. In order to prove these facts, we choose Stein coverings $\mathcal{U}_X=\{W_j,W_\lambda\}$ of X, $\mathcal{U}_{D_X^*}=\{V_j,V_\lambda^{(0)},V_\lambda^{(1)}\}$ of D_X^* and $\mathcal{U}_{D_S^*}=\{U_\alpha\}$ of D_S^*, and calculate various Cěch cohomology calsses with respect to these coverings. The coverings $\mathcal{U}_X=\{W_j,W_\lambda\}$, $\mathcal{U}_{D_X^*}=\{V_j,V_\lambda^{(0)},V_\lambda^{(1)}\}$ and $\mathcal{U}_{D_S^*}=\{U_\alpha\}$ are chosen as follows:

(4.17)

The covering $\mathcal{U}_X=\{W_j,W_\lambda\}$ is such one of X that consists of polycylinders having the following properties:

(1) $W_j\cap\Sigma t_X=\emptyset$, $W_\lambda\cap\Sigma t_X\neq\emptyset$
(2) $\nu_X^{-1}(W_\lambda) = V_\lambda^{(0)}\cup V_\lambda^{(1)}$, $V_\lambda^{(0)}\cap V_\lambda^{(1)}\neq\emptyset$, where $\nu_X: D_X^*\to X$ is the cpmposite of the normalization $n_X: D_X^*\to D_X$ and the inclusion $D\hookrightarrow X$.
(3) If $W_j\cap D_X\neq\emptyset$, then D_X has the equation $y_j=0$ on W_j where x_j, y_j are local coordinates on W_j.
(4) If D_X has the equation $x_\lambda y_\lambda=0$ on W_λ where x_λ, y_λ are local coordinates on W_λ.
(5) If $W_j\cap D_X\neq\emptyset$, then the map $\nu_X: D_X^*\to X$ is given by $u_j\to(x_j,y_j)=(u_j,0)$ on $V_j=\nu_X^{-1}(W_j)$, where u_j is a local coordinate on V_j.

(6) On $V_\lambda^{(0)}$ (resp. $V_\lambda^{(1)}$), the map $\nu_X : D_X^* \to X$ is given by $u_\lambda^{(0)} \to (x_\lambda, y_\lambda) = (u_\lambda^{(0)}, 0)$ (resp. $u_\lambda^{(1)} \to (x_\lambda, y_\lambda) = (0, u_\lambda^{(1)})$), where $u_\lambda^{(0)}$ (resp. $u_\lambda^{(1)}$) is a local coordinate on $V_\lambda^{(0)}$ (resp. on $V_\lambda^{(1)}$).

We define $\mathcal{U}_{D_X^*} := \{V_j, V_\lambda^{(0)}, V_\lambda^{(1)}\}$, which is a Stein covering of D_X^*. The covering $\mathcal{U}_{D_S^*} = \{U_\alpha\}$ is such a Stein covering of D_S^* for which $\mathcal{U}_{D_X^*} = \{g^{-1}(U_\alpha)\}$, where $g : D_X^* \to D_S^*$ is the lifting of the map $f_{|D_X} : D_X \to D_S$.

(i) By the definition of $[\theta'_X]$, there exists a Cěch 1-cocycle

$$\{\theta_{ij}\} \in Z^1(\mathcal{U}_X, \Theta_X(-log\, D_X)) \tag{4.18}$$

such that the cohomology class $[\theta'_X]$ is represented by the image of $\{\theta_{ij}\}$ in the Cěch cohomology group $\check{\mathrm{H}}^1(\mathcal{U}_X, \Theta_X)$. Let $\{\Omega_i\} \in Z^0(\mathcal{U}_X, \Omega_X^2)$ be a Cěch 0-cocycle which represents $\Omega_X \in H^0(X, \Omega_X^2)$. Then, $[\theta'_X]\lfloor\Omega_X \in H^0(X, \Omega_X^2)$ is represented by the Cěch 1-cocycle $\{\theta_{ij}\lfloor\Omega_j\}$ in $\check{\mathrm{H}}^1(\mathcal{U}_X, \Omega_X^1)$. Since θ_{ij} is tangent to X, it is represented as

$$\theta_{ij} = a_{ij}(\frac{\partial}{\partial x_j}) + b_{ij}y_j(\frac{\partial}{\partial y_j}) \tag{4.19}$$

$$(\text{resp. } \theta_{i\lambda} = a_{i\lambda}x_\lambda(\frac{\partial}{\partial x_\lambda}) + b_{i\lambda}y_\lambda(\frac{\partial}{\partial y_\lambda}))$$

on each $W_i \cap W_j$ (resp. $W_i \cap W_\lambda$, where a_{ij} and b_{ij} (resp. $a_{i\lambda}$ and $b_{i\lambda}$) are holomorphic functions on $W_i \cap W_j$ (resp. $W_i \cap W_\lambda$). Therefore, if we represent Ω_j (resp. Ω_λ) as

$$\Omega_j = c_j\, dx_j \wedge dy_j \quad \Omega_\lambda = c_\lambda\, dx_\lambda \wedge dy_\lambda$$

on each W_j (resp. W_λ), where c_j (resp. c_λ) is a holomorphic function on W_j (resp. W_λ). Then we have

$$\theta_{ij}\lfloor\Omega_j = a_{ij}c_j\, dy_j - b_{ij}c_jy_jdx_j$$

$$(\text{resp. } \theta_{i\lambda}\lfloor\Omega_\lambda = a_{i\lambda}c_\lambda\, dy_\lambda - b_{i\lambda}c_\lambda y_\lambda dx_\lambda).$$

Hence

$$\nu_X^*\theta_{ij}\lfloor\Omega_j = 0, \quad (\text{resp. } \nu_X^{(0)*}\theta_{i\lambda}\lfloor\Omega_\lambda = 0, \quad \nu_X^{(1)*}\theta_{i\lambda}\lfloor\Omega_\lambda = 0).$$

as required. Next we shall prove that $([\theta'_X], [\theta'_{D_S^*}])\lfloor(\omega_X, \omega_{D_S^*})$ belongs to the space in (4.16) for any $(\omega_X, \omega_{D_S^*})$, a pair of holomorphic 1-forms on X and D_S^* satisfying the condition $\nu_X * \omega_X = g^*\omega_{D_S^*}$. We represent $[\theta'_X]$ by a Cěch 1-cocycle $\{\theta_{ij}\}$ in (4.18) and (4.19) as before, and $[\theta'_{D_S^*}]$ by a Cěch 1-cycle

$$\{\xi_{ij}\} \in Z^1(\mathcal{U}_{D_S^*}, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)).$$

Since $\widehat{\omega\nu_X^*}([\theta_X]) = \widehat{\omega g^*}([\theta_{(D_S^*}])$ in $H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_S^*))$, there exists a 0-cochain

$$\{\theta_\alpha^*\} \in C^0(\mathcal{U}_{D_X^*}, \Theta_{D_X^*}(-\Sigma t_S^*)) \tag{4.20}$$

such that

$$\theta_\alpha^* - \theta_\beta^* = \omega\nu_X(\theta_{\nu_{X*}(\alpha)\nu_{X*}(\beta)}) - \omega g(\xi_{g_*(\alpha)g_*(\beta)}) \tag{4.21}$$

for any $\alpha, \beta \in I_{D_X^*}$ with $V_\alpha \cap V_\beta \neq \emptyset$, where $\nu_{X*} : I_{D_X^*} \longrightarrow I_X$ (resp. $g_* : I_{D_X^*} \longrightarrow I_{D_S^*}$) is the map between the index sets of the coverings $\mathcal{U}_{D_X^*}$ and $\mathcal{U}_{D_X}$ (resp. $\mathcal{U}_{D_X^*}$ and $\mathcal{U}_{D_S^*}$). We represent ω_X by a Cěch 0-cocycle $\{\omega_{X,i}\}$ in $\{\Omega_i\} \in Z^0(\mathcal{U}_X, \Omega_X^1)$, and $\omega_{D_S^*}$ by a Cěch 0-cocycle

$$\{\omega_{D_S^*}, i\} \in Z^0(\mathcal{U}_{D_S^*}, \Omega^1_{D_S^*}) \tag{4.22}$$

We put

$$\omega_\alpha^* := \nu_X^*\omega_{X,\nu_X^*(\alpha)} = g^*\omega_{D_S^*, g_*(\alpha)} \quad \alpha \in I_{D_X^*}, \text{ and}$$

$$\omega_{D_X^*} := \{\omega_\alpha^*\} \in Z^0(\mathcal{U}_{D_X^*}, \Omega^1_{D_X^*}).$$

Note that the cohomology class $[\theta'_X]\lfloor\Omega_X$ (resp. $[\theta'_{D_S^*}]\lfloor\omega_{D_S^*}$) is represented by $\{\theta_{\nu_{X*}(\alpha)\nu_{X*}(\beta)}\lfloor\Omega_{X,\nu_{X*}(\beta)}\}$ (resp. $\xi_{g_*(\alpha)g_*(\beta)}\lfloor\omega_{D_S^*, g_*(\beta)}$) in $\check{\mathrm{H}}^1(\mathcal{U}_X, \mathcal{O}_X)$ (resp. $\check{\mathrm{H}}^1(\mathcal{U}_{D_S^*}, \mathcal{O}_{D_S^*})$. By (4.21), we have

$$\begin{aligned}
&\nu_X^*(\theta_{\nu_{X*}(\alpha)\nu_{X*}(\beta)}\lfloor\omega_{X,\nu_{X*}(\beta)}) - g^*(\xi_{g_*(\alpha)g_*(\beta)}\lfloor\omega_{g_*(\beta)}) \\
&= \omega\nu_X(\theta_{\nu_{X*}(\alpha)\nu_{X*}(\beta)})\lfloor\nu_X^*\omega_{X,\nu_X^*(\beta)} - \omega g(\xi_{g_*(\alpha)g_*(\beta)})\lfloor g^*\omega_{D_S^*, g_*(\beta)} \\
&= (\omega\nu_X(\theta_{\nu_{X*}(\alpha)\nu_{X*}(\beta)}) - \omega g(\xi_{g_*(\alpha)g_*(\beta)}))\omega_\beta^* \\
&= \theta_\alpha^*\lfloor\omega_\alpha^* - \theta_\beta^*\lfloor\omega_\beta^*,
\end{aligned}$$

where $\omega\nu_X$ (resp. ωg) is the map

$$\Gamma(W_{\nu_{X^*}(\alpha)} \cap W_{\nu_{X^*}(\beta)}, \Theta_X(-logX)) \to \Gamma(V_\alpha \cap V_\beta, \Theta_{D_X^*}(-\Sigma t_X^*))$$
$$(\text{resp. } \Gamma(U_{g_*(\alpha)} \cap U_{g_*(\beta)}, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)) \to \Gamma(V_\alpha \cap V_\beta, \Theta_{D_X^*}(-\Sigma t_X^*)))$$

Therefore, since $\{\theta_\alpha^*\lfloor\omega_\alpha^*\} \in C^0(\mathcal{U}_{D_X^*}, \mathcal{O}_{D_X^*})$, we conclude that the cohomology class

$$([\theta'_X]\lfloor\omega_X, [\theta'_{D_S^*}]\lfloor\omega_{D_S^*}) \in H^1(X, \mathcal{O}_X) \oplus H^1(D_S^*, \mathcal{O}_{D_S^*})$$

belongs to the space in (4.16).

Q.E.D.

4.9 Remark In case X is regular,i.e., $H^0(X, \Omega^1) = 0$, the assertion (ii) of Lemma 4.8 is meaningless. Because, if $H^0(X, \Omega^1) = 0$, then

$$Ker\{H^0(\Omega^1_X) \oplus H^0(\Omega^1_{D^*_X}) \longrightarrow H^0(\Omega^1_{D^*_X})\} = 0,$$

since the map $H^0(\Omega^1_{D^*_S}) \longrightarrow H^0(\Omega^1_{D^*_X})$ is injective.

4.10 Theorem. *The map*

$$\tau^{(2)} : H^1(S, \Theta_S) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^0(\Omega^2_{X_\bullet}[1]), \mathbb{H}^1(\Omega^1_{X_\bullet}[1]))$$

is injective if all of the following conditions are satisfied:

(i) *The map*

$$H^0(X, \Theta_X(-logD_X)) \oplus H^0(D^*_S, \Theta_{D^*_S}(-\Sigma c^*_S - \Sigma t^*_S)) \longrightarrow H^0(D^*_X, \Theta_{D^*_X}(-\Sigma t^*_X))$$

(cf. Theorem 3.26, (ii) in Part I) *is surjective.*

(ii) *The infinitesimal Torelli concerning the cohomology $H^2(X, \mathbb{C})$ holds for X, that is, the homomorphism*

$$H^1(X, \Theta_X) \longrightarrow Hom_{\mathbb{C}}(H^0(X, \Omega^2_X), H^1(X, \Omega^1_X))$$

defined by taking cup-product of each element of $H^1(X, \Theta_X)$ with that of $H^0(X, \Omega^2_X)$ through contraction "$\lfloor$" is injective on the image of $H^1(S, \Theta_S)$ in $H^1(X, \Theta_X)$.

(iii) *The homomorphism*

(4.23)

$$\begin{aligned} \overline{\mu}^{(2)} &: H^0(D_X, \mathcal{N}_{D_X/X})/Im\,\{H^0(X, \Theta_X) \longrightarrow H^0((D_X, \mathcal{N}_{D_X/X})\} \\ &\longrightarrow Hom_{\mathbb{C}}(H^0(X, \Omega^2_X), H^0(D^*_X, \Omega^1_{D^*_X})/Im\,\{H^0(D^*_S, \Omega^1_{D^*_S}) \oplus H^0(X, \Omega^1_X)\} \end{aligned}$$

defined by taking contraction and pull-back is injective where

$$\mathcal{N}_{D_X/X} := \Theta_X/\Theta_X(-log\,D_X)$$

.

Proof. By the condition (i), any element of $H^1(S, \Theta_S)$ can be represented by a pair of cohomology classes

$$([\theta_X], [\theta_{D^*_S}]) \in H^1(X, \Theta_X(-logD_X)) \oplus H^1(D^*_S, \Theta_{D^*_S}(-\Sigma c^*_S - \Sigma t^*_S))$$

which satisfies the condition

$$\widehat{\omega\nu_X}^*([\theta_X]) = \widehat{\omega g}^*([\theta_{(D^*_S}]) \text{ in } H^1(D^*_X, \Theta_{D^*_X}(-\Sigma t^*_S))$$

(cf. Theorem 3.26, (ii) in Part I), where $\widehat{\omega\nu_X}^*$ (resp. $\widehat{\omega g}^*$) denotes the map

$$H^1(X, \Theta_X(-logD_X)) \to H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$
$$(\text{resp. } H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)) \to H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

induced from the sheaf homiomorphism

$$\Theta_X(-logD_X) \to \nu_{X*}\Theta_{D_X^*}(-\Sigma t_X^*)$$
$$(\text{resp. } \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*) \to g_*\Theta_{D_X^*}(-\Sigma t_X^*))$$

(cf. Theorem 3.19 in Part I). What we have to show is that if $\tau^{(2)}(([\theta_X], [\theta_{D_S^*}])) = 0$ in $Hom_{\mathbb{C}}(\mathbb{H}^0(\Omega^2_{X_\bullet}[1]), \mathbb{H}^1(\Omega^1_{X_\bullet}[1]))$, and if the condition (ii), (iii) are fullfilled, then

$$[\theta_X] = 0 \text{ in } H^1(X, \Theta_X(-logD_X)), \text{ and}$$
$$[\theta_{D_S^*}] = 0 \text{ in } H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)).$$

First, we note that if we could prove that

$$[\theta_X] = 0 \text{ in } H^1(X, \Theta_X(-logD_X)),$$

it follows from Proposition 4.1 that $[\theta_{D_S^*}] = 0$ in $H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*))$, since $\widehat{\omega g}^*([\theta_{(D_S^*}]) = \widehat{\omega\nu_X}^*([\theta_X]) = 0$ in $H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_S^*))$. In what follows we shall prove that $[\theta_X] = 0$ in $H^1(X, \Theta_X(-logD_X))$ if $\tau^{(2)}(([\theta_X], [\theta_{D_S^*}])) = 0$ in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^0(\Omega^2_{X_\bullet}[1]), \mathbb{H}^1(\Omega^1_{X_\bullet}[1]))$ and all of the conditions in the theorem are satisfied. If $\tau^{(2)}(([\theta_X], [\theta_{D_S^*}])) = 0$, by Lemma 4.8 (i), the homomorphism

$$[\theta'_X]\llcorner : H^0(X, \Omega^2_X) \to Ker\{H^1(X, \Omega^1_X) \to H^1(D_X^*, \Omega^1_{D_X^*})\}$$

defined by taking *cup-product* of each element of $H^0(X, \Omega^2)$ with the cohomology class $[\theta'_X]$ through *contraction* is zero map. Then by the condition (ii) of the theorem, we conclude that $[\theta'_X] = 0$ in $H^1(X, \Theta_X)$. Then, by the long exact sequence of cohomology groups derived from the short exact sequence of $\mathcal{O}_X$-modules

$$(4.24) \qquad 0 \to \Theta_X(-log\, D_X) \to \Theta_X \to \mathcal{N}_{D_X/X} \to 0,$$

there exists an element $s_{D_X} \in H^0(D_X, \mathcal{N}_{D_X/X})$ whose image in $H^1(X, \Theta_X(-log\, D_X)$ is θ_X. We denote by $\overline{s_{D_X}}$ the image of s_{D_X} in

$$H^0(D_X, \mathcal{N}_{D_X/X})/Im\{H^0(X, \Theta_X) \to H^0(D_X, \mathcal{N}_{D_X/X})\}.$$

If we regard this quotient space as a subspace of $H^1(X, \Theta_X(-log\, D_X))$, then we have:

Claim.

$$\overline{\mu^{(2)}(\overline{s_{D_X}})}(\Omega_X) = \tau^{(2)}(([\theta_X],[\theta_{D_S^*}]))'(\Omega_X)$$

for any $\Omega_X \in H^0(X,\Omega_X^2) \simeq \mathbb{H}^0(\Omega^2_{X_\bullet}[1])$. (*For the notation* $\tau^{(2)}(([\theta_X],[\theta_{D_S^*}]))'$, *see* Lemma 4.8 (i) (b)).

Proof of Claim. We describe the various cohomology classes, using the Čech cohomology with respect to the coverings $\mathcal{U}_X = \{W_j, W_\lambda\}$ of X, $\mathcal{U}_{D_X^*} = \{V_j, V_\lambda^{(0)}, V_\lambda^{(1)}\}$ of D_X^* in (4.17). Let $[\theta_X]$ be represented by a Čech 1-cocycle

$$\{\theta_{ij}\} \in Z^1(\mathcal{U}_X, \Theta_X(-log\, D_X)).$$

We take an element Ω_X of $H^0(X,\Omega_X^2)$, and represent it by a Čech 0-cocycle

$$\{\Omega_i\} \in Z^0(\mathcal{U}_X, \Omega_X^2),$$

and represent s_{D_X} by a Čech 0-cocycle

$$\{s_i\} \in Z^0(D_X \cap \mathcal{U}_X, \mathcal{N}_{D_X/X}).$$

We take $\theta_j \in \Gamma(W_j, \Theta_X)$ (resp. $\theta_\lambda \in \Gamma(W_\lambda, \Theta_X)$) such that $\theta_j d_{j|D_X} = s_j$ (resp. $\theta_\lambda d_{\lambda|D_X} = s_\lambda$) for any j (resp. λ), where $d_j = 0$ (resp. $d_\lambda = 0$) is the defining equation of D_X in W_j (resp. W_λ). We define $\nu_X^*(s_{D_X} \lfloor \Omega_{X|D_X})$ by

$$\{\nu_X^*(\theta_j \lfloor \Omega_j), \nu_X^{(0)*}(\theta_\lambda \lfloor \Omega_\lambda), \nu_X^{(1)*}(\theta_\lambda \lfloor \Omega_\lambda)\} \in C^0(\mathcal{U}_{D_X^*}, \Omega^1_{D_X^*}),$$

which does not depend on the choice of $\{\theta_j, \theta_\lambda\}$. Indeed, if $\{\theta'_j, \theta'_\lambda\}$ is anther one such that $\theta'_j d_{j|D_X} = s_j$ (resp. $\theta'_\lambda d_{\lambda|D_X} = s_\lambda$) for any j (resp. λ). Then, by the exact sequence (4.23), $\theta_j - \theta'_j \in \Gamma(W_j, \Theta_X(-log\, D_X))$ (resp. $\theta_\lambda - \theta'_\lambda \in \Gamma(W_\lambda, \Theta_X(-log\, D_X))$. Hence $\theta_j - \theta'_j$ (resp. $\theta_\lambda - \theta'_\lambda$) can be represented as

$$\theta_j - \theta'_j = a_j(\frac{\partial}{\partial x_j}) + b_j y_j(\frac{\partial}{\partial y_j})$$

$$(\text{resp. } \theta_\lambda - \theta'_\lambda = a_\lambda x_\lambda(\frac{\partial}{\partial x_\lambda}) + b_\lambda y_\lambda(\frac{\partial}{\partial y_\lambda}))$$

on each W_j (resp. W_λ), where a_j and b_j (resp. a_λ and b_λ) are holomorphic functions on W_j (resp. W_λ). Therefore, if we represent Ω_j (resp. Ω_λ) as

$$\Omega_j = c_j\, dx_j \wedge dy_j \quad \Omega_\lambda = c_\lambda\, dx_\lambda \wedge dy_\lambda$$

on each W_j (resp. W_λ), where c_j (resp. c_λ) is a holomorphic function on W_j (resp. W_λ). Then we have

$$(\theta_j - \theta'_j) \lfloor \Omega_j = a_j c_j\, dy_j - b_j c_j y_j dx_j$$

$$(\text{resp. } (\theta_\lambda - \theta'_\lambda)\lfloor \Omega_\lambda = a_\lambda x_\lambda c_\lambda\, dy_\lambda - b_\lambda c_\lambda y_\lambda dx_\lambda).$$

Hence

$$\nu_X^*(\theta_j \lfloor \Omega_j) - \nu_X^*(\theta'_j \lfloor \Omega_j) = \nu_X^*((\theta_j - \theta'_j)\lfloor \Omega_j) = 0,$$

$$(\text{resp. } \nu_X^{(0)*}(\theta_\lambda \lfloor \Omega_\lambda) - \nu_X^{(0)*}(\theta_\lambda \lfloor \Omega_\lambda) = \nu_X^{(0)*}((\theta_\lambda - \theta'_\lambda)\lfloor \Omega_\lambda) = 0,$$

$$\nu_X^{(1)*}(\theta_\lambda \lfloor \Omega_\lambda) - \nu_X^{(1)*}(\theta_\lambda \lfloor \Omega_\lambda) = \nu_X^{(1)*}((\theta_\lambda - \theta'_\lambda)\lfloor \Omega_\lambda) = 0)$$

as required. Now, we show that

$$\nu_X^*(s_{D_X} \lfloor \Omega_{X|D_X}) \in Z^0(\mathcal{U}_{D_X^*}, \Omega^1_{D_X^*}). \tag{4.25}$$

Indeed, if $W_i \cap W_j \neq \emptyset$ (resp. $W_i \cap W_\lambda \neq \emptyset$), we have $\theta_{i|D_X} = \theta_{j|D_X}$ (resp. $\theta_{i|D_X} = \theta_{\lambda|D_X}$) on $W_i \cap W_j \neq \emptyset$ (resp. on $W_i \cap W_\lambda \neq \emptyset$), and so $\theta_i - \theta_j \in \Gamma(W_i \cap W_j, \Theta_X(-log\, D_X))$ (resp. $\theta_i - \theta_\lambda \in \Gamma(W_i \cap W_\lambda, \Theta_X(-log\, D_X))$ from which it follows that

$$\nu_X^*(\theta_i \lfloor \Omega_i) - \nu_X^*(\theta_j \lfloor \Omega_j) = \nu_X^*((\theta_j - \theta_j)\lfloor \Omega_j = 0,$$

$$(\text{resp. } \nu_X^{(0)*}(\theta_i \lfloor \Omega_i) - \nu_X^{(0)*}(\theta_\lambda \lfloor \Omega_\lambda) = \nu_X^{(0)*}((\theta_i - \theta_\lambda)\lfloor \Omega_\lambda) = 0,$$

$$\nu_X^{(1)*}(\theta_i \lfloor \Omega_i) - \nu_X^{(1)*}(\theta_i \lfloor \Omega_\lambda) = \nu_X^{(1)*}((\theta_i - \theta_\lambda)\lfloor \Omega_\lambda) = 0$$

on $V_i \cap V_j \neq \emptyset$ (resp. on $V_j \cap V_\lambda^{(\alpha)} \neq \emptyset$, $\alpha = 0, 1$). We denote by $\overline{\nu_X^*(s_{D_X} \lfloor \Omega_{X|D_X})}$ the image of $\nu_X^*(s_{D_X} \lfloor \Omega_{X|D_X})$ in

$$H^0(D_X^*, \Omega^1_{D_X^*})/Im\,\{H^0(X, \Omega^1_X) \oplus H^0(D_S^*, \Omega^1_{D_S^*}) \to H^0(D_X^*, \Omega^1_{D_X^*}) \tag{4.26}$$

Then, by the definition of $\overline{\mu}^{(2)}$ in (4.23), we have

$$\overline{\mu}^{(2)}(\overline{s_{D_X}})(\Omega_X) = \overline{\nu_X^*(s_{D_X} \lfloor \Omega_{X|D_X})} \tag{4.27}$$

for any $\Omega_X \in H^0(X, \Omega^2_X)$, where $\overline{s_{D_X}}$ is the image of s_{D_X} in

$$H^0(D_X, \mathcal{N}_{D_X/X})/Im\,\{H^0(X, \Theta_X) \to H^0(D_X, \mathcal{N}_{D_X/X})\}$$

We consider the element

$$(0, 0, \nu_X^*(s_{D_X} \lfloor \Omega_{X|D_X})) \in K^1(\Omega^1_{X_\bullet}[1])$$
$$:= C^1(\mathcal{U}_X, \Omega^1_X) \oplus C^1(\mathcal{U}_{D_S^*}, \Omega^1_{D_S^*}) \oplus C^0(\mathcal{U}_{D_X^*}, \Omega^1_{D_X^*})$$

By the definition of $D^{(1)}[1] : K^1(\Omega^1_{X_\bullet}[1]) \to K^2(\Omega^1_{X_\bullet}[1])$ (cf.(4.14)) and by (4.15), we have

$$D^{(1)}[1]((0, 0, \nu_X^*(s_{D_X} \lfloor \Omega_{X|D_X})) = (0, 0, \delta^{(1,0)}_{(011)}\nu_X^*(s_{D_X} \lfloor \Omega_{X|D_X})) = 0.$$

Hence $(0,0,\nu_X^*(s_{D_X}\lfloor\Omega_{X|D_X}))$ is a 1-cocycle of the complex $K^1(\Omega^1_{X_\bullet}[1])$. Next we consider the element

$$(\theta_X\lfloor\Omega_X,0,0)\in K^1(\Omega^1_{X_\bullet}[1]):=C^1(\mathcal{U}_X,\Omega^1_X)\oplus C^1(\mathcal{U}_{D_S^*},\Omega^1_{D_S^*})\oplus C^0(\mathcal{U}_{D_X^*},\Omega^1_{D_X^*})$$

where

$$\theta_X\lfloor\Omega_X:=\{\theta_{ij}\lfloor\Omega_j,\theta_{i\lambda}\lfloor\Omega_\lambda\}\in Z^1(\mathcal{U}_X,\Omega^1_X).$$

Since

$$\theta_{ij}\in\Gamma(W_i\cap W_j,\Theta_X(-log\,D_X))\ \ (\text{resp. } \theta_{i\lambda}\in\Gamma(W_i\cap W_\lambda,\Theta_X(-log\,D_X)),$$

we have $\nu_X^*(\theta_{ij}\lfloor\omega_j)=0$ (resp. $\nu_X^*(\theta_{i\lambda}\lfloor\omega_\lambda)=0$) on $V_i\cap V_j$ (resp. on $V_i\cap V_\lambda^{(\alpha)}$, $\alpha=0,1$). Then, by the definition of $D^{(1)}[1]:K^1(\Omega^1_{X_\bullet}[1])\longrightarrow K^2(\Omega^1_{X_\bullet}[1]$ again, we have

$$D^{(1)}[1]((\theta_X\lfloor\Omega_X,0,0))=(-\delta^{(1,1)}_{(011)}(\theta_X\lfloor\Omega_X),0,d^{(1,1)*}_{(001),2}(\theta_X\lfloor\Omega_X))=0,$$

since $d^{(1,1)*}_{(001),2}=\nu_X^*$. Therefore $(\theta_X\lfloor\Omega_X,0,0)$ also defines a 1-cocycle of the complex $K^\bullet(\Omega^1_{X_\bullet}[1])$ whose cohomology class $[(\theta_X\lfloor\Omega_X,0,0)]$ in $\mathbb{H}^1(\Omega^1_{X_\bullet}[1])$ is nothing but $\tau^{(2)}(([\theta_X],[\theta_{D_S^*}]))(\Omega_X)$, that is,

$$\tau^{(2)}(([\theta_X],[\theta_{D_S^*}]))(\Omega_X)=[(\theta_X\lfloor\Omega_X,0,0)].$$

We are now going to show that $\tau^{(2)}([\theta_X],[\theta_{D_S^*}])$ and $(0,0,\nu_X^*(s_{D_X}\lfloor\Omega_{X|D_X})$ are cohomologous in $\mathbb{H}^1(\Omega^1_\bullet[1])$ on the assumption that θ_X is the image of s_{D_X} in $H^1(X,\Theta_X(-log\,D_X))$. Here we should recall that we regard the quotient space

$$H^0(D_X,\mathcal{N}_{D_X/X})/Im\,\{H^0(X,\Theta_X)\longrightarrow H^0(D_X,\mathcal{N}_{D_X/X})\}$$

(cf. (4.23)) as a subspace of $H^1(X,\Theta_X(-log\,D_X))$. The image of s_{D_X} is represented by a 1-cocycle $\{\theta_i-\theta_j\}$ if we take the same $\{\theta_i,\theta_\lambda\}$ as before. Since the image of $s_{D_X}=\{s_i,s_\lambda\}$ is $\theta_X=\{\theta_{ij},\theta_{i\lambda}\}$ in $H^1(X,\Theta_X(-log\,D_X))$, there is a 0-cochain $\{\theta_i',\theta_\lambda'\}\in C^0(\mathcal{U}_X,\Theta_X(-log\,D_X))$ such that

$$\theta_{ij}-(\theta_i-\theta_j)=\theta_i'-\theta_j',\ \ \theta_{i\lambda}-(\theta_i-\theta_\lambda)=\theta_i'-\theta_\lambda'.$$

Hence we have

(4.28)

$$\begin{aligned}&D^{(0)}[1]((\{(\theta_i+\theta_i')\lfloor\Omega_i,(\theta_\lambda+\theta_\lambda')\lfloor\Omega_\lambda\},0,0))\\&=(-\delta^{(1,0)}_{(001)}\{(\theta_i+\theta_i')\lfloor\Omega_i,(\theta_\lambda+\theta_\lambda')\lfloor\Omega_\lambda\},0,d^{(1,0)*}_{(001),2}\{(\theta_i+\theta_i')\lfloor\Omega_i,(\theta_\lambda+\theta_\lambda')\lfloor\Omega_\lambda\}\\&=(-\theta_X\lfloor\Omega_X,0,\nu_X^*(s_{D_X}\lfloor\Omega_{X|D_X})\end{aligned}$$

$$= (0, 0, \nu_X^*(s_{D_X} \lfloor (\Omega_{X|D_X}))) - (\theta_X \lfloor \Omega_X, 0, 0)$$

Thus $(\theta_X \lfloor \Omega_X, 0, 0)$ and $(0, 0, \nu_X^*(s_{D_X} \lfloor (\Omega_{|D_X}))$ are cohomologous in $\mathbb{H}^1(\Omega^1_{X_\bullet}[1])$. Since the image of $(\theta_X \lfloor \Omega_X, 0, 0)$ in

$$Ker\{H^1(X, \Omega^1_X) \to H^1(D_X^*, \Omega^1_{D_X^*})\}$$

is zero, by the definition of $\tau^{(2)}(([\theta_X], [\theta_{D_S^*}])'$ (cf. Lemma 4.8 (i) (b)), (4.28) shows that

$$\tau^{(2)}(([\theta_X], [\theta_{D_S^*}])'(\Omega_X) = \overline{\mu}^{(2)}(\overline{s_{D_X}})(\Omega_X) = \overline{\nu_X^*(s_{D_X} \lfloor \Omega_{X|D_X})} \tag{4.29}$$

Then, it follows $\tau^{(2)}((\theta_X, \theta_{D_S^*}))'(\Omega_X) = \overline{\mu^{(2)}}(\overline{s_{D_X}})(\Omega_X)$.

q.e.d.for Claim

Since we have assumed $\tau^{(2)}((\theta_X, \theta_{D_S^*})) = 0$ in

$$\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^0(\Omega^2_{X_\bullet}[1], \mathbb{H}^1(\Omega^1_{X_\bullet}[1]))$$

, $\tau^{(2)}((\theta_X, \theta_{D_S^*}))'(\Omega_X) = 0$ for any $\Omega_X \in H^0(X, \Omega^2_X)$ $\simeq \mathbb{H}^0(\Omega^2_{X_\bullet}[1])$. Then

$$\overline{\mu^{(2)}}(\overline{s_{D_X}})(\Omega_X) = 0$$

for any $\Omega_X \in H^0(X, \Omega^2_X)$ by the claim. Then, by the condition (iii) of the theorem, we conclude that $\theta_X = \overline{s_{D_X}} = 0$.

Q.E.D.

4.11 Theorem. *For an algebraic surface S with ordinary singularities such that X is irregular, i.e., $q(X) = \dim_{\mathbb{C}} H^0(X, \Omega^1_X) > 0$, the map*

$$\tau^{(1)} : H^1(S, \Theta_X) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^0(\Omega^1_{X_\bullet}[1], \mathbb{H}^1\mathcal{O}_{X_\bullet}[1]))$$

is injective if all of the following conditions are satisfied:

(i) *The map*

$$H^0(X, \Theta_X(-log D_X)) \oplus H^0(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)) \longrightarrow H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

(cf. Theorem 3.26, (ii) in Part I) *is surjective.*

(ii) *The homomorhism*

$$Ker\,\{H^1(X,\Theta_X)\oplus H^1(D_S^*,\Theta_{D_S^*}(-\Sigma c_{D_S^*}))\to H^1(D_X^*,\Theta_{D_X^*})\}$$
$$\to \mathrm{Hom}_{\mathbb{C}}(Ker\,\{H^0(\Omega_X^1)\oplus H^0(\Omega_{D_S^*}^1)\to H^0(\Omega_{D_X^*}^1)\},$$
$$Ker\,\{H^1(\mathcal{O}_X)\oplus H^1(\mathcal{O}_{D_S^*}\to H^1(\mathcal{O}_{D_X^*})\}$$

defined by taking contraction is injective on the image of

$$Ker\,\{H^1(X,\Theta_X(-log D_X))\oplus H^1(D_S^*,\Theta_{D_S^*}(-\Sigma c_S^*-\Sigma t_S^*))$$
$$\to H^1(D_X^*,\Theta_{D_X^*}(-\Sigma t_X^*))$$

$$in\quad Ker\,\{H^1(X,\Theta_X)\oplus H^1(D_S^*,\Theta_{D_S^*}(-\Sigma c_S^*))\to H^1(D_X^*,\Theta_{D_X^*})\}$$

(iii) *The homomorphism* $\overline{\mu}^{(1)}$:

$$\frac{H^0(D_X,\mathcal{N})_{D_X/X}\oplus H^0(\Sigma t_S^*,N_{\Sigma t_S^*/D_S^*})}{Im\,\{H^0(X,\Theta_X)\oplus H^0(D_S^*,\Theta_{D_S^*}(-\Sigma c_S^*))\to H^1(D_X^*,\Theta_{D_X^*}(-\Sigma t_X^*))\}}$$
$$\to \mathrm{Hom}_{\mathbb{C}}(Ker\,\{H^0(\Omega_X^1)\oplus H^0(\Omega_{D_S^*}^1)\to H^0(\Omega_{D_X^*}^1)\},$$
$$\frac{Ker\{H^0(\mathcal{O}_{D_X^*})\oplus H^0(\mathbb{C}_{\Sigma t_X})\oplus H^0(\mathbb{C}_{\Sigma t_S^*})\to H^0(\mathbb{C}_{\Sigma t_X^*})\}}{Im\{H^0(\mathcal{O}_X)\oplus H^0(\mathcal{O}_{D_S^*})\oplus H^0(\mathbb{C}_{\Sigma t_S})\}})$$

defined by taking contraction and pull-back is injective, where $N_{\Sigma t_S^*/D_S^*}$ *denotes the normal bundle of the divisor* Σt_S^* *in* D_S^*.

Proof. Let

$$([\theta_X],[\theta_{D_S^*}])\in H^1(X,\Theta_X(-log\,D_X))\oplus H^1(D_S^*,\Theta_{D_S^*}(-\Sigma c_S^*-\Sigma t_S^*))$$

be an element such that

$$\widehat{\omega\nu_X}^*([\theta_X])=\widehat{\omega g}^*([\theta_{(D_S^*}])\text{ in }H^1(D_X^*,\Theta_{D_X^*}(-\Sigma t_S^*)).$$

By the same reasoning as in the proof of the preceeding theorem, it suffices to show that $[\theta_X]=0$ if $\tau^{(1)}(([\theta_X],[\theta_{D_S^*}]))$ is zero, and if the conditions (ii), (iii) are fullfiled. From the condition that $\tau^{(1)}(([\theta_X],[\theta_{D_S^*}])$ is zero and the condition (ii), the images of $[\theta_X]$ and $[\theta_{D_S^*}]$ in $H^1(X,\Theta_X)$ and $H^1(D_X^*,\Theta_{D_X^*}(-\Sigma c_{D_S^*}))$, respectively, are zero. In order to represent various cohomology classes as Čech cohomology classes, we adopt the same system of Stein open coverings $\mathcal{U}_X=\{W_j,W_\lambda\}$ of X, $\mathcal{U}_{D_X^*}=\{V_j,V_\lambda^{(0)},V_\lambda^{(1)}\}$ of D_X^* and $\mathcal{U}_{D_S^*}=\{U_\alpha\}$ of D_S^* as before. Let θ_X (resp. $\theta_{D_S^*}$) be represented by a Čech 1-cochain

$$\{\Theta_{ij}\}\in Z^1(\mathcal{U}_X,\Theta_X(-log\,D_X))$$
$$(\text{resp. }\{\theta_{ij}\}\in Z^1(\mathcal{U}_{D_S^*},\Theta_{D_S^*}(-\Sigma c_{D_S^*}-\Sigma t_{D_S^*}))$$

Since $\theta_X = 0$ (resp. $\theta_{D_S^*} = 0$) in $H^1(X, \Theta_X)$ (resp. in $H^1(D_S^*, \Theta_{D_S^*}(-\Sigma \mathfrak{c}_{D_S^*}))$, there exists a cochain $\{\Theta_j\} \in C^0(\mathcal{U}_X, \Theta_X)$ (resp. $\{\theta_i\} \in C^0(\mathcal{U}_{D_S^*}, \Theta_{D_S^*}(-\Sigma \mathfrak{c}_{D_S^*}))$ such that

$$\Theta_{ij} = \Theta_i - \Theta_j \ \ (\text{resp. } \theta_{ij} = \theta_i - \theta_j)$$

for every $i, j \in I_X$ (resp. $i, j \in I_{D_S^*}$) with $W_i \cap W_j \neq \emptyset$ (resp. $U_i \cap U_j \neq \emptyset$). We denote by s_{D_X} (resp. $s_{\Sigma t_{D_S^*}}$) the cohomology class of $H^0(D_X, \mathcal{N}_{D_X/X})$ (resp. of $H^0(\Sigma t_{D_S^*}, N_{\Sigma t_{D_S^*}}))$ defined by the Cěch 0-cycle

$$\{\Theta_i s_{i|D_X \cap W_i}\} \in Z^0(D_X \cap W_i, \mathcal{N}_{D_X/X}/D_S^*)$$
$$(\text{resp. } \{\theta_i t_{i|\Sigma t_{D_S^*} \cap U_i}\} \in Z^0(\Sigma t_{D_S^*}, N_{\Sigma t_{D_S^*}}))$$

where $s_i = 0$ (resp. $t_i = 0$) is the defining equation of D_X (resp. of $\Sigma t_{D_S^*}$) in W_i (resp. U_i). Then the image of s_{D_X} (resp. $s_{\Sigma t_{D_S^*}}$) in $H^1(X, \Theta_X(-log\, D_X))$ (resp. in $H^1(D_S^*, \Theta_{D_S^*}(-\Sigma \mathfrak{c}_S^* - \Sigma t_S^*))$ is θ_X (resp. $\theta_{D_S^*}$). The condition $\widehat{\omega\nu_X}^*([\theta_X]) = \widehat{\omega g}^*([\theta_{D_S^*}])$ in $H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_S^*))$ implies that there exists a 0-cochain

$$\{\theta_i^*\} \in C^0(\mathcal{U}_{D_X^*}, \Theta_{D_X^*}(-\Sigma t_{D_X^*}))$$

such that

$$\delta\{\theta_i^*\} = \widehat{\omega\nu_X}^*(\{\Theta_i - \Theta_j\}) - \widehat{\omega g}^*(\{\theta_i - \theta_j\}) \tag{4.30}$$

where δ denotes the Cěch cohomology operator.

Now, by the short exact sequence in (3.31) in Part I, the sequence of

$$\begin{aligned} 0 &\longrightarrow \Gamma(U_i, \Theta_S) \\ &\longrightarrow \Gamma(f^{-1}(U_i), \Theta_X(-log\, D_X)) \oplus \Gamma^0(\nu_S^{-1}(U_i), \Theta_{D_S^*}(-\Sigma \mathfrak{c}_{D_S^*} - \Sigma t_{D_S^*})) \\ &\qquad\qquad \longrightarrow \Gamma(\nu^{-1}(U_i), \Theta_{D_X^*}(-\Sigma t_{D_S^*})) \longrightarrow 0 \end{aligned}$$

is exact for every $U_i \in \mathcal{U}_S$. Therefore there exist 0-cochains

$$\{\Theta_i'\}_{i \in I_X} \in C^0(\mathcal{U}_X, \Theta_X(-log\, D_X))$$

and

$$\{\theta_i'\} \in C^0(\mathcal{U}_{D_S^*}, \Theta_{D_S^*}(-\Sigma \mathfrak{c}_{D_S^*} - \Sigma t_{D_S^*}))$$

such that

$$\widehat{\omega\nu_X}^*(\{\Theta_i'\}) - \widehat{\omega g}^*(\{\theta_i'\}) = \{\theta_i^*\}. \tag{4.31}$$

Then by (4.30) and (4.31) we have

$$\widehat{\omega\nu_X}^*(\{(\Theta_i - \Theta_i') - (\Theta_j - \Theta_j')\}) = \widehat{\omega g}^*(\{(\theta_i - \theta_i') - (\theta_j - \theta_j')\}).$$

Replacing $\{\Theta_i\}$ (resp. $\{\theta_i\}$) by $\{\Theta_i - \Theta'_i\}$ (resp. by $\{\theta_i - \theta'_i\}$), we may assume from the begining that $\{\Theta_i\}_{i\in I_X} \in C^0(\mathcal{U}_X, \Theta_X(-log\, D_X))$ and $\{\theta_i\} \in C^0(\mathcal{U}_{D^*_S}, \Theta_{D^*_S}(-\Sigma \mathfrak{c}_{D^*_S} - \Sigma t_{D^*_S}))$ satisfy

$$\widehat{\omega\nu_X}^*((\Theta_{\nu_{X*}(\alpha)} - \Theta_{\nu_{X*}(\beta)})) - \widehat{\omega g}^*((\theta_{g_*(\alpha)} - \theta_{g_*(\beta)})) = 0 \tag{4.32}$$

for every $\alpha, \beta \in I_{D^*_X}$ with $V_\alpha \cap V_\beta \neq \emptyset$. We take an element

$$(\omega_X, \omega_{D^*_S}) \in Ker\{H^0(\Omega^1_X) \oplus H^0(\Omega^1_{D^*_S}) \to H^0(\Omega^1_{D^*_X})\tag{4.33}$$

We represent ω_X (resp. $\omega_{D^*_S}$) by a Cěch 0-cochain

$$\{\omega_{X,i}\} \in Z^0(\mathcal{U}_X, \Omega^1_X) \ \text{(resp. } \{\omega_{D^*_S,i}\} \in Z^0(\mathcal{U}_{D^*_S}, \Omega^1_{D^*_S}))$$

By (4.33), $\{\omega_{X,i}\}$ and $\{\omega_{D^*_S,i}\}$ satisfy the condition

$$\nu^*_X \omega_{X,\nu_{X*}(\alpha)} = g^*\omega_{D^*_S, g_*(\alpha)} \tag{4.34}$$

for every $\alpha \in I_{D^*_X}$. We put

$$\omega^*_\alpha = \nu^*_X \omega_{X,\nu_{X*}(\alpha)} = g^*\omega_{D^*_S, g_*(\alpha)}, \quad \text{and}$$
$$\omega_{D^*_X} = \{\omega^*_\alpha\} \in Z^0(\mathcal{U}_{D^*_X}, \Omega^1_{D^*_X}) \simeq \check{\mathrm{H}}^0(\mathcal{U}_{D^*_X}, \Omega^1_{D^*_X})$$

By (4.34), we have

$$\begin{aligned}
&\nu^*_X(\Theta_{\nu_{X*}(\alpha)} \lfloor \omega_{X,\nu_{X*}(\alpha)}) - g^*(\theta_{g_*(\alpha)} \lfloor \omega_{D^*_S,g_*(\alpha)}\} \\
&\qquad\qquad -\{\nu^*_X(\Theta_{\nu_{X*}(\beta)} \lfloor \omega_{X,\nu_{X*}(\beta)} - g^*(\theta_{g_*(\beta)} \lfloor \omega_{D^*_S,g_*(\beta)}\} \\
&= \nu^*_X((\Theta_{\nu_{X*}(\alpha)} - \Theta_{\nu_{X*}(\beta)}) \lfloor \omega_{X,\nu_{X*}(\alpha)}) - g^*((\theta_{g_*(\alpha)} - \theta_{g_*(\beta)}) \lfloor, \omega_{D^*_S,g_*(\alpha)}) \\
&= \widehat{\omega\nu_X}^*((\Theta_{\nu_{X*}(\alpha)} - \Theta_{\nu_{X*}(\beta)})) \lfloor \omega^*_\alpha - \widehat{\omega g}^*((\theta_{g_*(\alpha)} - \theta_{g_*(\beta)})) \lfloor \omega^*_\alpha \\
&= \{\widehat{\omega\nu_X}^*((\Theta_{\nu_{X*}(\alpha)} - \Theta_{\nu_{X*}(\beta)})) - \widehat{\omega g}^*((\theta_{g_*(\alpha)} - \theta_{g_*(\beta)})) \lfloor \omega^*_\alpha = 0\}
\end{aligned}$$

for every $\alpha, \beta \in I_{D^*_X}$ with $V_\alpha \cap V_\beta \neq \emptyset$. Hence

$$\{\nu^*_X(\Theta_{\nu_{X*}(\alpha)} \lfloor \omega_{X,\nu_{X*}(\alpha)}) - g^*(\theta_{g_*(\alpha)} \lfloor \omega_{D^*_S,g_*(\alpha)})\}$$

define an element of $H^0(D^*_X, \mathcal{O}_{D^*_X})$ which we denote by

$$\nu^*_X(s_{D_X} \lfloor \omega_{X|D_X}) - g^*(s_{\Sigma t^*_S} \lfloor \omega_{D^*_S|\Sigma t^*_S}).$$

Obviously the element

$$(\nu_X^*(s_{D_X} \lfloor \omega_{X|D_X}) - g^*(s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*}), \ (s_{D_X} \lfloor \omega_{X|D_X})_{|\Sigma t_X}, \ (s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})_{|\Sigma t_S^*})$$
$$\in H^0(D_X^*, \mathcal{O}_{D_X^*}) \oplus H^0(\Sigma t_X, \mathbb{C}_{\Sigma t_X}) \oplus H^0(\Sigma t_S^*, \mathbb{C}_{\Sigma t_S^*})$$

belongs to

$$Ker \{H^0(D_X^*, \mathcal{O}_{D_X^*}) \oplus H^0(\Sigma t_X, \mathbb{C}_{\Sigma t_X}) \oplus H^0(\Sigma t_S^*, \mathbb{C}_{\Sigma t_S^*}) \to H^0(\Sigma t_X^*, \mathbb{C}_{\Sigma t_X^*})\}.$$

If we take another $\{\Theta_i'\} \in C^0(\mathcal{U}_X, \Theta_X)$ and $\{\theta_i'\} \in C^0(\mathcal{U}_{D_S^*}, \Theta_{D_S^*}(\Sigma c_S^*))$ such that

$$\Theta_{ij} = \Theta_i' - \Theta_j' \quad \text{and } \theta_{ij} = \theta_i' - \theta_j',$$

then we have

$$\Theta_i - \Theta_j = \Theta_i' - \Theta_j' \quad \text{and } \theta_i - \theta_j = \theta_i' - \theta_j',$$

and so

$$\Theta_i - \Theta_i' = \Theta_j - \Theta_j' \quad \text{and } \theta_i - \theta_i' = \theta_j - \theta_j',$$

which means that $\{\Theta_i - \Theta_i'\}$ and $\{\theta_i - \theta_i'\}$ are global sections of Θ_X and $\Theta_{D_S^*}(-\Sigma c_S^*)$, respectively. Therefore

$$\{\nu_X^*(\Theta_{\nu_{X*}(\alpha)} \lfloor \Omega_{\nu_{X*}(\alpha)}) - g^*(\theta_{g_*(\alpha)} \lfloor \omega_{g_*(\alpha)})\}$$
$$-\{\nu_X^*(\Theta'_{\nu_{X*}(\alpha)} \lfloor \Omega_{\nu_{X*}(\alpha)}) - g^*(\theta'_{g_*(\alpha)} \lfloor \omega_{g_*(\beta)})\}$$

belongs to the image of $H^0(X, \mathcal{O}_X) \oplus H^0(D_S^*, \mathcal{O}_{D_S^*}))$ in $H^0(D_X^*, \mathcal{O}_{D_X^*})$. Hence, for $([\theta_X], [\theta_{D_X^*}])$, the element

$$(\nu_X^*(s_{D_X} \lfloor \omega_{X|D_X}) - g^*(s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*}), (s_{D_X} \lfloor \omega_{X|D_X})_{|\Sigma t_X}, (s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})_{|\Sigma t_S^*})$$

is only determined as a class of the quotient space

$$\frac{Ker\{H^0(\mathcal{O}_{D_X^*}) \oplus H^0(\mathbb{C}_{\Sigma t_X}) \oplus H^0(\mathbb{C}_{\Sigma t_S^*}) \to H^0(\mathbb{C}_{\Sigma t_X^*})}{Im\{H^0(\mathcal{O}_X) \oplus H^0(\mathcal{O}_{D_S^*}) \oplus H^0(\mathbb{C}_{\Sigma t_S})\}}.$$

In what follows we shall indicate the elements of various quotient spaces by drawing *lines* over their symbols. We define

$$\begin{aligned} &\overline{\mu}^{(1)}(\overline{s_{D_X}}, \overline{s_{\Sigma t_S^*}})(\omega_X, \omega_{D_S^*}) \\ &:= ((\nu_X^*(s_{D_X} \lfloor (\omega_{X|D_X}) - g^*(s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*}), (s_{D_X} \lfloor \omega_{X|D_X})_{|\Sigma t_X}, \\ &\qquad (s_{\Sigma t_S^*} \lfloor (\omega_{D_S^*|\Sigma t_S^*})_{|\Sigma t_S^*}) \end{aligned} \tag{4.35}$$

$$\textit{modulo} \quad Im\{H^0(\mathcal{O}_X) \oplus H^0(\mathcal{O}_{D_S^*}) \oplus H^0(\mathbb{C}_{\Sigma t_S})\}$$

where $\overline{s_{D_X}}$ (resp. $\overline{s_{\Sigma t_S^*}}$) denote the image of s_{D_X} (resp. $s_{\Sigma t_S^*}$) in

$$H^0(D_X, \mathcal{N}_{D_X/X})/Im\{H^0(X, \Theta_X) \to H^0(D_X, \mathcal{N}_{D_X/X})\}$$

$$(\text{resp. in } H^0(\Sigma t_S^*, N_{\Sigma t_S^*/D_S^*})/Im\{H^0(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^*)) \\ \to H^0(\Sigma t_S^*, N_{\Sigma t_S^*/D_S^*})\})$$

Claim. *For any*

$$(\omega_X, \omega_{D_S^*}) \in Ker\{H^0(\Omega_X^1) \oplus H^0(\Omega_{D_S^*}^1) \to H^0(\Omega_{D_X^*}^1)\},$$

we have

$$\overline{\mu}^{(1)}(\overline{s_{D_X}}, \overline{s_{\Sigma t_S^*}})(\omega_X, \omega_{D_S^*}) = \tau^{(1)}((\theta_X, \theta_{D_S^*}))([\omega_X], [\omega_{D_S^*}])'$$

For the notation $\tau^{(1)}((\theta_X, \theta_{D_S^*}))([\omega_X], [\omega_{D_S^*}])'$, see Lemma 4.8 (ii),(d).)

Proof of Claim. We consider the element

$$(0, 0, \nu_X^*(s_{D_X} \lfloor \omega_{X|D_X}) - g^*(s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*}), \\ (s_{D_X} \lfloor \omega_{X|D_X})_{|\Sigma t_X}, (s_{\Sigma t_S^*} \lfloor \omega_{D_S^*} | \Sigma t_S^*)_{|\Sigma t_S^*}) \tag{4.36}$$

By the definition of $D^{(1)}[1] : K^1(\mathcal{O}_{X_\bullet}[1]) \to K^2(\mathcal{O}_{X_\bullet}[1])$ (cf. (4.14)', (4.15)' and (4.15)')), we have

$$\begin{aligned} &D^{(1)}[1](0, 0, \nu_X^*(s_{D_X} \lfloor \omega_{X|D_X}) - g^*(s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*}), \\ &\qquad (s_{D_X} \lfloor \omega_{X|D_X})_{|\Sigma t_X}), s_{D_X} \lfloor (s_{\Sigma t_S^*} \lfloor (\omega_{D_S^*|\Sigma t_S^*}) \\ &= (0, 0, \delta_{(011)}^{(0,0)}(\{\nu_X^*(s_{D_X} \lfloor \omega_{X|D_X}) - g^*(s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})), \\ &\qquad d_{(011),1}^{(0,0)*}(\nu_X^*(s_{D_X} \lfloor \omega_{X|D_X}) - g^*(s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})) \\ &\qquad -d_{(101),2}^{(0,0)*}((s_{D_X} \lfloor \omega_{X|D_X})_{|\Sigma t_X}) + d_{(110),3}^{(0,0)*}((s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})_{|\Sigma t_S^*}). \end{aligned}$$

Since

$$(\delta_{011)}^{(0,0)}\{\nu_X^*(s_{D_X} \lfloor \omega_{X|D_X}) - g^*(s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})) = 0, \quad \text{and}$$

$$\begin{aligned} &d_{(011),1}^{(0,0)*}(\nu_X^*(s_{D_X} \lfloor \omega_{X|D_X}) - g^*(s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})) \\ &\qquad -d_{(101),2}^{(0,0)*}((s_{D_X} \lfloor \omega_{X|D_X})_{|\Sigma t_X}) + d_{(110),3}^{(0,0)*}((s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})_{|\Sigma t_S^*}) \\ &= \nu_{X|\Sigma t_X^*}^*((s_{D_X} \lfloor \omega_{X|D_X})_{|\Sigma t_X}) - g_{|\Sigma t_X^*}^*((s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})_{|\Sigma t_S^*}) \\ &\qquad -\nu_{X|\Sigma t_X^*}^*((s_{D_X} \lfloor \omega_{X|D_X})_{|\Sigma t_X}) + g_{|\Sigma t_X^*}^*((s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})_{|\Sigma t_S^*}) = 0, \end{aligned}$$

the element in (4.36) is a 1-cocycle of the complex $K^\bullet(\mathcal{O}_{X_\bullet}[1])$. Next we consider the element

$$(\theta_X \lfloor \omega_X, \theta_{D_S^*} \lfloor \omega_{D_S^*}), 0, 0, 0) \in K^1(\mathcal{O}_X[1]).$$

By the definition of $D^{(1)}[1] : K^1(\mathcal{O}_{X_\bullet}[1]) \longrightarrow K^2(\mathcal{O}_{X_\bullet}[1])$ (cf. (4.15)$'$ and (4.15)$''$), we have

$$D^{(1)}[1](\theta_X \lfloor \omega_X, \theta_{D_S^*} \lfloor \omega_{D_S^*}), 0, 0, 0) \in K^1(\mathcal{O}_X[1])$$
$$= (-\delta^{(01)}_{(001)}(\theta_X \lfloor \omega_X), -\delta^{(01)}_{(010)}(\theta_{D_S^*} \lfloor \omega_{D_S^*}),$$
$$d^{(0,1)*}_{(001),2}(\theta_X \lfloor \omega_X) - d^{(0,1)*}_{(010),3}(\theta_{D_S^*} \lfloor \omega_{D_S^*}), 0)$$

Since $\theta_X = \{\Theta_{ij} = \Theta_i - \Theta_j\}$, $\theta_{D_S^*} = \{\theta_{ij} = \theta_i - \theta_j\}$, $\omega_X = \{\omega_{X,i}\}$ and $\omega_{D_S^*} = \{\omega_{D_S^*,i}\}$ satisfy the relation in (4.32) and (4.36), we have

$$\delta^{(01)}_{(001)}(\theta_X \lfloor \omega_X) = \delta^{(01)}_{(010)}(\theta_{D_S^*} \lfloor \omega_{D_S^*}) = 0, \quad \text{and}$$

$$\{d^{(0,1)*}_{(001),2}(\theta_X \lfloor \omega_X) - d^{(0,1)*}_{(010),3}(\theta_{D_S^*} \lfloor \omega_{D_S^*})\}_{\alpha\beta}$$
$$= \nu_X^*((\Theta_{\nu_{X*}(\alpha)} - \Theta_{\nu_{X*}(\beta)}) \lfloor \omega_{X,\nu_{X*}(\beta)}) - g^*((\theta_{g_*(\alpha)} - \theta_{g_*(\beta)}) \lfloor \omega_{D_S^*, g_*(\beta)})$$
$$= \widehat{\omega\nu_X}^*(\Theta_{\nu_{X*}(\alpha)} - \Theta_{\nu_{X*}(\beta)}) \lfloor \omega_\alpha^* - \widehat{\omega g}^*(\theta_{g_*(\alpha)} - \theta_{g_*(\beta)}) \lfloor \omega\beta^*$$
$$= \{\widehat{\omega\nu_X}^*(\Theta_{\nu_{X*}(\alpha)} - \Theta_{\nu_{X*}(\beta)}) - \widehat{\omega g}^*(\theta_{g_*(\alpha)} - \theta_{g_*(\beta)})\} \lfloor \omega_\beta^* = 0$$

for every $\alpha, \beta \in I_{D_X^*}$ with $V_\alpha \cap V_\beta \neq \emptyset$, where

$$\omega_\alpha^* := \nu_X^* \omega_{X,\nu_{X*}(\alpha)} = g^* \omega_{D_S^*, g_*(\alpha)} \quad \text{and} \quad \omega_\beta^* := \nu_X^* \omega_{X,\nu_{X*}(\beta)} = g^* \omega_{D_S^*, g_*(\beta)}$$

as before. By the definition of $\tau^{(1)}([\theta_X], [\theta_{D_X^*}])$ (cf. Theorem 3.17 in Part I)

$$\tau^{(1)}([\theta_X], [\theta_{D_S^*}])(\omega_X, \omega_{D_S^*})$$
$$[(\theta_X \lfloor \omega_X, \theta_{D_S^*} \lfloor \omega_{D_S^*}, \{\theta_\alpha^* \lfloor \omega_\alpha^*\}, 0, 0)],$$
$$[(\{\Theta_{\alpha\beta} \lfloor \Omega_\beta\}, \{\theta_{\alpha\beta} \lfloor \omega_\beta\}, \{\theta_\alpha^* \lfloor \omega_\alpha^*\}, 0, 0)],$$

where $\{\theta_\alpha^*\} \in C^0(\mathcal{U}_{D_X^*}, \Theta_{D_X^*}(-\Sigma t_X^*))$ (resp. $\omega_{D_X}^* = \{\omega_\alpha^*\} \in C^0(\mathcal{U}_{D_X^*}, \Omega^1_{D_X^*})$) is such that

$$\delta\{\theta_\alpha^*\} = \widehat{\omega\nu_X}^*(\{\Theta_{\alpha\beta}\}) - \widehat{\omega g^*}(\{\theta_{\alpha\beta}\})$$
$$(resp. \nu_X^* \omega_X = g^* \omega_{D_S}^* = \omega_{D_X}^*),$$

and "[ ]" denotes cohomology classes in $\mathbb{H}^1(\mathcal{O}_{X_\bullet}[1])$. In the case with which we are now being concerned, since we assume that $\widehat{\omega\nu_X}^*(\{\Theta_{ij}\}) - \widehat{\omega g}^*(\{\theta_{ij}\}) = 0$ from the begining, we have $\{\theta_i^*\} = 0$. Hence

$$\tau^{(1)}(\theta_X, \theta_{D_S^*})(\omega_X, \omega_{D_S^*})$$
$$= [(\theta_X \lfloor \omega_X, \theta_{D_S^*} \lfloor \omega_{D_S^*}, 0, 0, 0)] \tag{4.37}$$
$$= [(\{\Theta_{ij} \lfloor \omega_{X,j}\}, \{\theta_{ij} \lfloor \omega_{D_X^*,j}\}, 0, 0, 0)].$$

We claim that

$$(4.48)\qquad [(\theta_X \lfloor \omega_X, \theta_{D_S^*} \lfloor \omega_{D_S^*}, 0, 0, 0]$$

and

$$(4.39)\qquad [(0, 0, \nu_X^*(s_{D_X} \lfloor \omega_{X|D_X}) - g^*(s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})), \\ (s_{D_X} \lfloor \omega_{X|D_X})_{|\Sigma t_X}, (s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})]$$

are cohomologous in $\mathbb{H}^1(\mathcal{O}_{X_\bullet}[1])$. Indeed, by the definition of $D^{(1)}[1]$ (cf. (4.15)$'$), we have

$$D^{(0)}[1](\{\Theta_i \lfloor \omega_{X,i}\}, \{\theta_i \lfloor \omega_{D_S^*,i}\}, 0)$$
$$= (-\delta_{(001)}^{(00)}\{\Theta_i \lfloor \omega_{X,i}\}, -\delta_{(010)}^{(00)}\{\theta_i \lfloor \omega_{D_S^*,i}\},$$
$$d_{(001),2}^{(0,0)*}\{\Theta_i \lfloor \omega_{X,i}\} - d_{(010),3}^{(0,0)*}\{\theta_i \lfloor \omega_{D_S^*,i}\}, d_{(001),1}^{(0,0)*}\{\Theta_i \lfloor \omega_{X,i}\}, d_{(010),1}^{(0,0)*}\{\theta_i \lfloor \omega_{D_S^*,i})\})$$
$$= (-\delta\{\Theta_i \lfloor \omega_{X,i}\}, -\delta\{\theta_i \lfloor \omega_{D_S^*,i}\}, \\ \nu_X^*\{\Theta_i \lfloor \omega_{X,i}\} - g^*\{\theta_i \lfloor \omega_{D_S^*,i}\}, \{\Theta_i \lfloor \omega_{X,i}\}_{|\Sigma t_X}, \{\theta_i \lfloor \omega_{D_S^*,i}\}_{\Sigma t_S^*})$$
$$= (0, 0, \nu_X^*(s_{D_X} \lfloor \omega_{X|D_X}) - g^*(s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})), \\ (s_{D_X} \lfloor \omega_{X|D_X})_{|\Sigma t_X}, (s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*}) \\ -(\theta_X \lfloor \omega_X, \theta_{D_X^*} \lfloor \omega_{D_X^*}, 0, 0, 0)]$$

Thus the element in (4.38) and (4.39) are cohomologus to each others in $\mathbb{H}^1(\mathcal{O}_{X_\bullet}[1])$. By (4.37), $\tau^{(1)}([\theta_X], [\theta_{D_S^*}])(\omega_X, \omega_{D_S^*}) \in \mathbb{H}^1(\mathcal{O}_{X_\bullet}[1])$ is represented by the element in (4.39) whose image in

$$Ker\,\{H^1(\mathcal{O}_X) \oplus H^1(\mathcal{O}_{D_S^*}) \to H^1(\mathcal{O}_{D_X^*})\}$$

is zero. Therefore, by the definition of $\tau^{(1)}([\theta_X], [\theta_{D_S^*}])'(\omega_X, \omega_{D_S^*})$, it is equal to

$$\overline{((\nu_X^*(s_{D_X} \lfloor (\omega_{X|D_X}) - g^*(s_{\Sigma t_S^*} \lfloor \omega_{D_S^*|\Sigma t_S^*})), s_{D_X} \lfloor (\omega_{X|D_X})_{|\Sigma t_X},} \\ \overline{s_{\Sigma t_S^*} \lfloor \omega_{D_S^*}|\Sigma t_S^*)},$$

where the *overline* denotes the image of an element of the space

$$Ker\,\{H^0(\mathcal{O}_{D_X^*}) \oplus H^0(\mathbb{C}_{\Sigma t_X}) \oplus H^0(\mathbb{C}_{\Sigma t_S^*}) \to H^0(\mathbb{C}_{\Sigma t_X^*})\}$$

to its quotient space by the subspace

$$Im\,\{H^0(\mathcal{O}_X) \oplus H^0(\mathcal{O}_{D_S^*}) \oplus H^0(\mathbb{C}_{\Sigma t_S}) \to H^0(\mathcal{O}_{D_X^*}) \oplus H^0(\mathbb{C}_{\Sigma t_X}) \oplus H^0(\mathbb{C}_{\Sigma t_S^*}))\}$$

Now by the definition of $\overline{\mu}^{(1)}(\overline{s_{D_X}}, \overline{s_{\Sigma t_S^*}})(\omega_X, \omega_{D_S^*})$ in (4.35), we have

$$\overline{\mu}^{(1)}(\overline{s_{D_X}}, \overline{s_{\Sigma t_S^*}})(\omega_X, \omega_{D_S^*}) = \tau^{(1)\prime}((\theta_X, \theta_{D_S^*}))(\omega_X, \omega_{D_S^*})$$

as requied.

q.e.d.for Claim

Now, it follows $\tau^{(1)}(\theta_X, \theta_{D_S^*})' = 0$ from the assumption $\tau^{(1)}(\theta_X, \theta_{D_S^*}) = 0$ because of Lemma 4.7, (ii). Then, by the claim above, we have

$$\overline{\mu}^{(1)}(\overline{s_{D_X}}, \overline{s_{\Sigma t_S^*}})(\omega_X, \omega_{D_S^*}) = 0,$$

and so by the condition (iii) of the theorem, we coclude that $\theta_X = \overline{s_{D_X}} = 0$. This complete the proof of the theorem.

Q.E.D.

4.12 Theorem. *For an algebraic surface S with ordinary singularities such that X is irregular, i.e., $q(X) = \dim_{\mathbb{C}} H^0(X, \Omega_X^1) > 0$, the map*

$$\oplus_{i=1}^2 \tau^{(i)} : H^1(S, \Theta_S) \to \oplus_{p=1}^2 Hom_{\mathbb{C}}(\mathbb{H}^0(\Omega_{X_\bullet}^p[1]), \mathbb{H}^{p-1}(\Omega_{X_\bullet}^{p-1}[1])\}$$

defined by taking cup-product and contraction "$\lfloor$" is injective if all of the following conditions are satisfied:

(i) *The map*

$$H^0(X, \Theta_X(-logD_X)) \oplus H^0(D_S^*, \Theta_{D_S^*}(-\Sigma \mathfrak{c}_S^* - \Sigma t_S^*)) \to H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

(cf. Theorem 3.26, (ii) in Part I) *is surjective.*

(ii) *The infinitesimal Torelli concerning the cohomology $H^2(X, \mathbb{C})$ holds for X, that is, the homomorphism*

$$H^1(X, \Theta_X) \to Hom_{\mathbb{C}}(H^0(X, \Omega_X^2), H^1(X, \Omega_X^1))$$

defined by coupling through contraction $\lfloor$ is injective on the image of $H^1(S, \Theta_S)$ in $H^1(X, \Theta_X)$.

(iii) *The homomorphism $\overline{\mu}^{(1)}$:*

$$\frac{H^0(D_X, \mathcal{N}_{D_X/X}) \oplus H^0(\Sigma t_S^*, N_{\Sigma t_S^*/D_S^*})}{Im\{H^0(X, \Theta_X) \oplus H^0(D_S^*, \Theta_{D_S^*}(-\Sigma \mathfrak{c}_S^*)) \to H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))\}}$$
$$\to \mathrm{Hom}_{\mathbb{C}}(Ker\{H^0(\Omega_X^1) \oplus H^0(\Omega_{D_S^*}^1) \to H^0(\Omega_{D_X^*}^1)\},$$
$$\frac{Ker\{H^0(\mathcal{O}_{D_X^*}) \oplus H^0(\mathbb{C}_{\Sigma t_X}) \oplus H^0(\mathbb{C}_{\Sigma t_S^*}) \to H^0(\mathbb{C}_{\Sigma t_X^*})}{Im\{H^0(\mathcal{O}_X) \oplus H^0(\mathcal{O}_{D_S^*}) \oplus H^0(\mathbb{C}_{\Sigma t_S})\}})$$

defined by taking contraction and pull-back is injective where $N_{\Sigma t_{D_S^*}/D_S^*}$ denote the normal bundle of the divisor $\Sigma t_{D_S^*}$ in D_S^*.

Proof. By the condition (i) any element of $H^1(S, \Theta_X)$ can be represenred by a pair of cohomology classes

$$(\theta_X, \theta_{D_S^*}) \in H^1(X, \Theta_X(-logD_X)) \oplus H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*))$$

which is subjective to the condition

$$\widehat{\omega\nu_X}^*([\theta_X]) = \widehat{\omega g}^*([\theta_{(D_S^*}]) \text{ in } H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_S^*))$$

(cf. Theorem 3.26, (ii) in Part I). What we have to show is that if $\tau^{(2)}((\theta_X, \theta_{D_S^*}))$ in $Hom_{\mathbb{C}}(\mathbb{H}^0(\Omega^2_{X_\bullet}[1]), \mathbb{H}^1(\Omega^1_{X_\bullet}[1])$, and if the condition (ii), (iii) are fullfilled, then $\theta_X = 0$ in $H^1(X, \Theta_X(-logD_X))$ and $\theta_{D_S^*} = 0$ in $H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*))$.

We denote by θ'_X the image of θ_X in $H^1(X, \Theta_X)$, and by $\theta'_{D_S^*}$ the image of $\theta_{D_S^*}$ in $H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*))$. If $\tau^{(2)}((\theta_X, \theta_{D_S^*}))$ in

$$Hom_{\mathbb{C}}(\mathbb{H}^0(\Omega^2_{X_\bullet}[1]), \mathbb{H}^1(\Omega^1_{X_\bullet}[1]),$$

then by Lemma 4.7 (i), the homomorphism

$$\theta'_X \llcorner : H^0(X, \Omega^2_X) \to Ker\{H^1(X, \Omega^1_X) \to H^1(D_X^*, \Omega^1_{D_X^*})\}$$

defined by taking *cup-product* with the cohomology class θ'_X is zero. Then by the condition (ii),we conclude that $\theta'_X = 0$ in $H^1(X, \Theta_X)$. From the following commutative diagram

$$\begin{array}{ccc}
H^1(\Theta_X(-logD_X)) \oplus H^1(\Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)) & \xrightarrow{\widehat{\omega f} - \widehat{\omega g}} & H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_S^*)) \\
\downarrow & & \downarrow \\
H^1(X, \Theta_X) \oplus H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^*)) & \xrightarrow{\widehat{\omega f} - \widehat{\omega g}} & H^1(D_X^*, \Theta_{D_X^*})
\end{array}$$

and the fact that $\widehat{\omega f}(\theta_X) = \widehat{\omega g}(\theta_{D_S^*})$, it follows that $\widehat{\omega f}(\theta'_X) = \widehat{\omega g}(\theta'_{D_S^*})$. THen, since $\theta'_X = 0$, we have $\widehat{\omega g}(\theta'_{D_S^*}) = 0$. Therefore, since

$$\widehat{\omega g} : H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^*)) \to H^1(D_X^*, \Theta_{D_X^*})$$

is injective (cf. Proposition 4.1), we conclude that $\theta'_{D_S^*}) = 0$. Then the homomorphism in Lemma 4.7 (ii)-(c)

$$\begin{aligned}
(\theta'_X, \theta'_{D_S^*}) \llcorner : & Ker\{H^0(\Omega^1_X) \oplus H^0(\Omega^1_{D_X^*}) \to H^1(\Omega^1_{D_X^*})\} \\
& \to Ker\{H^1(\mathcal{O}_X) \oplus H^1(\mathcal{O}_{D_S^*}) \to H^1(\mathcal{O}_{D_X^*})\}
\end{aligned}$$

defined by taking *cup-product* with the pair of cohomology classes $(\theta'_X, \theta'_{D_S^*})$ through the *contraction* " $\lfloor$ " is zero map. Hence the homomorphism in Lemma 4.7 (ii)-(d)

$$\tau^{(1)}((\theta_X, \theta_{D_S^*})' : Ker\,\{H^0(\Omega^1_X) \oplus H^0(\Omega^1_{D_S^*}) \to H^0(\Omega^1_{D_X^*})\}$$
$$\to \frac{Ker\{H^0(\mathcal{O}_{D_X^*}) \oplus H^0(\mathbb{C}_{\Sigma t_X}) \oplus H^0(\mathbb{C}_{\Sigma t_S^*}) \to H^0(\mathbb{C}_{\Sigma t_X^*})\}}{Im\{H^0(\mathcal{O}_X) \oplus H^0(\mathcal{O}_{D_S^*}) \oplus H^0(\mathbb{C}_{\Sigma t_S})\}}$$

is defined, and Lemma 4.7 (ii) this map is zero map under the assumption that $\tau^{(1)}((\theta_X, \theta_{D_S^*}) = 0$ in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^0(\Omega^2_{X_\bullet}[1]), \mathbb{H}^1(\Omega^1_{X_\bullet}[1])$. Since θ'_X, the image of θ_X in $H^1(X, \Theta_X)$ (resp. $\theta'_{D_S^*}$, the image of $\theta_{D_S^*}$ in $H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^*)))$, is zero, there exists an element

$$s_{D_X} \in H^0(D_X, \mathcal{N}_{D_X/N}) \text{ (resp. } s_{\Sigma t_{D_S^*}} \in H^0(\Sigma t_{D_S^*}, N_{\Sigma t_{D_S^*}/D_S^*}))$$

such that its image in $H^1(X, \Theta_X(-log\, X))$ (resp. in $H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^*)))$ coinsides with θ_X (resp. $\theta'_{D_S^*}$). Here we should remember that we have proved in Theorem 4.10 the equalty

$$\overline{\mu}^{(1)}(\overline{s_{D_X}}, \overline{s_{\Sigma t_S^*}})(\omega_X, \omega_{D_S^*}) = \tau^{(1)\prime}((\theta_X, \theta_{D_S^*}))(\omega_X, \omega_{D_S^*})$$

holds in Claim for any

$$(\omega_X, \omega_{D_S^*}) \in \mathrm{Ker}\{H^0(\Omega^1_X) \oplus H^0(\Omega^1_{D_S^*}) \to H^0(\Omega^1_{D_X^*})\},$$

where $\overline{s_{D_X}}$ (resp. $\overline{s_{\Sigma t_S^*}}$) denotes the image of s_{D_X} (resp. $s_{\Sigma t_S^*}$) in

$$H^0(D_X, \mathcal{N}_{D_X/X})/\mathrm{Im}\,\{H^0(X, \Theta_X) \to H^0(D_X, \mathcal{N}_{D_X/X})\}$$

(resp. $H^0(\Sigma t_{D_S^*}, N_{\Sigma t_{D_S^*}/D_S^*}))$

modulo $\mathrm{Im}\,\{H^0(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^*)) \to H^0(\Sigma t_{D_S^*}, N_{\Sigma t_{D_S^*}/D_S^*}))\})$.

Therefore, since $\tau^{(1)\prime}((\theta_X, \theta_{D_S^*}))(\omega_X, \omega_{D_S^*}) = 0$, we have

$$\overline{\mu}^{(1)}(\overline{s_{D_X}}, \overline{s_{\Sigma t_S^*}})(\omega_X, \omega_{D_S^*}) = 0.$$

Then by the condition (iii) in the theorem, we coclude $\overline{s_{D_X}} = \overline{s_{\Sigma t_S^*}} = 0$, and so $theta_X$, the image of $\overline{s_{\Sigma t_S^*}}$ (resp. $\theta_{\Sigma t_{D_S^*}}$, the image of $\overline{s_{\Sigma t_S^*}}$, is zero as requied.

Q.E.D.

§5 Examples

5.1 Example. Let S be an irreducible hypersurface with ordinary singularities in the 3-dimensional complex projective space $\mathbb{P}^3(\mathbb{C})$ of degree n, and let D_S be the singular locus of S, which we call the *double curve* of S. In what follows we denote $\mathbb{P}^3(\mathbb{C})$ by P for simplicity. The classical formula due to Enriques

([1]), the numerical characters of the normal model X, which turns out to be non-singular, of S are given as follows:

$$
(5.1.1)\quad
\begin{aligned}
p_a(X) &= \binom{n-1}{3} - (n-4)m + \pi(D_S) - 1,\\
c_1^2(X) &= n(n-4)^2 - (5n-24)m + 4\pi(D_S) - 4 + t,\\
c_2(X) &= n(n^2-4n+6) - (7n-24)m + 8\pi(D_S) - 8 - t,\\
p_g(X) &= dim_{\mathbb{C}}\, \mathcal{L}_{n-4}(-D_S),
\end{aligned}
$$

where m denotes the degree of D_S in $\mathbb{P}^3(\mathbb{C})$, $\pi(D_S) = 1 - \chi(D_S, \mathcal{O}_{D_S})$ the virtual genus of D_S, $t = \#\Sigma t_S$, the cardinal number of the triple point set Σt_S, and

$$
\begin{aligned}
\mathcal{L}_{n-4}(-D_S) = {} & \text{homogeneous polynomials of degree } n-4\\
& \text{in four variables which vanish on } D_S
\end{aligned}
$$

In what follows we use the same notation as in the diagram (1.3) and (1.4) in Part I for the surface S. We define the sheaf $\mathcal{N}_{D_S}$ to be the quotient sheaf $\Theta_P/\Theta_P(-log\, D_S)$ where Θ_P denotes the sheaf of germs of holomorphic tangent vector fields on P, and $\Theta_P(-log\, D_S)$ the subsheaf of Θ_P consisting of the derivations of $\mathcal{O}_P$ which send the sheaf $\mathcal{I}(D_S)$ into itself. Note that the sheaf $\mathcal{N}_{D_S}$ coincides with the sheaf N_Δ ($\Delta = D_S$) defined by Kodaira in [6]. We assume that:

(5.1.2)

The double curve D_S of S belongs to an analytic family $\mathfrak{f} := \{D_{t_1}\}_{t_1 \in M_1}$ of locally trivial displacements of D_S in P such that

(1) the parameter space M_1 is non-singular, and

(2) the characteristic map

$$\sigma_o^{\mathfrak{f}} : T_o(M_1) \hookrightarrow H^0(D_S, \mathcal{N}_{D_S/P})$$

at the point $o \in M_1$ with $D_o = D_S$ is surjective.

For the definition of an anlytic family of locally trivial displacements of analyitc subvarieties, see [11, Definition 8.1], and for the definition of the characteristic map $\sigma_o^{\mathfrak{f}}$, see [ibid. Definition 8.3].

Letting H be a hyperplane in P, we simply denote by $\mathcal{O}_P(k)$ ($k \in \mathbb{Z}$) the sheaf $\mathcal{O}_P([kH])$ where $[kH]$ denote the line bundle determined by the divisor kH on P. We denote by $\mathcal{O}_P(k - 2D_S)$ ($k \in \mathbb{Z}$) the subsheaf of $\mathcal{O}_P(k)$ consisting of those local holomorphic cross-sections of the line bundle $[kH]$ whose fiber coordinets vanish on D_S together with their partial derivatives.

5.1.1 Theorem. *If the condition*

$$(5.1.3)\qquad H^1(P, \mathcal{O}_P(n - 2D_S)) = 0$$

is fulfilled as well as the condition (5.1.2), then the surface S belongs to a maximal analytic family $\mathcal{S} = \{S_t\}_{t \in M}$ of locally trivial displacements of S in P such that

(1) *the parameter space M is non-singular, and*

(2) *the characteristic map*

$$\sigma_t^S : T_t(M_1) \longrightarrow H^0(S_t, \mathcal{N}_{S_t/P})$$

is surjective at every point $t \in M$, where $\mathcal{N}_{S_t/P} := \Theta_P/\Theta_P(-log\, S_t)$.

For the proof of the theorem above, see [6]. Note that the sheaf $\mathcal{N}_{S_t/P}$ coincides with the restriction of the sheaf Φ defined by Kodaira in [6].

Now we consider the following exact suquences:

$$0 \longrightarrow \Theta_S \longrightarrow \Theta_{P|S} \longrightarrow \mathcal{N}_{S/P} \longrightarrow 0, \tag{5.1.4}$$

$$0 \longrightarrow \Theta_P(-n) \longrightarrow \Theta_P \longrightarrow \Theta_{P|S} \longrightarrow 0, \tag{5.1.5}$$

$$0 \longrightarrow \Theta_P(-n) \longrightarrow \Theta_P(-log\, S) \longrightarrow \Theta_S \longrightarrow 0, \tag{5.1.6}$$

where $\Theta_{P|S} := \Theta_P \otimes_{\mathcal{O}_S} \mathcal{O}_S$ and $\Theta_P(-n) := \Theta_P \otimes_{\mathcal{O}_P} \mathcal{O}_P(-n)$. The exact sequence (5.1.4) is due to the fact $\Theta_S \simeq \Theta_P(-log\, S) \otimes_{\mathcal{O}_S} \mathcal{O}_S$.

From now on we assume that

$$n \geq 5 \tag{5.1.7}$$

Under the assumption (5.1.7), by Bott's theorem, we have

$$H^q(P, \Theta_P(-n) \simeq H^{3-q}(P, \Omega_P(n-4)) = 0 \quad \text{for } 0 \leq q \leq 2.$$

Therefore by (5.1.5) we have

$$H^0(P, \Theta_P) \simeq H^0(S, \Theta_{P|S}). \tag{5.1.8}$$

and,

$$H^1(S, \Theta_{P|S}) = 0. \tag{5.1.9}$$

By (5.1.6),

$$H^0(P, \Theta_P(-log\, S)) \simeq H^0(S, \Theta_S) = 0. \tag{5.1.10}$$

On the other hand, if $n \geq 5$, the logarithmic Kodaira dimension $\overline{\kappa}(P - S)$ is equal to 3. Therefore by Theorem 6 and the cororally to Proposition 4 in [5], we have $H^0(P, \Theta_P(-log\, S)) = 0$, and so by (5.1.10)

$$H^0(S, \Theta_S) = 0. \tag{5.1.11}$$

Then by (5.1.4), (5.1.9) and (5.1.11), we have

(5.1.12)
$$\begin{aligned} dim\, H^1(S,\Theta_S) &= dim\, H^0(S,\mathcal{N}_{S/P}) - dim\, H^0(P,\Theta_P) \\ &= dim\, H^0(S,\mathcal{N}_{S/P}) - 15. \end{aligned}$$

Note that (5.1.12) holds for every member S_t of the family $\mathcal{S} = \{S_t\}_{t\in M}$ in Theorem 5.1.1. We assume that

(5.1.13)

$dim\, H^0(S_t,\mathcal{N}_{S_t/P})$ is independent on $t \in M$ for every member S_t of the family $\mathcal{S} = \{S_t\}_{t\in M}$ in Theorem 5.1.1.

Then the following thorem follows from Theorem 5.1.1 and (5.1.12).

5.1.2 Theorem. *Assume that the conditions* (5.1.2), (5.1.3), (5.1.7) *and* (5.1.13) *are fulfilled. Then the surface S belongs to the Kuranishi family $\mathcal{S}' = \{S_t\}_{t\in M'}$, or an effectively perametrized complex analytic family, of locally trivial deformations of S such that*

(1) *the parameter space M' is non-singular, and the dimension of it is equal to $dim\, H^0(C,\mathcal{N}_{S/P}) - 15$, and*

(2) *the characteristic map*

$$\sigma_t^{\mathcal{S}'} : T_t(M') \longrightarrow H^1(S_t,\Theta_{S_t})$$

is bijective at every point $t \in M'$.

In what follows, using Usui's result in [15], we shall show that the *infinitesimal mixed Torelli theorem* holds for the family $\mathcal{S}' = \{S_t\}_{t\in M'}$ in Theorem 5.1.2 if X is regular, and if the degree n of S is sufficiently large enough comparing to the double curve D_S of S. In order to state the sufficient conditions for the *infinitesimal mixed Torelli theorem* for the family $\mathcal{S}'$ to hold, we construct an embedding resolution of S.

Let $q_1 : P_1 \longrightarrow P$ be the blowing-up of P with center Σt_S, let E_1 be the exceptional divisor of q_1. X_1 and D_1 denote the proper transformations of S and D_S respctively. We set $f_1 := q_{1|X_1} : X_1 \longrightarrow X$, $q_1 := q_{1|D_1} : D_1 \longrightarrow D_S$, $T_1 := f_1^{-1}(\Sigma t_S)$. Next, let $q_2 : P_2 \longrightarrow P_1$ be the blowing-up of P_1 along D_1, let E_2 be the exceptional divisor of q_2. X_2, T_2 and E_1' denote the proper transforms of X_1, T_1 and E_1 by q_2, respectively. It is easy to see that D_1, X_2 and D_2 are smooth and that T_2 consists of the exceptional curves of the first kind on X_2. The surface obtained from X_2 by contracting T_2 coincides with the normalization X of S by virtue of Zariski's Main Theorem. We consider the following conditions:

(5.1.14) $\quad H^0(\Omega_P \otimes \omega_{D_S} \otimes \mathcal{O}_{D_S}(5-n)) = 0.$

(5.1.15) $\quad H^0(S^2(N_{D_1/P}) \otimes \mathcal{O}_{D_1}(4[E_1 \cdot D_1]) \otimes g_1^*\mathcal{O}_{D_S}(1-n)) = 0.$

where $S^2(N_{D_1/P})$ denotes the second symmetric power of the normal bundle $N_{D_1/P}$ of the divisor D_1 in P of degree 2.

(5.1.16) $H^0(\Omega_{P_1} \otimes \mathcal{O}_{D_1}(2[E_1 \cdot D_1]) \otimes g_1^*\mathcal{O}_{D_S}(5-n)) = 0.$

(5.1.17) $\mathcal{O}_{P_2}(-2E_2) \otimes g_1^*\mathcal{O}_{P_1}(-4E_1) \otimes (q_1 \circ q_2)^*\mathcal{O}_P(n-1)$ is ample.

(5.1.18) There exists an integer m satisfying the following conditions:

(1) $m \leq n-4$

(2) $|\mathcal{O}_X(m - D_X)|$ is fixed components free

(3) $H^0(\mathcal{O}_X(n-2m-3+D_X)) \neq 0.$

Note that these conditions are fulfilled by Serre's Theorem provided the degree n of S is suuficiently large enough comparing to D.

5.1.3 Theorem. ([16, Theorem 5.8]) *In the case that the degree n of S in $P = \mathbb{P}^3(\mathbb{C})$ is sufficiently large enough comparing to the singular locus D_S of S in the sense that the conditions from* (5.1.14) *through* (5.1.18) *are fulfilled, the infimitesimal Torelli concerning the cohomology $H^2(X,\mathbb{C})$ holds, that is, the homomorphism*

$$H^1(X,\Theta_X) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(H^0(X,\Omega_X^2), H^1(X,\Omega_X^1))$$

defined by taking cup-product and contraction is surjective.

Thus, in this case, the condition (ii) in Theorem 4.10 is fulfilled. Next we consider the condition (iii) in Theorem 4.10. We compute the cohomology $H^0(D_X, \mathcal{N}_{D_X/X})$. For this end we consider the exact sequence

(5.1.19) $$0 \longrightarrow \mathcal{N}_{D_X/X} \longrightarrow \mathcal{O}_{D_X}([D_X]) \longrightarrow \mathbb{C}_{\Sigma t_X} \longrightarrow 0.$$

Here the homomorphisms in this sequence are defined as follows:

Let p be a point of D_X, U a coordinate neighborhood of p in X, (x,y) a local coordinate with center p, and $x=0$ a local defining equation of D_X in U. Since $\mathcal{N}_{D_X/X} := \Theta_X/\Theta_X(-log\, D_X)$ by definition, a local cross-section of $\mathcal{N}_{D_X/X}$ over $U \cap D_X$ is represented by a local holomorphic vector field $\theta = a(\partial/\partial x) + b(\partial/\partial y)$ modulo $\Theta_X(-log\, D_X)$ where a, b are local holomorphic functions on U. The homomorphism $\mathcal{N}_{D_X/X} \longrightarrow \mathcal{O}_X([D_X])$ in (5.1.19) at p is defined as

$$\theta \ \mathit{modulo}\ \Theta_X(-log\, D_X) \longrightarrow \theta \cdot x \quad \mathit{modulo}\ \mathcal{I}_{D_X},$$

where $\mathcal{I}_{D_X}$ denotes the ideal sheaf of D_X in $\mathcal{O}_X$. If p is a point of Σt_X, the homomorphism $\mathcal{O}_{D_X}([D_X]) \longrightarrow \mathbb{C}_{\Sigma t_X}$ in (5.1.19) is defined as $\phi \longrightarrow \phi(p)$ where ϕ is a local holomorphic function which represents a local holomorphic section of $\mathcal{O}_{D_X}([D_X])$ around p. Next we consider the following sequence:

(5.1.20) $$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X([D_X]) \longrightarrow \mathcal{O}_{D_X}([D_X]) \longrightarrow 0.$$

5.1.4 Lemma.

$$K_X := \mathcal{O}_X((n-4) - D_X) \quad \textit{(Adjunction formula)}, \tag{5.1.21}$$

$$f_*\mathcal{O}_X(a - b\,D_X) \simeq \mathcal{O}_S(a - b\,D_S) \quad (a, b \in \mathbb{Z}), \tag{5.1.22}$$

where $f : X \longrightarrow S$ denotes the normalization map.

Proof. Let $F := \iota \circ f$, where $\iota : S \hookrightarrow P$ is the inclusion map. For each $p \in S$, we take a polycylindrical open neighborhood U of p in P and complex analytic local coordinates (x, y, z) with center p such that S is given by one of the equations (i) $z = 0$ (simple point), (ii) $yz = 0$ (ordinary double point), (iii) $xyz = 0$ (ordinary triple point), and (iv) $xy^2 - z^2 = 0$ (cuspidal point). We put $V := f^{-1}(U)$ for a simple, or cuspidal point p, $V^{(1)} \coprod V^{(2)} := f^{-1}(U)$ for a ordinary double point p, and $V^{(1)} \coprod V^{(2)} \coprod V^{(3)} := f^{-1}(U)$ for a ordinary triple point p. Taking sufficiently small U, we may assume that there are complex analytic local coordinates (u, v) on V and $(u^{(i)}, v^{(i)})$, $i = 1, 2, 3$, such that the map F is given by

$$(u, v) \longrightarrow (u, v, 0) = (x, y, z) \quad \text{if } p \text{ is a simple point,}$$

$$\begin{cases} (u^{(1)}, v^{(1)}) \longrightarrow (u^{(1)}, v^{(1)}, 0) = (x, y, z) \\ (u^{(2)}, v^{(2)}) \longrightarrow (u^{(2)}, 0, v^{(2)}) = (x, y, z) \end{cases} \quad \text{if } p \text{ is an ordinary double point,}$$

$$\begin{cases} (u^{(1)}, v^{(1)}) \longrightarrow (u^{(1)}, v^{(1)}, 0) = (x, y, z) \\ (u^{(2)}, v^{(2)}) \longrightarrow (u^{(2)}, 0, v^{(2)}) = (x, y, z) \\ (u^{(3)}, v^{(3)}) \longrightarrow (0, u^{(3)}, v^{(3)}) = (x, y, z) \end{cases} \quad \text{if } p \text{ is an ordinary triple point,}$$

$$(u, v) \longrightarrow (u^2, v, uv) = (x, y, z) \quad \text{if } p \text{ is a cuspidal point.}$$

Then the lemma follows from direct computation by using the local coordinate expression of f above.

Q.E.D.

By (5.1.21) and (5.1.22), we have

$$\begin{aligned} H^i(X, \mathcal{O}_X([D_X]) &\simeq H^{2-i}(X, \mathcal{O}_X([-D_X]) \otimes K_X) \\ &\simeq H^{2-i}(X, \mathcal{O}_X((n-4) - 2\,D_X)) \simeq H^{2-i}(S, \mathcal{O}_S((n-4) - 2D_S)) \\ &(0 \leq i \leq 2) \end{aligned} \tag{5.1.23}$$

To compute the last cohomology, we consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}_P(-4) \longrightarrow \mathcal{O}_P((n-4) - 2D_S) \longrightarrow \mathcal{O}_S((n-4) - 2D_S) \longrightarrow 0. \tag{5.1.24}$$

We assume that n is sufficiently large enough comparing to the double curve D_S of S so that the following condition is satisfied:

$$H^i(P, \mathcal{O}_P((n-4) - 2D_S)) = 0 \qquad (1 \leq i \leq 3). \tag{5.1.25}$$

Under this assumption, by (5.1.23) and (5.1.24), we have

(5.1.26) $$H^1(X, \mathcal{O}_X([D_X]) \simeq H^1(S, \mathcal{O}_S((n-4) - 2D_S)) = 0,$$

(5.1.27) $$H^0(X, \mathcal{O}_X([D_X]) \simeq H^2(S, \mathcal{O}_S((n-4) - 2\,D_S)) \simeq H^3(P, \mathcal{O}_P(-4)) \simeq H^0(P, \mathcal{O}_P) \simeq \mathbb{C}$$

Then, by the long exact sequence of cohomology derived from the short exact sequence in (5.1.20), we have

(5.1.28) $$H^0(D_X, \mathcal{O}_{D_X}([D_X]) \simeq H^1(X, \mathcal{O}_X),$$

and the exact sequence

(5.1.29) $$0 \longrightarrow H^1(D_X, \mathcal{O}_{D_X}([D_X])) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow H^2(X, \mathcal{O}_X([D_X])) \longrightarrow 0.$$

5.1.5 Proposition. *Under the assumption* (5.1.25), *we have*

(i) $dim_{\mathbb{C}}\, H^0(D_X, \mathcal{N}_{D_X/X}) \leq q(X)$,

(ii) *If* $q(X) = 0$,

$$\begin{aligned} &dim_{\mathbb{C}}\, H^1(D_X, \mathcal{N}_{D_X/X}) \\ &= \chi(D_X, \mathcal{O}_{D_X}([D_X])) - \#\Sigma t_X \\ &= dim_{\mathbb{C}}\, H^0((\mathcal{O}_P((n-4) - 2\,D_S)) - p_g(X) - \#\Sigma t_X \\ &= -dim_{\mathbb{C}}\, H^0(\mathcal{O}_P((n-4) - D_S)) \\ &\qquad + dim_{\mathbb{C}}\, H^0(\mathcal{O}_P((n-4) - 2\,D_S)) - \#\Sigma t_X \end{aligned}$$

Proof. (i) follows from (5.1.19) and (5.28). If $q(X) = 0$, by (5.1.28),

$$H^0(D_X, \mathcal{O}_{D_X}([D_X])) = 0.$$

Hence, by (5.1.19)

$$dim_{\mathbb{C}}\, H^1(D_X, \mathcal{N}_{D_X/X}) = \#\Sigma t_X + dim_{\mathbb{C}}\, H^1(D_X, \mathcal{O}_{D_X}([D_X])).$$

Then (ii) follows from (5.1.19), (5.1.28), (5.1.29), (5.1.23), (5.1.21) and (5.1.24).

Q.E.D.

Now we have the following theorem.

5.1.6 Theorem. *We assume that:*

(i) *The map*

$$H^0(X, \Theta_X(-logD_X)) \oplus H^0(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)) \longrightarrow H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

(cf. Theorem 3.26, (ii) in Part I) *is surjective.*

(ii) X *is regular.*

Then, if the degree of S in $P = \mathbb{P}^3(\mathbb{C})$ is sufficiently large enough comparing to the singular locus D_S of S in the sense that the conditions from (5.1.14) *through* (5.1.18) *and the condition* (5.1.25) *are fulfilled, the map*

$$\tau^{(2)} : H^1(S, \Theta_S) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^0(\Omega^2_{X_\bullet}[1]), \mathbb{H}^1(\Omega^1_{X_\bullet}[1]))$$

in (4.1) (cf. Theorem 4.9) *is injective, that is, the cohomological infinitesimal mixed Torelli theorem holds for the family $\mathcal{S}' = \{S_t\}_{t \in M'}$ of locally trivial displacements of S in $\mathbb{P}^3(\mathbb{C})$ in* Theorem 5.1.2.

5.2 Example. In case of the former example we have $H^0(D_X, \mathcal{N}_{D_X/X}) = 0$. Hence the condition (iii) in Theorem 4.10 is meaningless. In what follows we shall give an example of surfaces with ordinary singularities in $\mathbb{P}^3(\mathbb{C})$ for which the cohomlogical infinitesimal mixed Torelli holds even though $H^0(D_X, \mathcal{N}_{D_X/X}) \neq 0$. We denote by P the complex projective 3-space as in the former example. Let S_1, S_2 be non-singular surface of respective positive degrees r_1, r_2 in the complex projective space P such that they intersect transversally. Let f_i, $(i = 1, 2)$ be the homogeneous polynomial of degree r_i which defines the surface S_i in P. We may assume $r_1 \geq r_2$ because of symmetry. We choose and fix a positive integer n with $n \geq 2r_1$. Let S be a surface in P defined by the equation

$$f := Af_1^2 + 2Bf_1f_2 + Cf_2^2 = 0, \tag{5.2.1}$$

where A, B, C are homogeneous polynoimials of four variables of respective degrees $n - 2r_1$, $n - r_1 - r_2$, $n - 2r_2$. We exclude the case $n = 2r_1 = 2r_2$ where S is reducible. By Bertini's Theorem S is non-singular outside D_S if we choose sufficiently generic A, B, C. Let p be a point of D_S. We may assume that $A(p) \neq 0$, or $C(p) \neq 0$ for generic A, C. We assume $A(p) \neq 0$ because of symmetry.

(i) In the case $A(p) \neq 0$, $(B^2 - AC)(p) \neq 0$: If we put

$$X = f_1 + \frac{B + \sqrt{B^2 - AC}}{A} f_2, \qquad Y = f_1 - \frac{B + \sqrt{B^2 - AC}}{A} f_2,$$

then $f = AXY$.

(ii) In the case $A(p) \neq 0$, $(B^2 - AC)(p) = 0$: If we put

$$X := \frac{B^2 - AC}{A^2}, \qquad Y := f_2, \qquad Z := f_1 + \frac{B}{A} f_2,$$

then $f = -A(XY^2 - Z^2)$. Therefore S is a surface with ordinary singularities whose double curve in D_S for sufficiently generic A, B, C. The formula in (5.1.1) tell that the numerical characters of the non-singular normal model X of S and genus of the double cueve D_S of S are as follows:

$$
\begin{aligned}
&p_a(X) := \chi(X, \mathcal{O}_X) - 1 = \binom{n-1}{3} - (n-4)r_1r_2 + \frac{1}{2}r_1r_2(r_1+r_2-4),\\
&p_g(X) = \binom{n-1-r_1}{3} + \binom{n-1-r_2}{3} - \binom{n-r_1-r_2-1}{3},\\
&q(X) = p_g - p_a = 0,\\
&c_1^2(X) = n(n-4)^2 - (5n-24)r_1r_2 + 2r_1r_2(r_1+r_2-4),\\
&c_2(X) = n(n^2-4n+6) - (7n-24)r_1r_2 + 4r_1r_2(r_1+r_2-4),\\
&g(D_S) = \frac{1}{2}r_1r_2(r_1+r_2-4) + 1.
\end{aligned}
$$

In this case, since D_S is triple point free, the shaef $\mathcal{N}_{D_S/P}$ coinsides with the one of the germs of holomorphic cross-sections of the normal bundle of D_S in P. We define $\mathfrak{f}$ to be the set of all non-singular curves which are the complete intersections $S_1' \cdot S_2'$ of non-singular surfaces S_1' and S_2' of respective degrees n_1 and n_2 in P. The family $\mathfrak{f}$ is an analytic family of locally trivial displacements of D_S in P which satisfies the condition (5.1.2). By a simple calculation we get

$$\dim_{\mathbb{C}} H^0(D_S, N_{D_S/P}) = C(r_1) + C(r_2) - C(r_1 - r_2) - \delta_{r_1r_2} - 2 \tag{5.2.2}$$

where $\delta_{n_1n_2}$ is Kronecker's delta and

$$C(m) = \frac{1}{6}(m+3)(m+2)(m+1), \tag{5.2.3}$$

for a non-negative integer m. Now we consider the following diagram with rows and columbs consisting of exact short sequences of sheaves

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_P & \longrightarrow & \mathcal{O}_P & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_P(n-2D_S) & \longrightarrow & \Phi & \longrightarrow & N_{D_S/P} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_S(n-2D_S) & \longrightarrow & \mathcal{N}_{S/P} & \longrightarrow & N_{D_S/P} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5.2.4}
$$

where the horizontal exact sequence in the middle can be found in [6, Theorem 4] and Φ is the sheaf whose restriction to S is $\mathcal{N}_{S/P}$. Furthermore, we consider the following exact sequence:

(5.2.5)

$$0 \longrightarrow \mathcal{O}_P(n-r_1-r_2) \longrightarrow \mathcal{O}_P(n-2D_S) \longrightarrow \mathcal{O}_{S_1}(n-2D_S)\oplus\mathcal{O}_{S_2}(n-2D_S) \longrightarrow 0.$$

We have

(5.2.6) $$\mathcal{O}_P(n-r_i-2r_j) \simeq \mathcal{O}_{S_i}(n-2r_j) \qquad (i,j)=(1,2), \text{ or } (2,1)$$

(5.2.7) $$0 \longrightarrow \mathcal{O}_P(n-r_i-2r_j) \longrightarrow \mathcal{O}_P(n-2r_j) \longrightarrow \mathcal{O}_{S_i}(n-2r_j) \longrightarrow 0, \quad (i,j)=(1,2), \text{ or } (2,1)$$

By (5.2.5), (5.2.7) and the isomorphism in (5.2.6), we have

(5.2.8) $$H^1(P, \mathcal{O}_P(n-2D_S)) = 0.$$

Thus the condition (5.1.3) is fulfilled. Hence, by Theorem 5.1.1, the surface S belongs to a maximal analytic family $\mathcal{S} := \{S_t\}_{t\in M}$ of locally trivial displacements of S in P which satisfies (1) and (2) in Theorem 5.1.1. From now on we assume

(5.2.9) $\quad n \geq 5.$

Then we have

(5.2.10) $$H^0(S, \Theta_S) = 0, \quad \text{and}$$

(5.2.11) $$dim_{\mathbb{C}}\, H^1(S, \Theta_S) = dim_{\mathbb{C}}\, H^0(S, \mathcal{N}_{S/P}) - 15.$$

On the other hand, by the vertical exact sequence on the left hand side in the diagram (5.2.4) and (5.2.8), we have

$$H^1(S, \mathcal{O}_S(n-2D_S)) = 0.$$

Hence by the horizontal exact sequence at the bottom in the diagram (5.2.4), we have

(5.1.12)

$$dim_{\mathbb{C}}\, H^0(S, \mathcal{N}_{S/P}) = dim_{\mathbb{C}}\, H^0(S, \mathcal{O}_S(n-2D_S)+dim_{\mathbb{C}}\, H^0(D_S, N_{D_S/P}).$$

By the vertical exact sequence on the left hand side in the diagram (5.2.4), the short exact sequences (5.2.5), (5.2.7), and the isomorphisms in (5.2.6), we have

(5.2.13)

$$\begin{aligned} dim_{\mathbb{C}} H^0(S, \mathcal{O}_S(n-2D_S)) &= dim_{\mathbb{C}} H^0(P, \mathcal{O}_P(n-2D_S) - 1 \\ &= dim_{\mathbb{C}} H^0(P, \mathcal{O}_P(n-r_1-r_2)) + \Sigma_{i=1}^2 dim_{\mathbb{C}} H^0(S_i, \mathcal{O}_{S_i}(n-2D_S)) - 1 \\ &= dim_{\mathbb{C}} H^0(P, \mathcal{O}_P(n-r_1-r_2)) + dim_{\mathbb{C}} H^0(S_1, \mathcal{O}_{S_1}(n-2n_2)) \\ &\qquad + dim_{\mathbb{C}} H^0(S_2, \mathcal{O}_{S_2}(n-2n_1)) - 1 \\ &= dim_{\mathbb{C}} H^0(P, \mathcal{O}_P(n-n_1-n_2)) + dim_{\mathbb{C}}, H^0(P, \mathcal{O}_P(n-2n_2)) \\ &\quad - dim_{\mathbb{C}} H^0(P, \mathcal{O}_P(n-n_1-2n_2)) + dim_{\mathbb{C}} H^0(P, \mathcal{O}_P(n-2n_1)) \\ &\qquad - dim_{\mathbb{C}} H^0(P, \mathcal{O}_P(n-2n_1-n_2)) - 1 \\ &= C(n-r_1-r_2) + C(n-2r_1) + C(n-2r_2) \\ &\qquad - C(n-r_1-2r_2) - C(n-2r_1-r_2) - 1 \end{aligned}$$

By (5.2.12), (5.2.2) and (5.2.13), $dim_{\mathbb{C}} H^0(S, \mathcal{O}_S(n-2D_S))$ depends on only the integers n, r_1 and r_2. Hence the condition (5.1.13) is also fulfilled. Cosequently, we have the following:

5.2.1 Theorem. *If $n \geq 5$, then the surface S defined by the equation (5.2.1) in $\mathbb{P}^3(\mathbb{C})$ belongs to the Kuranishi family $\mathcal{S}' = \{S_t\}_{t \in M'}$ of locally trivial displacements of S such that:*

(1) *For every point $t \in M'$, S_t is a surface with ordinary singularities in $\mathbb{P}^3(\mathbb{C})$ whose double curve is a non-singular complete intersection,*
(2) *the parameter space M' is non-singular and*

$$\begin{aligned} dim_{\mathbb{C}} M' = C(n-r_1-r_2) &+ C(n-2r_1) + C(n-2r_2) \\ -C(n-r_1-2r_2) &- C(n-2r_1-r_2) \\ +C(r_1) + C(r_2) &- C(r_1-r_2) - \delta_{r_1 r_2} - 18 \end{aligned}$$

(3) *the characteristic map*

$$\sigma_t^{\mathcal{S}'} : T_t(M') \longrightarrow H^1(S_t, \Theta_{S_t})$$

is bijective at every point $t \in M'$.

From now on, we assume that

(5.2.14)
$$\begin{aligned} (n, r_1, r_2) \neq (3.1, 1),\ (4,1,1),\ (4,2,1),\ (5,1,1), (5,2,1), \\ (5,2,2),\ (6,2,2),\ (6,3,2),\ (7,3,3). \end{aligned}$$

Then, as shown in [16], the non-singular normal model X of S is regular, i.e., $q(X) = 0$, minimal algebraic surface of general type and condition (ii) of Theorem 4.10 is satisfied. Furthermore, the condition (i) of Theorem 4.10 is also satisfied. Indeed, by Hurwitz's formula, we have

$$\begin{aligned} g(D_X^*) &= g(D_X) - 1 + \frac{1}{2}(\#\Sigma c_S) \\ &= r_1r_2(r_1r_2 - 4) + 2 - 1 + r_1r_2(n - r_1 - r_2) \\ &= r_1r_2(n-4) + 1. \end{aligned}$$

Under the condition (5.2.14), we have $n \geq 6$ and so $g(D_X^*) \geq 3$ since $r_1 \geq r_2 \geq 1$. From this it follows that

$$H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*)) = 0. \tag{5.2.15}$$

Hence the condition (i) of Theorem 4.10 is filfilled as asserted. In order to see that Theorem 4.10 effectively works when we consider the infinitesimal mixed Torelli problem for the surface S defined by the equation (5.2.1) satisfying the condition (5.2.14), we are now going to find out a surface S with $H^0(X, \mathcal{N}_{D_X/X}) \neq 0$ among these surfaces. We denote by

$$h : H^1(S, \Theta_S) \to H^1(X, \Theta_X) \tag{5.2.16}$$

the composite of the homomorphism $H^1(S, \Theta_S) \to H^1(X, \Theta_X(-log\, D_X))$ in Corollary 4.5 and the one $H^1(X, \Theta_X(-log\, D_X)) \to H^1(X, \Theta_X)$. For the surface S under consideration, we have $H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*)) = 0$ (cf. (5.2.15)), and so the condition in Corollary 4.5 is fulfilled. Hence the homomorphism $H^1(S, \Theta_S) \to H^1(X, \Theta_X(-log\, D_X))$ is injective. Therefore $H^0(D_X, N_{D_X/X}) \neq 0$ if the homomorphism in (5.2.16) is not injective, because the kernel of the homomorphism

$$H^1(X, \Theta_X(-log\, D_X)) \to H^1(X, \Theta_X)$$

is the image of $H^0(D_X, N_{D_X/X})$ in $H^1(X, \Theta_X(-log\, D_X)$, where $N_{D_X/X}$ denotes the sheaf of germs of holomorphic cross-sections of the normal bundle of D_X in X. Regarding the injectivity of the homomorphism in (5.2.16), we have the following:

5.2.2 Theorem. *For the surface S defined in* (5.2.1) *satisfying the condition* (5.2.14), *the homomorphism $h : H^1(S, \Theta_S) \to H^1(X, \Theta_X)$ in* (5.2.16) *is not injective if and only if*

$$H^2(P, \Omega_P^1([S + K_P] - D_S)) \neq 0,$$

where $P = \mathbb{P}^3(\mathbb{C})$ and K_P denotes the canonical line bundle of P.

Proof. We denote by F the composite of $f : X \to S$ and the inclusion map $S \hookrightarrow P$, and define the sheaf $\mathcal{T}_F$ to be the quotint of $F^*\Theta_P$ by Θ_X. We consider the following short exact sequences:

$$0 \to \Theta_X \to F^*\Theta_P \to \mathcal{T}_F \to 0, \tag{5.2.17}$$

$$(5.2.18) \qquad 0 \longrightarrow \Theta_S \longrightarrow \Theta_P \otimes \mathcal{O}_S \longrightarrow \mathcal{N}_{S/P} \longrightarrow 0.$$

By [10, Proposition 9.1], we have

$$(5.2.19) \qquad \mathcal{N}_{S/P} \simeq F_* \mathcal{T}_F.$$

Since X is of general type,

$$(5.2.20) \qquad H^0(X, \Theta_X) = H^0(S, \Theta_S) = 0.$$

By the exact sequence

$$0 \longrightarrow \Theta_P(-S) \longrightarrow \Theta_P \longrightarrow \Theta_P \otimes \mathcal{O}_S \longrightarrow 0,$$

we have

$$(5.2.21) \qquad H^1(S, \Theta_P \otimes \mathcal{O}_S) = 0,$$

since $n \neq 4$. Therefore, by (5.2.17), (5.2.18), (5.2.19), (5.2.20) and (5.2.21), we have the following commutative diagram with exact rows and columbs:

(5.2.22)

$$\begin{tikzcd}
 & 0 & & \\
0 \arrow[r] & H^0(X, \mathcal{T}_F)/Im\, H^0(X, F^*\Theta_P) \arrow[u] \arrow[r] & H^1(X, \Theta_X) \arrow[r] & 0 \\
0 \arrow[r] & H^0(S, \mathcal{N}_{S/P})/Im\, H^0(S, \Theta_S \otimes \mathcal{O}_S) \arrow[u] \arrow[r] & H^1(S, \Theta_S) \arrow[u, Rightarrow, "h"] \arrow[r] & 0 \\
 & H^0(X, F^*\Theta_P)/Im\, H^0(S, \Theta_P \otimes \mathcal{O}_S) \arrow[u] & & \\
 & 0. \arrow[u] & &
\end{tikzcd}$$

Let $\sigma : P_1 \longrightarrow P$ be the blowing-up of P along the non-singular center D_S. Then we may regard X as the proper inverse image of S by the map σ and $F := \sigma_{|X} : X \longrightarrow P$, the restriction of σ to X. Now we have the following exact sequence:

$$(5.2.23) \qquad 0 \longrightarrow \sigma^*\Theta_P(-X) \longrightarrow \sigma^*\Theta_P \longrightarrow F^*\Theta_P \longrightarrow 0.$$

Since $H^1(P_1, \sigma^*\Theta_P) \simeq H^1(P, \sigma_*\sigma^*\Theta_P) = 0$,

(5.2.24)

$$H^0(X, F^*\Theta_P)/Im\, H^0(S, \Theta_P \otimes \mathcal{O}_S) \simeq H^0(X, F^*\Theta_P)/Im\, H^0(P_1, \sigma\Theta_P) \simeq H^1(P_1, (\sigma^*\Theta_P(-X)).$$

On the other hand, by [16, Proposition 1.2], we have

$$H^1(P_1, \sigma^*\Theta_P(-X)) \simeq H^2(P, \Omega^1_P([S + K_P] - D_S)). \tag{5.2.25}$$

Therefore, by (5.2.25), (5.2.24) and (5.2.22), we infer that the homomorphism $h : H^1(S, \Theta_S) \longrightarrow H^1(X, \Theta_X)$ is not injective if and only if $H^2(P, \Omega^1_P([S+K_P] - D_S)) \neq 0$.

Q.E.D.

We are now going to find out a surface S with $H^0(P, \Omega^1_P([S+K_P]-D_S)) \neq 0$ among such surfaces in (5.2.1) that satisfy the condition (5.2.12). We refer to the following exact sequence from [4, Theorem 8.13]:

$$0 \longrightarrow \Omega^1_P \longrightarrow \mathcal{O}_P(-1)^{\oplus 4} \longrightarrow \mathcal{O}_P \longrightarrow 0.$$

Tensoring $\mathcal{O}_P((n-4) - D_S)$ to this exact sequence, we have the following:

(5.2.26)

$$O \longrightarrow \Omega^1_P((n-4) - D_S) \longrightarrow \mathcal{O}_P((n-5) - D_S)^{\oplus 4} \longrightarrow \mathcal{O}_P((n-4) - D_S) \longrightarrow 0.$$

A resolution of the ideal shaef $\mathcal{I}_{D_S}$ of D_S in $\mathcal{O}_P$ by locally free sheaves is obtained as follows:

$$0 \longrightarrow \mathcal{O}_P(-r_1 - r_2) \longrightarrow \mathcal{O}_P(-r_1) \oplus \mathcal{O}_P(-r_2) \longrightarrow \mathcal{I}_{D_S} \longrightarrow 0.$$

From this we have the short exact sequence

(5.2.27)

$$0 \longrightarrow \mathcal{O}_P(n - r_1 - r_2 - k) \longrightarrow \oplus_{i=1}^2 \mathcal{O}_P(n - r_i - k) \longrightarrow \mathcal{O}_P((n-k) - D_S) \longrightarrow 0.$$

By this exact sequence for $k = 4$, we have

$$H^1(P, \mathcal{O}_P((n-4) - D_S) = 0.$$

Hence, by the long exact sequence of cohomology derived from (5.2.26), we have

$$0 \longrightarrow H^2(P, \Omega^1_P((n-4) - D_S)) \longrightarrow H^2(P, \mathcal{O}_P((n-5) - D_S))^{\oplus 4} \longrightarrow H^2(P, \mathcal{O}_P((n-4) - D_S)) \longrightarrow \cdots. \tag{5.2.28}$$

By the long sequence of cohomology derived from (5.2.27), we have

$$(5.2.29)\quad \begin{aligned} 0 \longrightarrow H^2(P,\mathcal{O}_P((n-k)-D_S)) &\longrightarrow H^3(P,\mathcal{O}_P(n-r_1-r_2-k)) \\ &\longrightarrow \oplus_{i=1}^2 H^3(P,\mathcal{O}_P(n-r_i-k)) \longrightarrow \cdots \quad (k=4,5) \end{aligned}$$

In order to find out (n,r_1,r_2) for which $H^2(P,\Omega^1_P((n-4-D_S))\neq 0$, we first look for (n,r_1,r_2) for which $H^3(P,\mathcal{O}_P(n-r_1-r_2-5))\neq 0$. By duality,

$$(5.2.30)\qquad H^3(P,\mathcal{O}_P(n-r_1-r_2-5))\simeq H^0(P,\mathcal{O}_P(r_1+r_2+1-n)).$$

Hence, $H^0(P,\mathcal{O}_P(n-r_1-r_2-5)\neq 0$ if and only if

$$r_1+r_2+1\geq n.$$

Since $n\geq 2r_1$ and $r_1\geq r_2$, this inequality implies

$$\begin{aligned} &r_1=r_2,\ \ 2r_1+1\geq n\geq 2r_1, \text{ or} \\ &r_1=r_2+1,\ \ n=2r_1. \end{aligned}$$

We exclude the case $n=2r_1$ and $r_1=r_2$, since in this case, S becomes reducible. Assume $(n,r_1,r_2)=(2r+1,r,r)$ $(r\geq 1)$. Then by (5.2.29) for $k=5$, we have

$$(5.2.31)\qquad H^2(P,\mathcal{O}_P((n-5)-D_S))\simeq \mathbb{C}.$$

On the other hand, for those cases, we have

$$H^3(P,\mathcal{O}_P(n-r_1-r_2-4))\simeq H^0(P,\mathcal{O}_P(r_1+r_2-n))\simeq H^0(P,\mathcal{O}_P(-1))=0.$$

Hence, by (5.2.29) for $k=4$, we have

$$(5.2.32)\qquad H^0(P,\mathcal{O}_P((n-4)-D_S)=0.$$

Consequently, for $(n,r_1,r_2)=(2r+1,r,r)$ $(r\geq 1)$, or $(2r,r,r-1)$ $(r\geq 2)$, by (5.2.26), we have

$$H^0(P,\Omega^1_P((n-4)-D_S)\simeq H^3(P,\mathcal{O}_P(n-r_1-r_2-5))^{\oplus 4}\simeq \mathbb{C}^{\oplus 4}.$$

Now we are going to compute the dimension of $H^0(D_X,\mathcal{N}_{D_X/X})$. Note that, in this case,

$$\mathcal{N}_{D_X/X}=\mathcal{O}_{D_X}([D_X])=N_{D_X/X},$$

since D_S is triple point free, where $N_{D_X/X}$ denotes the sheaf of germs of local holomorphic cross-sections of the normal bundle of D_X in X. Since X is regular, by (5.1.19), (5.1.20) and (5.1.23), we have

(5.2.33)

$$\begin{aligned} dim_{\mathbb{C}} H^0(D_X,\mathcal{N}_{D_X/X}) &= dim_{\mathbb{C}} H^0(X,\mathcal{O}_X([D_X]))-1 \\ &= dim_{\mathbb{C}} H^2(S,\mathcal{O}_S((n-4)-2D_S))-1 \end{aligned}$$

By the short exact sequences

$$0 \longrightarrow \mathcal{O}_P(-4) \longrightarrow \mathcal{O}_P((n-4)-2D_S) \longrightarrow \mathcal{O}_S((n-4)-2D_S) \longrightarrow 0,$$

and (5.2.5), (5.2.6) and (5.2.7) replaced n by $n-4$, we have

(5.2.34)

$$\begin{aligned}
&dim\, H^2(S, \mathcal{O}_S((n-4)-2D_S)) \\
&= dim_{\mathbb{C}}\, H^3(P, \mathcal{O}_P((n-4)-r_1-2r_2)) - dim_{\mathbb{C}}\, H^3(P, \mathcal{O}_P((n-4)-2r_1)) \\
&\quad + dim_{\mathbb{C}}\, H^3(P, \mathcal{O}_P((n-4)-r_2-2r_1)) - dim_{\mathbb{C}}\, H^3(P, \mathcal{O}_P((n-4)-2r_2)) \\
&\quad - dim_{\mathbb{C}}\, H^3(P, \mathcal{O}_P((n-4)-r_1-r_2)) + 1 \\
&= dim_{\mathbb{C}}\, H^0(P, \mathcal{O}_P(r_1+2r_2-n)) - dim_{\mathbb{C}}\, H^0(P, \mathcal{O}_P(2r_1-n)) \\
&\quad + dim\, H^0(P, \mathcal{O}_P(2r_1+r_2-n)) - dim\, H^0(P, \mathcal{O}_P(2r_2-n)) \\
&\quad - dim\, H^0(P, \mathcal{O}_P(r_1+r_2-n)) + 1 \\
&= \begin{cases} 2C(r-1)+1 & \text{if } (n, r_1, r_2) = (2r+1, r, r) \quad (r \geq 1) \\ C(r-1)+C(r-2) & \text{if } (n, r_1, r_2) = (2r, r, r-1) \quad (r \geq 2) \end{cases}
\end{aligned}$$

Then, by (5.2.33), we can know $dim_{\mathbb{C}}\, H^0(D_X, \mathcal{N}_{D_X/X})$. We summarize the results as follows:

5.2.3 Proposition. *Among surfaces S defined in* (5.2.1) *satisfying the condition* (5.2.14), *the surfaces for which $H^0(D_X, \mathcal{N}_{D_X/X}) \neq 0$ are only those of types $(n, r_1, r_2) = (2r+1, r, r)$ $(r \geq 1)$ and $(n, r_1, r_2) = (2r, r, r-1)$ $(r \geq 2)$. For those surfaces we have*

$$dim\, H^0(D_X, \mathcal{N}_{D_X/X}) =$$

$$\begin{cases} \dfrac{1}{3}(r+2)(r+1)r & \text{if } (n, r_1, r_2) = (2r+1, r, r)\ (r \geq 1) \\[2ex] \dfrac{1}{6}(r+2)(r+1) + \dfrac{1}{6}(r+1)r(r-1) - 1 & \text{if } (n, r_1, r_2) = (2r, r, r-1)\ (r \geq 2) \end{cases}$$

Now we are going to show that the condition (iii) in Theorem 4.10 is satisfied by the surface S of type $(2r+1, r, r)$ and $(2r, r, r-1)$ with $r \geq 4$. The condition $r \geq 4$ is in order that S satisfies the condition (5.2.14). We remember that the normal model of the surface S is an irregular, minimal algebraic surface of general type. Since $H^0(X, \Theta_X) = H^0(X, \Omega^1_X) = 0$, and since S is triple point free, the condiotion (iii) in Theorem 4.10 for the surface S is restated as follows:

(5.2.35) The homomorphism

$$\overline{\mu}^{(2)} : H^0(D_X, N_{D_X/X})$$

$$\longrightarrow \mathrm{Hom}_{\mathbb{C}}(H^0(X,\Omega_X^2), H^0(D_X,\Omega_{D_X}^1)/Im\, H^0(D_S,\Omega_{D_S}^1))$$

defined by taking *contraction* and the *pull-back* is injective.

The strategy to see that this condition is satisfied is to reduce the condition on X to the one on S.

5.2.4 Proposition.

(i) $H^0(D_X, N_{D_X/X})$

$$\simeq \mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S),\omega_S)/Im\,\mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-D_S),\omega_S)$$

(ii) $H^0(X,\Omega_X^2) \simeq H^0(S,\omega_S(-D_S))$,

(iii) $H^0(D_X,\Omega_{D_X}^1) \simeq \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S),\omega_S)/Im\,\mathrm{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X,\omega_S)$

(iv) $H^0(D_S,\Omega^1_{D_S}) \simeq \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S),\omega_S)/Im\,\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S,\omega_S)$

(v) $H^0(D_X,\Omega_{D_X}^1)/Im\,H^0(D_S,\Omega_{D_S}^1)$

$$\simeq \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S,\omega_S)/Im\,\mathrm{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X,\omega_S)$$

where $\omega_S := \mathcal{O}_S(n-4)$ *is the dualizing sheaf of* S *(*n *= the degree of* S *in* $\mathbb{P}^3(\mathbb{C})$*), and* $f: X \longrightarrow S$ *the normalization map.*

Proof. In what follows we denote by the symbol $\vee$ the dual objects of various things such as cohomology groups, locally free sheaves, e.t.c..

(i) By the duality and the adjunction formula, we have

$$H^0(D_X,N_{D_X/X}) \simeq H^1(D_X,N_{D_X/X}^\vee\otimes\mathfrak{K}_{D_X})^\vee \simeq H^1(D_X,K_{X|D_X})^\vee, \tag{5.2.36}$$

$$H^1(D_X,K_{X|D_X})^\vee \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(K_{X|D_X},K_X), \tag{5.2.37}$$

where $\mathfrak{K}_{D_X}$ denotes the canonical bundle of D_X, and K_X that of X. Hence we have

$$H^0(D_X,N_{D_X/X}) \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(K_{X|D_X},K_X). \tag{5.2.38}$$

On the other hand, since $\mathrm{Hom}_{\mathcal{O}_X}(K_{X|D_X},K_X) = \mathrm{Ext}^1_{\mathcal{O}_X}(K_X,K_X) = 0$, the short exact sequence

$$0 \longrightarrow K_X(-D_X) \longrightarrow K_X \longrightarrow K_{X|D_X} \longrightarrow 0$$

implies the short exact sequence

$$\begin{aligned} 0 \longleftarrow \mathrm{Ext}^1_{\mathcal{O}_X}(K_{X|D_X},K_X) \longleftarrow \mathrm{Hom}_{\mathcal{O}_X}(K_X(-D_X),K_X) \\ \longleftarrow \mathrm{Hom}_{\mathcal{O}_X}(K_X,K_X) \longleftarrow 0 \end{aligned} \tag{5.2.39}$$

By (5.2.38) and (5.2.39), we have

$$H^0(D_X, N_{D_X/X}) \simeq \mathrm{Hom}_{\mathcal{O}_X}(K_X(-D_X), K_X)/\mathrm{Im}\,\mathrm{Hom}_{\mathcal{O}_X}(K_X, K_X). \tag{5.2.40}$$

In order to relate the right-hand-side of (5.2.40) to the cohomology concerning S, we consider the dual of (5.2.39). Then, since $H^1(X, K_X) \simeq H^1(X, \mathcal{O}_X) = 0$, we have the following exact sequence:

$$0 \to H^1(D_X, K_{X|D_X}) \to H^2(X, K_X(-D_X)) \to H^2(X, K_X) \to 0. \tag{5.2.41}$$

Comparing (5.2.39) with (5.2.41), we have

(5.2.42)
$$\begin{aligned}
&\{\mathrm{Hom}_{\mathcal{O}_X}(K_X(-D_X), K_X)/\mathrm{Im}\,\mathrm{Hom}_{\mathcal{O}_X}(K_X, K_X)\}^\vee \\
&\simeq Im\,\{H^1(D_X, K_{X|D_X}) \to H^2(X, K_X(-D_X))\} \\
&\simeq Im\,\{H^1(D_X, (f^*\omega_S)(-D_X)_{|D_X}) \to H^2(X, (f^*\omega_S)(-2D_X))\} \\
&\simeq Im\,\{H^1(D_S, \omega_S(-D_S)_{|D_S}) \to H^2(S, \omega_S(-2D_S))\}
\end{aligned}$$

Here the second isomorphism follows from the adjunction formula

$$K_X := (f^*\omega_S)(-D_X) \simeq f^*(\omega_S(-D_S))$$

and the third one follows from (5.1.22) and the commutative diagram with exact rows

(5.2.43)
$$\begin{array}{ccccc}
\longrightarrow & H^1(X, K_X) & \longrightarrow & H^1(X, K_{X|D_X}) & \\
& \uparrow & & \uparrow & \\
\longrightarrow & H^1(S, \omega_S(-D_S)) & \longrightarrow & H^1(D_S, \omega_S(-D_S)_{|D_S}) & \\
\longrightarrow & H^2(X, K_X(-D_X)) & \longrightarrow & H^2(X, K_X) & \longrightarrow 0 \\
& \uparrow & & \uparrow & \\
\longrightarrow & H^2(S, \omega_S(-2D_S)) & \longrightarrow & H^2(S, \omega_S(-D_S)) & \longrightarrow 0,
\end{array}$$

where the horizontal exact sequence at the bottom follows from

$$0 \to \omega_S(-2D_S) \to \omega_S(-D_S) \to \omega_S(-D_S)_{|D_S} \to 0.$$

Taking the dual of the horizontal exact sequence at the bottom in (5.2.43), we have

(5.2.44)
$$\begin{aligned}
\cdots \leftarrow \mathrm{Ext}^1_{\mathcal{O}_S}(\omega_S(-D_S)_{|D_S}, \omega_S) \leftarrow \mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S), \omega_S) \\
\leftarrow \mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-D_S), \omega_S) \leftarrow 0.
\end{aligned}$$

Comparing the horizontal exact sequence at the bottom in (5.2.43) with (5.2.44), we have

$$(5.2.45)\qquad \begin{aligned} &Im\{H^1(D_S,\omega_S(-D_S)_{|D_S}) \longrightarrow H^2(S,\omega_S(-2D_S))\}^\vee \\ &\simeq \mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S),\omega_S)/Im\,\mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-D_S),\omega_S). \end{aligned}$$

Cosequently, by (5.2.40), (5.2.42) and (5.2.45), we have the assertion (i).

(ii) The assertion (ii) follows from the *adunction formmula.*

(iii) By duality, we have

$$(5.2.46)\qquad H^0(D_X,\Omega^1_{D_X})^\vee \simeq H^1(D_X,\mathcal{O}_{D_X}),\quad \text{and}$$

$$(5.2.47)\qquad H^1(D_X,\mathcal{O}_{D_X})^\vee \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{D_X},K_X).$$

By the short exact sequence of $\mathcal{O}_X$-modules

$$(5.2.48)\qquad 0 \longrightarrow \mathcal{O}_X(-D_X) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{D_X} \longrightarrow 0,$$

we have the exact sequence

$$(5.2.49)\qquad \begin{aligned} 0 \longleftarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{D_X},K_X) &\longleftarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-D_X),K_X) \\ &\longleftarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,K_X) \longleftarrow 0. \end{aligned}$$

Hence we have

(5.2,50)

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_{D_X},K_X) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-D_X),K_X)/Im\,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,K_X).$$

By (5.2.46), (5.2.47) and (5.2.50), we have

$$(5.2.51)\qquad H^0(D_X,\Omega^1_{D_X}) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-D_X),K_X)/Im\,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,K_X).$$

Since X is regular, i.e., $H^1(X,\mathcal{O}_X)=0$, the exact sequence

$$0 \longrightarrow H^1(D_X,\mathcal{O}_{D_X}) \longrightarrow H^2(X,\mathcal{O}_X(-D_X)) \longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow 0$$

follows from (5.2.48), which is dual to the exact sequence in (5.2.49). Comparing (5.2.49) with the exact sequence above, we have

(5.2.52)

$$\begin{aligned} &\{\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-D_X),K_X)/Im\,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,K_X)\}^\vee \\ &\simeq Im\{H^1(D_X,\mathcal{O}_{D_X}) \longrightarrow H^2(X,\mathcal{O}_X(-D_X))\} \\ &\simeq Im\{H^1(D_S,g_*\mathcal{O}_{D_X}) \longrightarrow H^2(S,f_*\mathcal{O}_X(-D_X))\}. \end{aligned}$$

Here the second isomorphism follows from fact that the maps $g : D_X \longrightarrow D_S$ and $f : X \longrightarrow S$ are finite maps. Taking the image of the short exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow f_*\mathcal{O}_X(-D_X) \longrightarrow f_*\mathcal{O}_X \longrightarrow g_*\mathcal{O}_{D_X} \longrightarrow 0.$$

From this, the long exact sequences

$$\cdots \longleftarrow \mathrm{Ext}^1_{\mathcal{O}_S}(g_*\mathcal{O}_{D_X}, \omega_S) \longleftarrow \mathrm{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X(-D_X), \omega_S) \longleftarrow \mathrm{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X, \omega_S) \longleftarrow 0,$$

and

$$\cdots \longrightarrow H^1(D_S, g_*\mathcal{O}_{D_X}) \longrightarrow H^2(S, f_*\mathcal{O}_X(-D_X)) \longrightarrow H^2(S, f_*\mathcal{O}_X) \longrightarrow 0,$$

follows, which are dual to each other. Therefore we have

$$\begin{aligned} &[Im\,\{H^1(D_S, g_*\mathcal{O}_{D_X}) \longrightarrow H^2(S, f_*\mathcal{O}_X(-D_X)\}]^\vee \\ &\simeq \mathrm{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X(-D_X), \omega_S)/Im\,\mathrm{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X, \omega_S). \end{aligned} \tag{5.2.53}$$

Consequently, by (5.2.51), (5.2.52), (5.2.53) and (5.1.22), we have the assertion (iii).

(iv) By duality, we have

$$H^0(D_S, \Omega^1_{D_S})^\vee \simeq H^1(D_S, \mathcal{O}_{D_S}), \tag{5.2.54}$$

$$H^1(D_S, \mathcal{O}_{D_S})^\vee \simeq \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_{D_S}, \omega_S). \tag{5.2.55}$$

By the short exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow \mathcal{O}_S(-D_S) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{D_S} \longrightarrow 0, \tag{5.2.56}$$

we have the exact sequence

$$\begin{aligned} &0 \longleftarrow \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_{D_S}, \omega_S) \longleftarrow \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S) \\ &\qquad \longleftarrow \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \omega_S) \longleftarrow 0. \end{aligned} \tag{5.2.57}$$

Hence we have

$$\mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_{D_S}, \omega_S) \simeq \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S)/Im\,\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \omega_S). \tag{5.2.58}$$

By (5.2.54), (5.2.55) and (5.2.58), we have the assertion (iv)

(v) Consider the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S)/Im\,\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \omega_S)$$

(5.2.59) $$\to \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S)/Im \operatorname{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X, \omega_S)$$
$$\to \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \omega_S)/\mathrm{Im} \operatorname{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X, \omega_S) \to 0.$$

Here, by the assertions (ii) and (iii) of the proposition, we identify

$$\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S)/Im \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \omega_S)$$
$$(resp.\ \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S)/Im \operatorname{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X, \omega_S))$$

with $H^0(D_S, \Omega^1_{D_S})$ (resp. $H^0(D_X, \Omega^1_{D_X})$). Now let us recall that the space

$$\{\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \omega_S)/Im \operatorname{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X, \omega_S)\}$$

can be identified with a subspace of $H^0(D_X, \Omega^1_{D_X})$. Since $g : D_X \to D_S$ is a double conering map, there is an isomorphism σ on D_X with $\sigma^2 = id_{D_X}$. Therefore, if we denote by $H^0(D_X, \Omega^1_{D_X})^+$ (resp. $H^0(D_X, \Omega^1_{D_X})^-$) the subspace of $H^0(D_X, \Omega^1_{D_X})$ consisting of differential 1-forms ω on D_X with $\sigma^*\omega = \omega$ (resp. $\sigma^*\omega = -\omega$), then we have

$$H^0(D_X, \Omega^1_{D_X}) = H^0(D_X, \Omega^1_{D_X})^+ \oplus H^0(D_X, \Omega^1_{D_X})^-, \quad \text{and}$$
$$H^0(D_X, \Omega^1_{D_X})^+ \simeq H^0(D_S, \Omega^1_{D_S}).$$

Now, by the assertion (iii) and (iv) and (5.2.59), we infer that

$$\{\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \omega_S)/Im \operatorname{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X, \omega_S)\} \cap H^0(D_X, \Omega^1_{D_X})^+ = \{0\},$$

and the homomorphism

(5.2.60)

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \omega_S)/Im \operatorname{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X, \omega_S) &\to H^0(D_X, \Omega^1_{D_X})/H^0(D_X, \Omega^1_{D_X})^+ \\ &\simeq H^0(D_X, \Omega^1_{D_X})/Im\, H^0(D_S, \Omega^1_{D_S})\end{aligned}$$

is surjective. Consequently, we conclude the homomorphism in (5.2.60) is an isomorphism.

Q.E.D.

For any $\phi \in \operatorname{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S), \omega)$ and $\omega \in H^0(S, \omega_S(-D_S))$ we define $\phi \circ \omega \in \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S)$ by

$$(\phi \circ \omega)(s) := \phi(\omega \otimes s) \qquad \text{for } s \in \mathcal{O}_S(-D_S),$$

and put

$$V_\phi := \{\phi \circ \omega \in \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S) \,|\, \omega \in H^0(S, \omega_S(-D_S))\}.$$

With this notation, by Proposition 5.2.4, we can restate the condition in (5.2.35) as follows:

(5.2.61) If

$$V_\phi \cap Im\{ \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \omega_S) \longrightarrow \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S) \}$$
$$\hookrightarrow Im \{ \operatorname{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X, \omega_S) \longrightarrow \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S) \} ,$$

then

$$\phi \in Im \{ \operatorname{Hom}_{\mathcal{O}_S}(\omega_S(-D_S), \omega_S) \longrightarrow \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-2D_S), \omega_S) \}.$$

In what follows we shall restate the condition (5.2.61) in terms of the homogeneous polynomial rings for the algebraic surface with ordinary singularities defined by the equation (5.2.1). We denote the homogeneous coordinates of $\mathbb{P}^3(\mathbb{C})$ by $(X_0 : X_1 : X_2 : X_3)$, the polynomial ring of variables X_0, X_1, X_2, X_3 by $\mathbb{C}[X_0 : X_1 : X_2 : X_3]$, or simply by $\mathbb{C}[X]$, the homogeneous part of degree k of the graded polynomial ring $\mathbb{C}[X]$ by $\mathbb{C}[X]_k$, and that of the homogeneous ideal I(S) generated by the polynomial which defines S in $\mathbb{C}[X]$ by $\mathrm{I(S)}_k$.

5.2.5 Proposition.

In what follows f_1, f_2 are the same ones as in the equation (5.2.1).

(i)
$$\begin{aligned} &\operatorname{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S), \omega_S) \\ &\simeq \{ (\phi_1, \phi_2, \phi_3) \in \mathbb{C}[X]_{2r_1} \oplus \mathbb{C}[X]_{r_1+r_2} \oplus \mathbb{C}_{2r_2}[X] \\ &\qquad \textit{modulo } \mathrm{I(S)}_{2r_1} \oplus \mathrm{I(S)}_{r_1+r_2} \oplus \mathrm{I(S)}_{2r_2} \\ &\qquad\qquad \mid \phi_1 f_2 - \phi_2 f_1 \in \mathrm{I(S)}, \phi_3 f_1 - \phi_2 f_2 \in \mathrm{I(S)} \}. \end{aligned}$$

(ii)
$$\begin{aligned} &\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S) \\ &\simeq \{ (F, G) \in \mathbb{C}[X]_{n-4+r_1} \oplus \mathbb{C}[X]_{n-4+r_2} \\ &\qquad \textit{modulo } \mathrm{I(S)}_{n-4+r_1} \oplus \mathrm{I(S)}_{n-4+r_2} \\ &\qquad\qquad \mid f_2 F - f_1 G \in I(S) \}, \end{aligned}$$

(iii) *By the isomorphism in (i) above, we represent an element*

$$\phi \in \operatorname{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S), \omega_S)$$

by

$$(\phi_1, \phi_2, \phi_3) \in \mathbb{C}[X]_{2r_1} \oplus \mathbb{C}[X]_{r_1+r_2} \oplus \mathbb{C}_{2r_2}$$

with $\quad \phi_1 f_2 - \phi_2 f_1 \in \mathrm{I(S)}$ *and* $\phi_3 f_1 - \phi_2 f_2 \in \mathrm{I(S)}$.

Then the subspace V_ϕ of $\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S)$ *defined in* (5.2.61) *can be represented as*

$$V_\phi = \{ (\omega_1\phi_1 + \omega_2\phi_2, \omega_1\phi_2 + \omega_2\phi_3) \in \mathbb{C}[X]_{n-4+r_1} \oplus \mathbb{C}[X]_{n-4+r_2}$$
$$\textit{modulo } \mathrm{I(S)}_{n-4+r_1} \oplus \mathrm{I(S)}_{n-4+r_2}$$
$$\mid (\omega_1, \omega_2) \in \mathbb{C}[X]_{n-4-r_1} \oplus \mathbb{C}[X]_{n-4-r_2} \},$$

where we identify $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S)$ *with the space on the right-hand-side in (ii) by the isomorphism there.*

(iv) *By the isomorphism in (ii) above, we identify* $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S)$ *with the space on the right-hand-side in (ii). Then we have*

(a) $Im\{ \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \omega_S) \longrightarrow \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S) \}$

$$= \{ (f_1\omega, f_2\omega) \in \mathbb{C}[X]_{n-4+r_1} \oplus \mathbb{C}[X]_{n-4+r_2}$$
$$\textit{modulo } \mathrm{I(S)}_{n-4+r_1} \oplus \mathrm{I(S)}_{n-4+r_2} \mid \omega \in \mathbb{C}[X]_{n-4} \}$$

(b) $Im\{ \mathrm{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X, \omega_S) \longrightarrow \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S) \}$

$$= \{ ((\omega_1 f_1 + \omega_2 f_2) f_1, (\omega_1 f_1 + \omega_2 f_2) f_2) \in \mathbb{C}[X]_{n-4+r_1} \oplus \mathbb{C}[X]_{n-4+r_2}$$
$$\textit{modulo } \mathrm{I(S)}_{n-4+r_1} \oplus \mathrm{I(S)}_{n-4+r_2}$$
$$\mid (\omega_1, \omega_2) \in \mathbb{C}[X]_{n-4-r_1} \oplus \mathbb{C}[X]_{n-4-r_2} \}$$

(v) *By the isomorphism (i) above, we identify* $\mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S), \omega_S)$ *with the space on the right-hand-side in (i) by the isomorphism there. Then we have*

$$Im\{ \mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-D_S), \omega_S) \longrightarrow \mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S), \omega_S) \}$$
$$= \{ (Ff_1, \tfrac{1}{2}(Ff_2 + Gf_1), Gf_2) \in \mathbb{C}[X]_{2r_1} \oplus \mathbb{C}[X]_{r_1+r_2} \oplus \mathbb{C}[X]_{2r_2}$$
$$\textit{modulo } \mathrm{I(S)}_{2r_1} \oplus \mathrm{I(S)}_{r_1+r_2} \oplus \mathrm{I(S)}_{2r_2}$$
$$\mid (F, G) \in \mathbb{C}[X])_{r_1} \oplus \mathbb{C}[X]_{r_2} \ \ \textit{with} \ \ f_2F - f_1G \in \mathrm{I(S)} \}$$

Proof. (i) A locally free resolution of the sheaf $\mathcal{O}_P(-2D_S)$ ($P = \mathbb{P}^3(\mathbb{C})$) is obtained as follows:

$$\begin{aligned} &0 \longrightarrow \mathcal{O}_P(-2r_1 - r_2) \oplus \mathcal{O}_P(-r_1 - 2r_2) \\ (5.2.62) \qquad &\xrightarrow{\alpha_1} \mathcal{O}_P(-2r_1) \oplus \mathcal{O}_P(-r_1 - r_2) \oplus \mathcal{O}_P(-2r_2) \\ &\xrightarrow{\beta} \mathcal{O}_P(-2D_S) \longrightarrow 0, \ \text{(exact)} \end{aligned}$$

where the map α_1 and β_1 are defined by

$$\alpha_1(a, b) = (af_2, -af_1 - bf_2, bf_1)$$
$$\text{for } (a, b) \in \mathcal{O}_P(-2r_2) \oplus \mathcal{O}_P(-r_1 - 2r_2)$$

and

$$\beta_1(A,B,C) = Af_1^2 + Bf_1f_2 + Cf_2^2$$
$$\text{for } (A,B,C) \in \mathcal{O}_P(-2r_1) \oplus \mathcal{O}_P(-r_1-r_2) \oplus \mathcal{O}_P(-2r_2)$$

Note that we consider here f_i as a cross-section of $\mathcal{O}_P(r_i)$ for $i = 1,2$. Tensoring $\omega_S \simeq \mathcal{O}_S(n-4)$ to the exact sequence in (5.2.62), we have

$$
\begin{aligned}
&0 \longrightarrow \omega_S(-2r_1-r_2) \oplus \omega_S(-r_1-2r_2) \\
&\xrightarrow{\alpha_1'} \omega_S(-2r_1) \oplus \omega_S(-r_1-r_2) \oplus \omega_S(-2r_2) \\
&\xrightarrow{\beta'} \omega_S(-2D_S) \longrightarrow 0, \quad \text{(exact)}
\end{aligned} \tag{5.2.63}
$$

Hence we have the exact sequence

$$
\begin{aligned}
0 \longleftarrow &\operatorname{Hom}_{\mathcal{O}_S}(\omega_S(-2r_1-r_2) \oplus \omega_S(-r_1-2r_2), \omega_S) \\
&\longleftarrow \operatorname{Hom}_{\mathcal{O}_S}(\omega_S(-2r_1) \oplus \omega_S(-r_1-r_2) \oplus \omega_S(-2r_2), \omega_S) \\
&\longleftarrow \operatorname{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S), \omega_S) \longleftarrow 0,
\end{aligned}
$$

which is isomorphic to

$$
\begin{aligned}
0 \longleftarrow &\Gamma(S, \mathcal{O}_S(2r_1+r_2)) \oplus \Gamma(S, \mathcal{O}_S(r_1+2r_2)) \\
&\longleftarrow \Gamma(S, \mathcal{O}_S(2r_1)) \oplus \Gamma(S, \mathcal{O}_S(r_1+r_2)) \oplus \Gamma(S, \mathcal{O}_S(2r_2)) \\
&\longleftarrow \operatorname{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S), \omega_S) \longleftarrow 0.
\end{aligned} \tag{5.2.64}
$$

Now we will clarify what the map

$$
\begin{aligned}
&\Gamma(S, \mathcal{O}_S(2r_1)) \oplus \Gamma(S, \mathcal{O}_S(r_1+r_2)) \oplus \Gamma(S, \mathcal{O}_S(2r_2)) \\
&\qquad \longrightarrow \Gamma(S, \mathcal{O}_S(2r_1+r_2)) \oplus \Gamma(S, \mathcal{O}_S(r_1+2r_2))
\end{aligned}
$$

above is. Assume that

$$(\phi_1, \phi_2, \phi_3) \in \Gamma(S, \mathcal{O}_S(2r_1)) \oplus \Gamma(S, \mathcal{O}_S(r_1+r_2)) \oplus \Gamma(S, \mathcal{O}_S(2r_2))$$

corresponds to

$$(\psi_1, \psi_2) \in \Gamma(S, \mathcal{O}_S(2r_1+r_2)) \oplus \Gamma(S, \mathcal{O}_S(r_1+2r_2))$$

through this map. Then, since $(a,b) \in \omega_S(-2r_1-r_2) \oplus \omega_S(-r_1-2r_2)$ is assigned to

$$(af_2, -af_1-bf_2, bf_1) \in \omega_S(-2r_1) \oplus \omega_S(-r_1-r_2) \oplus \omega_S(-2r_2)$$

by the map α_1' in (5.2.63), we have

$$
\begin{aligned}
\psi_1 a + \psi_2 b &= \phi_1(af_2) + \phi_2(-af_1-bf_2) + \phi_3(bf_1) \\
&= (\phi_1 f_2 - \phi_2 f_1)a + (\phi_3 f_1 - \phi_2 f_2)b
\end{aligned}
$$

for any $(a, b) \in \omega_S(-2r_1 - r_2) \oplus \omega_S(-r_1 - 2r_2)$. Hence we have

$$\psi_1 = \phi_1 f_2 - \phi_2 f_1 \quad \text{and} \quad \psi_2 = \phi_3 f_1 - \phi_2 f_2.$$

Therefore, by (5.2.64), we conclude that

$$\begin{aligned}
&\operatorname{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S), \omega_S) \\
&\simeq Ker\{\, \Gamma(S, \mathcal{O}_S(2r_1)) \oplus \Gamma(S, \mathcal{O}_S(r_1 + r_2)) \oplus \Gamma(S, \mathcal{O}_S(2r_2)) \\
&\qquad\qquad \longrightarrow \Gamma(S, \mathcal{O}_S(2r_1 + r_2) \otimes \Gamma(S, \mathcal{O}_S(r_1 + 2r_2))\,\} \\
&\simeq \{\, (\phi_1, \phi_2, \phi_3) \in \mathbb{C}[X]_{2r_1} \oplus \mathbb{C}[X]_{r_1+r_2} \oplus \mathbb{C}[X]_{2r_2} \\
&\qquad\qquad \mid \phi_1 f_2 - \phi_2 f_1 \in I(S), \phi_3 f_1 - \phi_2 f_2 \in \mathrm{I(S)}\,\}.
\end{aligned}$$

Here we identify $\Gamma(S, \mathcal{O}_S(r))$ $(r \in \mathbb{N})$ with $\mathbb{C}[X]_r$ *modulo* I(S), the ideal of S in $\mathbb{C}[X]_r$.

(ii) A locally free resolution of the sheaf $\mathcal{O}_P(-D_S)$ is obtained as follows:

(5.2.65)

$$0 \longrightarrow \mathcal{O}_P(-r_1 - r_2) \xrightarrow{\alpha_2} \mathcal{O}_P(-r_1) \oplus \mathcal{O}_P(-r_2) \xrightarrow{\beta_2} \mathcal{O}_P(-D_S) \longrightarrow 0 \quad \text{(exact)},$$

where the maps α_2 and β_2 are defined by

$$\alpha_2(c) = (cf_2, -cf_1) \qquad \text{for } c \in \mathcal{O}_P(-r_1 - r_2),$$

and

$$\beta_2(A, B) = Af_1 + Bf_2 \qquad \text{for } (A, B) \in \mathcal{O}_P(-r_1) \oplus \mathcal{O}_P(-r_2).$$

Restricting the eaxct sequence in (5.2.65) to S, we have

(5.2.66)

$$0 \longrightarrow \mathcal{O}_S(-r_1 - r_2) \xrightarrow{\alpha_2'} \mathcal{O}_S(-r_1) \oplus \mathcal{O}_S(-r_2)' \xrightarrow{\beta_2'} \mathcal{O}_S(-D_S) \longrightarrow 0 \quad \text{(exact)},$$

From this it follows that

$$\begin{aligned}
&\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S) \\
&\simeq Ker\{\, \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-r_1) \oplus \mathcal{O}_S(-r_2), \omega_S) \qquad (5.2.67) \\
&\qquad\qquad \longrightarrow Hom_{\mathcal{O}_S}(\mathcal{O}_S(-r_1 - r_2), \omega_S)\,\} \\
&\simeq Ker\{\, \Gamma(S, \mathcal{O}_S(n - 4 + r_1)) \oplus \Gamma(S, \mathcal{O}_S(n - 4 - r_2)) \\
&\qquad\qquad \longrightarrow \Gamma(S, \mathcal{O}_S(n - 4 + r_1 + r_2))\,\}.
\end{aligned}$$

Now we consider what the map

$$\Gamma(S, \mathcal{O}_S(n-4+r_1)) \oplus \Gamma(S, \mathcal{O}_S(n-4-r_2)) \longrightarrow \Gamma(S, \mathcal{O}_S(n-4+r_1+r_2))$$

is. Assume that

$$(F, G) \in \Gamma(S, \mathcal{O}_S(n-4+r_1) \oplus \Gamma(S, \mathcal{O}_S(n-4-r_2))$$

is assigned to $H \in \Gamma(S, \mathcal{O}_S(n-4+r_1+r_2))$ by this map. Then, since $a \in \mathcal{O}_S(-r_1-r_2)$ is assigned to $(af_2, -af_1) \in \mathcal{O}_S(-r_1) \oplus \mathcal{O}_S(-r_2)$ by the map α_2' in (5.2.66), we have

$$(af_2)F - (af_1)G = Ha$$

for any $a \in \mathcal{O}_S(-r_1-r_2)$. Hence we have

$$H = f_2F - f_1G.$$

Therefore, by (5.2.67)

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S) \\
&\simeq \{\, (F, G) \in \mathbb{C}[X]_{n-4+r_1} \oplus \mathbb{C}[X]_{n-4+r_2} \\
&\qquad\qquad \textit{modulo } \mathrm{I(S)}_{n-4-r_1} \oplus \mathrm{I(S)}_{n-4+r_2} \\
&\qquad\qquad\qquad\qquad \mid f_2F - f_1G \in I(S) \,\}.
\end{aligned}$$

(iii) By the sheaf exact sequence

$$0 \longrightarrow \mathcal{O}_P(-4) \longrightarrow \mathcal{O}_P((n-4) - D_S) \longrightarrow \omega_S(-D_S) \longrightarrow 0,$$

we have

(5.2.68)
$$\begin{aligned}
H^0(S, \omega_S(-D_S)) &\simeq H^0(P, \mathcal{O}_P((n-4) - D_S)) \\
&\simeq \{\, \omega_1 f_1 + \omega_2 f_2 \mid (\omega_1, \omega_2) \in \mathbb{C}[X]_{n-4-r_1} \oplus \mathbb{C}[X]_{n-4-r_2} \,\}
\end{aligned}$$

Let $(\phi_1, \phi_2, \phi_3) \in \mathbb{C}[X]_{2r_1} \oplus \mathbb{C}[X]_{r_1+r_2} \oplus \mathbb{C}[X]_{2r_2}$ be the element corresponding to $\omega \in H^0(S, \omega_S(-D_S))$ by the isomorphism in (i) of the proposition. Then $\phi_1 f_2 - \phi_2 f_1 \in I(S)$ and $\phi_3 f_1 - \phi_2 f_2 \in I(S)$. Considering ω as an element of $H^0(S, \omega_S(-D_S))$, let $\omega_1 f_1 + \omega_2 f_2 \in \mathbb{C}[X]_{n-4}$ be the element corresponding to ω by the isomorphism in (5.2.68), where $(\omega_1, \omega_2) \in \mathbb{C}[X]_{n-4-r_1} \oplus \mathbb{C}[X]_{n-4-r_2}$. Let $(F, G) \in \mathbb{C}[X])_{n-4+r_1} \oplus \mathbb{C}[X]_{n-4+r_2}$ be the element corresponding to $\phi \circ \omega \in \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S)$ *modulo* $\mathrm{I(S)}_{n-4+r_1} \oplus I(S)_{n-4+r_2}$ by the isomorphism in

(ii) of the proposition. Then $f_2F - f_1G \in \mathrm{I(S)}$. Now, for any $af_1 + bf_2 \in \mathcal{O}_S(-D_S)$ with $(a,b) \in \mathcal{O}_S(-r_1) \oplus \mathcal{O}_S(-r_2)$, we have

$$\begin{aligned}
&(\phi \circ \omega)(af_1 + bf_2) \\
&= \phi(\omega \otimes (af_1 + bf_2)) = \phi((\omega_1 f_1 + \omega_2 f_2) \otimes (af_1 + bf_2)) \\
&= \phi((a\omega_1)f_1^2 + (a\omega_2 + b\omega_1)f_1f_2 + (b\omega_2)f_2^2) \\
&= a\omega_1\phi_1 + (a\omega_2 + b\omega_1)\phi_2 + b\omega_2\phi_3 \\
&= (\omega_1\phi_1 + \omega_2\phi_2)a + (\omega_1\phi_2 + \omega_2\phi_3)b
\end{aligned}$$

Therefore we have

$$(\omega_1\phi_1 + \omega_2\phi_2)a + (\omega_1\phi_2 + \omega_2\phi_3)b = Fa + Gb.$$

Since this equality holds for any $(a,b) \in \mathcal{O}_S(-r_1) \oplus \mathcal{O}_S(-r_2)$, we conclude that

$$\begin{aligned}
F &\equiv \omega_1\phi_1 + \omega_2\phi_2 \quad \textit{modulo} \;\; \mathrm{I(S)}_{n-4+r_1}, \;\; \text{and} \\
G &\equiv \omega_1\phi_2 + \omega_2\phi_3 \quad \textit{modulo} \;\; \mathrm{I(S)}_{n-4+r_2},
\end{aligned}$$

which implies the assertion (iii).

(iv) (a) Since

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \omega_S) \simeq H^0(S, \omega_S) \simeq H^0(S, \mathcal{O}_S(n-4)),$$

we identify $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \omega_S)$ with $H^0(S, \mathcal{O}_S(n-4))$. Let $(F,G) \in \mathbb{C}[X]_{n-4+r_1} \oplus \mathbb{C}[X]_{n-4+r_2}$ with $f_2F - f_1G \in \mathrm{I(S)}$ be the element corresponding to the image of $\omega \in \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \omega_S)$ in $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S)$ by the isomorphism in (ii). Then, for any element $af_1 + bf_2 \in \mathcal{O}_S(-D_S)$ with $(a,b) \in \mathcal{O}_S(-r_1) \oplus \mathcal{O}_S(-r_2)$, we have

$$\begin{aligned}
\omega(af_1 + bf_2) &= (af_1 + bf_2) \otimes \omega \\
&= a(f_1\omega) + b(f_2\omega) \\
&= aF_{|S} + bG_{|S},
\end{aligned}$$

where $F_{|S}$ and $G_{|S}$ are considered as cross-sections of $\mathcal{O}_S(n-4+r_1) \oplus \mathcal{O}_S(n-4+r_2)$. Since this equality holds for any $(a,b) \in \mathcal{O}_S(-r_1) \oplus \mathcal{O}_S(-r_2)$, we have

$$\begin{aligned}
F &\equiv f_1\omega \quad \textit{modulo} \;\; \mathrm{I(S)}_{n-4+r_1}, \quad \text{and} \\
G &\equiv f_2\omega \quad \textit{modulo} \;\; I(S)_{n-4+r_2},
\end{aligned}$$

which implies the assertion (iv) (a).

(iv) (b) : We consider the following commutative diagram consisting of exact rows and columbs:

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & \mathcal{O}_{D_S} & & & & 0 & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & f_*\mathcal{O}_X & \longrightarrow & \omega_D \otimes \omega_S^\vee & \longrightarrow & 0 \\
 & & \uparrow & & \| & & \uparrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & \mathcal{O}_S(-D_S) & \longrightarrow & f_*\mathcal{O}_X & \longrightarrow & g_*\mathcal{O}_{D_X} & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & \operatorname{Ker}\gamma & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0, & &
\end{array}
\tag{5.2.69}
$$

where the first row is due to J. Robert ([9]), the second row follows from the short exact sequence

$$0 \to \mathcal{O}_X(-D_X) \to \mathcal{O}_X \to \mathcal{O}_{D_X} \to 0$$

and the fact $\mathcal{O}_S(-D_S) \simeq f_*\mathcal{O}_X(-D_X)$, and the map γ is the one induced from the identity map on $f_*\mathcal{O}_X$. By the second row in the diagram (5.2.69), we have

$$
\begin{aligned}
0 \to \operatorname{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X, \omega_S) \to \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S), \omega_S) \\
\to \operatorname{Ext}^1_{\mathcal{O}_S}(g_*\mathcal{O}_{D_X}, \omega)) \to \cdots
\end{aligned}
\tag{5.2.70}
$$

The exact sequence which is dual to (5.2.70) is

(5.2.71)

$$
\begin{array}{ccccccc}
0 \leftarrow & H^2(S, f_*\mathcal{O}_X) & \leftarrow & H^2(S, \mathcal{O}_S(-D_S)) & \leftarrow & H^1(S, g_*\mathcal{O}_{D_X}) & \leftarrow \cdots \\
 & \simeq\uparrow & & \simeq\uparrow & & & \\
0 \leftarrow & H^2(X, \mathcal{O}_X) & \leftarrow & H^2(X, \mathcal{O}_X(-D_X)) & \leftarrow \cdots . & &
\end{array}
$$

On the other hand, the dual map of $H^2(X, \mathcal{O}_X(-D_X)) \to H^2(X, \mathcal{O}_X)$ in the diagram (5.2.71) is

(5.2.72)

$$\begin{array}{ccccc} H^2(X,\mathcal{O}_X)^\vee & \longrightarrow & H^2(X,\mathcal{O}_X(-D_X))^\vee & & \\ \simeq\uparrow & & \simeq\uparrow & & \\ H^0(X,K_X) & \longrightarrow & H^0(X,\mathcal{O}_X([D_X])\otimes K_X) & \xrightarrow{\simeq} & H^0(X,f^*\omega_S) \\ \simeq\uparrow & & \simeq\uparrow & & \\ H^0(S,\omega_S(-D_S)) & \longrightarrow & H^0(S,\omega_S\otimes f_*\mathcal{O}_X). & & \end{array}$$

Therefore, by (5.2.70), (5.2.71) and (5.2.72), we have the commutative diagram

$$(5.2.73)\qquad \begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X,\omega_S) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S),\omega_S) \\ \simeq\uparrow & & \simeq\uparrow \\ H^0(S,\omega_S(-D_S)) & \longrightarrow & H^0(S,\omega_S\otimes f_*\mathcal{O}_X). \end{array}$$

By the isomorphism on the left-hand-side of (5.2.72), we identify $\mathrm{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X, \omega_S)$ with $H^0(S,\omega_S(-D_S))$, which is identified with the space

$$\{\ \omega_1 f_1+\omega_2 f_2\ \ \textit{modulo}\ \ \mathrm{I(S)}_{n-4}\ \mid\ (\omega_1,\omega_2)\in\mathbb{C}[X]_{n-4-r_1}\oplus\mathbb{C}[X]_{n-4-r_2}\ \}$$

Let $(F,G)\in\mathbb{C}[X]_{n-4+r_1}\oplus\mathbb{C}[X]_{n-4+r_2}$ *modulo* $\mathrm{I(S)}_{n-4+r_1}\oplus\mathrm{I(S)}_{n-4+r_2}$ with $f_2F-f_1G\in\mathrm{I(S)}$ be the element corresponding to the image of $\omega_1 f_1+\omega_2 f_2$ (*modulo* $\mathrm{I(S)}_{n-4}$) $\in\mathrm{Hom}_{\mathcal{O}_S}(f_*\mathcal{O}_X,\omega_S)$ in $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S(-D_S),\omega_S)$ by the isomorphism in (ii) of the proposition. Then, for any element $af_1+bf_2\in\mathcal{O}_S(-D_S)$ with $(a,b)\in\mathcal{O}_S(-r_1)\oplus\mathcal{O}_S(-r_2)$, we have

$$\begin{aligned} &(\omega_1 f_1+\omega_2 f_2)(af_1+bf_2)\\ &=(af_1+bf_2)\otimes(\omega_1 f_1+\omega_2 f_2)\\ &=a\,(\omega_1 f_1+\omega_2 f_2)f_1+b\,(\omega_1 f_1+\omega_2 f_2)f_2\\ &=aF_{|S}+b\,G_{|S}. \end{aligned}$$

Therefore, by the same reasoning as before, we have

$$\begin{aligned} F&\equiv(\omega_1 f_1+\omega_2 f_2)f_1\ \ \textit{modulo}\ \ \mathrm{I(S)}_{n-4+r_1}\ \ \text{and}\\ G&\equiv(\omega_1 f_1+\omega_2 f_2)f_1\ \ \textit{modulo}\ \ \mathrm{I(S)}_{n-4+r_2}, \end{aligned}$$

which imply the assertion (iv) (b).

(v) By the similar reasoning to prove the assertion (ii), we can identify

$\mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-D_S),\omega_S)$ with the space

$$\{\ (F,G)\in\mathbb{C}[X]_{r_1}\oplus\mathbb{C}[X]_{r_2}\ \mid\ f_2F-f_1G\in I(S)\ \}.$$

We identify $\mathrm{Hom}_{\mathcal{O}_X}(\omega_S(-2D_S),\omega_S)$ with the left-hand-side in (i). Under these identifications, let

$$\begin{aligned}&(\phi_1,\phi_2,\phi_3)\in\mathbb{C}[X]_{2r_1}\oplus\mathbb{C}[X]_{r_1+r_2}\oplus\mathbb{C}[X]_{2r_2}\\ &\textit{modulo}\ \ \mathrm{I(S)}_{2r_1}\oplus I(S)_{r_1+r_2}\oplus\mathrm{I(S)}_{2r_2}\\ &\text{with } \phi_1f_2-\phi_2f_1\in\mathrm{I(S)}\text{ and }\phi_3f_1-\phi_2f_2\in\mathrm{I(S)}\end{aligned}$$

be the element of $\mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S),\omega_S)$ which is assigned to $(F,G)\in\mathbb{C}[X]_{r_1}\oplus\mathbb{C}[X]_{r_2}$ with $f_2F-f_1G\in I(S)$ by the isomorphism $\mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-D_S),\omega_S)\to\mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S),\omega_S)$. Then, since the sheaf homomorphism $\omega_S(-2D_S)\to\omega_S(-D_S)$ is given by

$$\begin{aligned}&af_1^2+bf_1f_2+cf_2^2\mapsto(af_1+\frac{1}{2}bf_2)f_1+(\frac{1}{2}bf_1+cf_2)f_2\\ &\text{for } (a,b,c)\in\mathcal{O}_S(n-4-2r_1)\oplus\mathcal{O}_S(n-4-r_1-r_2)\oplus\mathcal{O}_S(n-4-2r_2)\end{aligned}$$

we have

$$\begin{aligned}a\phi_1+b\phi_2+c\phi_3&=(af_1+\frac{1}{2}bf_2)F+(\frac{1}{2}bf_1+cf_2)G\\ &=a(Ff_1)+\frac{1}{2}b(Ff_2+Gf_1)+c(Gf_2).\end{aligned}$$

Since this equality holds for any

$$(a,b,c)\in\mathcal{O}_S(n-4-2r_1)\oplus\mathcal{O}_S(n-4-r_1-r_2)\oplus\mathcal{O}_S(n-4-2r_2),$$

we have

$$\begin{aligned}&(\phi_1,\phi_2,\phi_3)\equiv(Ff_1,\tfrac{1}{2}(Ff_2+Gf_1),Gf_2)\\ &\quad\textit{modulo}\ \mathrm{I(S)}_{2r_1}\oplus\mathrm{I(S)}_{r_1+r_2}\oplus\mathrm{I(S)}_{2r_2},\end{aligned}$$

which implies the assertion (v).

Q.E.D.

5.2.6 Lemma. *If* $(n,r_1,r_2)=(2r+1,r,r)$ $(r\geq 1)$, *or* $(n,r_1,r_2)=(2r,r,r-1)$ $(r\geq 2)$, *then we have*

$$\begin{aligned}&Im\{\ \mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-D_S),\omega_S)\to\mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S),\omega_S)\ \}\\ &=\{\ \lambda(f_1^2,f_1f_2,f_2^2)\in\mathbb{C}[X]_{2r_1}\oplus\mathbb{C}[X]_{r_1+r_2}\oplus\mathbb{C}[X]_{2r_2}\end{aligned}$$

$$\textit{modulo}\ \mathrm{I(S)}_{2r_1} \oplus \mathrm{I(S)}_{r_1+r_2} \oplus \mathrm{I(S)}_{2r_2}\ \mid\ \lambda \in \mathbb{C}^*\ \}$$

where we identify $\mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S), \omega_S)$ *with the space*

$$\begin{aligned} \{\ (\phi_1, \phi_2, \phi_3) \in \mathbb{C}[X]_{2r_1} \oplus \mathbb{C}[X]_{r_1+r_2} \oplus \mathbb{C}[X]_{2r_2} \\ \textit{modulo}\ \mathrm{I(S)}_{2r_1} \oplus \mathrm{I(S)}_{r_1+r_2} \oplus \mathrm{I(S)}_{2r_2} \\ \mid\ \phi_1 f_2 - \phi_2 f_1 \in \mathrm{I(S)},\ \ \phi_3 f_1 - \phi_2 f_2 \in \mathrm{I(S)}\ \} \end{aligned}$$

(cf. Proposition 5.2.5 (i)).

Proof. By Proposition 5.2.5 (v), an element of

$$Im\,\{\ \mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-D_S), \omega_S) \longrightarrow \mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S), \omega_S)\ \}$$

is represented as

$$\begin{aligned} (Ff_1, \frac{1}{2}(Ff_2 + Gf_1), Gf_2) \in \mathbb{C}[X]_{2r_1} \oplus \mathbb{C}[X]_{r_1+r_2} \oplus \mathbb{C}[X]_{2r_2} \\ \textit{modulo}\ \ \mathrm{I(S)}_{2r_1} \oplus \mathrm{I(S)}_{r_1+r_2} \oplus \mathrm{I(S)}_{2r_2}, \end{aligned}$$

where $(F, G) \in \mathbb{C}[X]_{r_1} \oplus \mathbb{C}[X]_{r_2}$, with $f_2F - f_1G \in \mathrm{I(S)}$. We have

$$\deg(f_2F - f_1G) \le r_1 + r_2 = \begin{cases} 2r - 1 & \text{if } n = 2r, r_1 = r, r_2 = r - 1 \\ 2r & \text{if } n = 2r + 1, r_1 = r_2 = 2r \end{cases}$$

Hence, in both cases, we have

$$\deg(Af_1^2 + Bf_1f_2 + Cf_2^2) > \deg(f_2F - f_1G).$$

Therefore, since $f_2F - f_1G \in \mathrm{I(S)}$, we have

$$f_2F - f_1G \equiv 0.$$

Since $(f_1, f_2) = 1$ in $\mathbb{C}[X]$, and since $\deg F = \deg f_1$ and $\deg G = \deg f_2$, the above identity implies that there exists $\lambda \in \mathbb{C}^*$ such that

$$(F, G) = \lambda(f_1, f_2)$$

Consequently , we conclude that any element of

$$Im\,\{\ \mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-D_S), \omega_S) \longrightarrow \mathrm{Hom}_{\mathcal{O}_S}(\omega_S(-2D_S), \omega_S)\ \}$$

is represented as

$$\lambda(f_1^2, f_1f_2, f_2^2) \in \mathbb{C}[X]_{2r_1} \oplus \mathbb{C}[X]_{r_1+r_2} \oplus \mathbb{C}[X]_{2r_2}$$

$$modulo\ \mathrm{I(S)}_{2r_1} \oplus \mathrm{I(S)}_{r_1+r_2} \oplus I(S)_{2r_2} \qquad \text{for } \lambda \in \mathbb{C}^*.$$

Q.E.D.

By Proposition 5.2.5 and Lemma 5.2.6, in the case of $(n, r_1, r_2) = (2r+1, r, r)$, or $(n, r_1, r_2) = (2r, r, r-1)$, the condition (5.2.61) is restated as follows:

(5.2.74) Assume that we are given an element

$$(\phi_1, \phi_2, \phi_3) \in H^0(\mathcal{O}_S(2r_1)) \oplus H^0(\mathcal{O}_S(r_1+r_2)) \oplus H^0(\mathcal{O}_S(2r_2))$$

satisfying the condition

$$\phi_1 f_2 - \phi_2 f_1 \in \mathrm{I(S)}, \text{ and } \phi_3 f_1 - \phi_2 f_2 \in \mathrm{I(S)}$$

For this element (ϕ_1, ϕ_2, ϕ_2), if

$$\{ (\omega_1\phi_1 + \omega_2\phi_2, \omega_1\phi_2 + \omega_2\phi_3) \\ \mid (\omega_1, \omega_2) \in H^0(\mathcal{O}_S(n-4-r_1) \oplus H^0(\mathcal{O}_S(n-4-r_2)) \}$$

is included in

$$\{ ((\omega_1 f_1 + \omega_2 f_2) f_1, (\omega_1 f_1 + \omega_2 f_2) f_2 \\ \mid (\omega_1, \omega_2) \in H^0(\mathcal{O}_S(n-4-r_1) \oplus H^0(\mathcal{O}_S(n-4-r_2)) \},$$

is the element (ϕ_1, ϕ_2, ϕ_3) represented as

$$(\phi_1, \phi_2, \phi_3) = \lambda(f_1^2, f_1 f_2, f_2^2)$$

for some $\lambda \in \mathbb{C}^*$?

From now on we consider this problem.

5.2.7 Lemma. *An element*

$$(\phi_1, \phi_2, \phi_3) \in H^0(\mathcal{O}_S(2r_1)) \oplus H^0(\mathcal{O}_S(r_1+r_2)) \oplus H^0(\mathcal{O}_S(2r_2))$$

satisfies the conditions

$$\phi_1 f_2 - \phi_2 f_1 \in \mathrm{I(S)} \quad \textit{and} \quad \phi_3 f_1 - \phi_2 f_2 \in \mathrm{I(S)}$$

if and only if there exists as element

$$(\phi, \psi, \lambda) \in H^0(\mathcal{O}_S(2r_1+r_2-n)) \oplus H^0(\mathcal{O}_S(r_1+2r_2-n)) \oplus H^0(\mathcal{O}_S)$$

such that:

$$\left\{\begin{aligned}\phi_1 &= (\lambda f_1 - \psi C)f_1 + (\phi B)f_1 + (\phi C)f_2, \\ \phi_2 &= \lambda f_1 f_2 - (\psi C)f_2 - (\phi A)f_1, \quad \textit{and} \\ \phi_3 &= (\psi A)f_1 + (\psi B)f_2 + (\lambda f_2 - \phi A)f_2\end{aligned}\right. \tag{5.2.75}$$

where $f = Af_1^2 + 2Bf_1f_2 + Cf_2^2$ ($A \in \mathbb{C}[X]_{n-2r_1}$, $B \in \mathbb{C}[X]_{n-r_1-r_2}$, $C \in \mathbb{C}[X]_{n-2r_2}$) *is the defining equation of* S *in* $\mathbb{P}^3(\mathbb{C})$.

Proof. The proof of "If part" is just a direct calculation. We will prove "only if part". Assume that $\phi_1 f_2 - \phi_2 f_1 \in \mathrm{I(S)}$ and $\phi_3 f_1 - \phi_2 f_2 \in \mathrm{I(S)}$. Then there exists an element $(\phi, \psi) \in H^0(\mathcal{O}_S(2r_1 + r_2 - n)) \oplus H^0(\mathcal{O}_S(r_1 + 2r_2 - n))$ such that:

$$\phi_1 f_2 - \phi_2 f_1 = (\phi A)f_1^2 + (\phi B)f_1 f_2 + (\phi C)f_2^2, \tag{5.2.76}$$

$$\phi_3 f_1 - \phi_2 f_2 = (\psi A)f_1^2 + (\psi B)f_1 f_2 + (\psi C)f_2^2. \tag{5.2.77}$$

By (5.2.76) we have

$$\{\ \phi_1 - (\phi B)f_1 - (\phi C)f_2\ \}f_2 = \{\ (\phi A)f_1 + \phi_2\ \}f_1. \tag{5.2.78}$$

Simce $(f_1, f_2) = 1$ in $\mathbb{C}[X]$, the above equality implies that there exists an element $\phi' \in H^0(\mathcal{O}_S(r_1))$ such that

$$(\phi A)f_1 + \phi_2 = \phi' f_2. \tag{5.2.79}$$

From (5.2.78) and (5.2.79) it follows that

$$\left\{\begin{aligned}\phi_1 &= \phi' f_1 + (\phi B)f_1 + (\phi C)f_2 \\ \phi_2 &= \phi' f_2 - (\phi A)f_1.\end{aligned}\right. \tag{5.2.80}$$

On the other hand, by (5.2.77), we have

$$\{\ \phi_3 - (\psi A)f_1 - (\psi B)f_2\}f_1 = \{\ (\psi C)f_2 + \phi_2\ \}f_2. \tag{5.2.81}$$

Since $(f_1, f_2) = 1$ in $\mathbb{C}[X]$, the above equality implies that there exists an element $\psi' \in H^0(\mathcal{O}_S(r_2))$ such that

$$(\psi C)f_2 + \phi_2 = \psi' f_1. \tag{5.2.82}$$

From (5.2.81) and (5.2.82) it follows that

$$\left\{\begin{aligned}\phi_3 &= \psi' f_2 + (\psi A)f_1 + (\psi C)f_2 \\ \phi_2 &= \psi' f_1 - (\psi C)f_2.\end{aligned}\right. \tag{5.2.83}$$

From the second equations in (5.2.80) and (5.2.83), we have

$$\phi' f_2 - (\phi A) f_1 = \psi' f_1 - (\psi C) f_2,$$

that is,

$$(\phi' + \psi C) f_2 = (\psi' + \phi A) f_1.$$

Since $(f_1, f_2) = 1$ in $\mathbb{C}[X]$, this equality implies that there exists an element $\lambda \in H^0(\mathcal{O}_S)$ such that

$$\begin{cases} \phi' + (\psi C) = \lambda f_1 \\ \psi' + \phi A = \lambda f_2 \end{cases}$$

Substituting $\phi' = \lambda f_1 - \psi C$ and $\psi' = \lambda f_2 - \phi A$ into (5.2.80) and (5.2.83), we have the expressions in (5.2.75).

Q.E.D.

5.2.8 Proposition. *In the case* $(n, r_1, r_2) = (2r+1, r, r)$ (*resp.* $(n, r_1, r_2) = (2r, r, r-1)$, (5.2.73)), *which is equivalent to* (5.2.35), *holds if* $r \geq 3$ (*resp.* $r \geq 4$).

Proof. By Lemma 5.2.7, the element (ϕ_1, ϕ_2, ϕ_3) in (5.2.74) is expressed as in (5.2.75). From the "if part" in (5.2.74) it follows that for any

$$(\omega_1, \omega_2) \in H^0(\mathcal{O}_S(n-4-r_1)) \oplus H^0(\mathcal{O}_S(n-4-r_2))$$

there exists an element

$$(\omega_1', \omega_2') \in H^0(\mathcal{O}_S(n-4-r_1)) \oplus H^0(\mathcal{O}_S(n-4-r_2))$$

such that; as an element of $\mathcal{O}_P(n+r_1-4)$,

(5.2.84)

$$\begin{aligned} &\omega_1\phi_1 + \omega_2\phi_2 \\ &= \{\, \omega_1(\lambda f_1 - \psi C + \phi B) + \omega_2(\lambda f_2 - \phi A)\} f_1 + \{\omega_1(\phi C) - \omega_2(\psi C)\} f_2 \\ &= (\omega_1' f_1 + \omega_2' f_2) f_1 + \Phi f \end{aligned}$$

for some $\Phi \in H^0(\mathcal{O}_P(r_1 - 4))$, where f is the polynomial defining the surface S (cf. (5.2.1)). Since $(f_1, f_2) = 1$ in $\mathbb{C}[X]$, (5.2.84) implies that

$$f_1 \mid \omega_1(\phi C) - \omega_2(\psi C) - \Phi C f_2.$$

We may assume that $f_2 \nmid C$, and so

$$f_1 \mid \omega_1\phi - \omega_2\psi - \Phi f_2.$$

Therefore, there exists an element $\Psi \in H^0(\mathcal{O}_P(r_2 - 4))$ such that:

$$\omega_1\phi - \omega_2\psi = \Psi f_1 + \Phi f_2,$$

that is,

$$\begin{aligned} &\omega_1\phi - \omega_2\psi \in (f_1, f_2) \\ (5.2.85)\qquad & \\ &\text{for any} \quad (\omega_1, \omega_2) \in H^0(\mathcal{O}_P(n-4-r_1)) \oplus H^0(\mathcal{O}_P(n-4-r_2)) \end{aligned}$$

By (5.2.85) we have $\omega_1\phi \in (f_1, f_2)$ (resp. $\omega_2\psi \in (f_1, f_2)$) for any $\omega_1 \in H^0(\mathcal{O}_P(n-4-r_1))$ (resp. for any $\omega_2 \in H^0(\mathcal{O}_P(n-4-r_2))$).

i) In the case $n = 2r+1$, $r_1 = r_2 = r$: we have $\deg\phi = \deg\psi = r-1$, $\deg\omega_1 = \deg\omega_2 = r-3$ and $\deg f_1 = \deg f_2 = r$, so $\omega_1 \notin (f_1, f_2)$ and $\phi \notin (f_1, f_2)$ (resp. $\omega_2 \notin (f_1, f_2)$ and $\psi \notin (f_1, f_2)$). If $\phi \neq 0$, or $\psi \neq 0$, we have a contradiction to that (f_1, f_2) is a prime ideal. Hence we have $\phi = \psi = 0$. Consequently, we have

$$(\phi_1, \phi_2, \phi_3) = \lambda\,(f_1, f_1 f_2, f_2^2) \qquad \text{for } \lambda \in \mathbb{C}$$

as required.

ii) In the case $n = 2r$, $r_1 = r$, $r_2 = r-1$: we have $\deg\phi = r-1$, $\deg\psi = r-2$, $\deg\omega_1 = r-4$, and $\deg\omega_2 = r-3$, so $\omega_1 \notin (f_1, f_2)$ and $\phi \notin (f_1, f_2)$ (resp. $\omega_2 \notin (f_1, f_2)$ and $\psi \in (f_1, f_2)$). Hence, by the same reason as in the case (i), we have also the same comclusion.

Q.E.D.

5.2.9 Theorem. *Under the condition* (5.2.14), *the infinitesimal mixed Torelli holds for an algebraic surface S with ordinary singularities defined by the equation* (5.2.1).

Proof:. Note that the condition in (5.2.14) implies $n \geq 6$. First, we claim that the map

$$H^0(X, \Theta_X(-logD_X)) \oplus H^0(D_S^*, \Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*)) \to H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

(cf. Theorem 3.26, (ii) in Part I) is surjective. Indeed, by Hurwitz's formula, we have

$$\begin{aligned} g(D_X^*) = g(D_X) &= 2g(D_S) - 1 + \frac{1}{2}\#\Sigma\mathfrak{c}_S \\ &= r_1 r_2(r_1 + r_2 - 4) + 1 + r_1 r_2(n - r_1 - r_2) \\ &= r_1 r_2(n-4) + 1 \end{aligned}$$

Therefore, since $n \geq 6$ and $r_1 \geq r_2 \geq 1$, we have $g(D_X^*) \geq 3$, which implies

$$H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*)) = 0.$$

As shown in [16], X is a non-singular algebraic surface of general type for which the infinitesimal Torelli with respect to the cohomology $H^2(X, \mathbb{C})$ holds, that is, the condition (ii) in Theorem 4.10 is satisfied. Since X is of general type, we have $H^0(X, \Theta_X) = 0$. Therefore, by the exact sequence of sheaves in (4.24), we have

$$0 \to H^0(D_X, \mathcal{N}_{D_X/X}) \to H^1(X, \Theta_X(-log\, D_X)) \to H^1(X, \Theta_X). \tag{5.2.86}$$

We denote by

$$h : H^1(S, \Theta_S) \to H^1(X, \Theta_X) \tag{5.2.87}$$

the composite of the homomorphism $H^1(S, \Theta_S) \to H^1(X, \Theta_X(-log\, D_X))$ which is derived from the short exact sequence in Theorem 3.19 in Part I and

$$H^1(X, \Theta_X(-log\, D_X)) \to H^1(X, \Theta_X)$$

in (5.2.86). By Corollary 4.5 the homomorphism

$$H^1(S, \Theta_S) \to H^1(X, \Theta_X(-log\, D_X))$$

is injective, and if $(n, r_1, r_2) \neq (2r+1, r, r)$ $(r \geq 1)$, $(2r, r, r-1)$, $(r \geq 2)$, by Proposition 5.2.3 and (5.2.86), the homomorphism

$$H^1(X, \Theta_X(-log\, D_X)) \to H^1(X, \Theta_X)$$

is also injective. Hence the homomorphism h in (5.2.87) is injective for these cases. This means that the infinitesimal Toelli for X with respect to the cohomology $H^2(X, \mathbb{C})$ implies the infinitesimal mixed Torelli for S. Concerning the case $(n, r_1, r_2) = (2r+1, r, r)$ $(r \geq 1)$, or $(2r, r, r-1)$ $(r \geq 2)$, the condition (5.2.14) implies $r \geq 3$ for the former case and $r \geq 4$ for the latter case. Therefore, by Proposition 5.2.8, the condition (iii) in Theorem 4.10 is satisfied. Consequently, by Theorem 4.10, we conclude that the infinitesimal Torelli holds for such S that is defined by the equation in (5.2.1).

Q.E.D.

5.3 Example

We shall give a toy example for which the sufficient condition for the map

$$\tau^{(1)} : H^1(S, \Theta_S) \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^0(\Omega_{X_\bullet}), \mathbb{H}^1(\mathcal{O}_{X_\bullet}))$$

to be injective in Theorem 4.12 holds. Let C_i, $(i = 1, 2)$, be non-singular curves defined over $\mathbb{C}$, or compact Riemman surfaces, with the genuses $g(C_i) \geq 2$, and let $g : C_1 \to C_0$ be a double ramified covering map where C_0 is a curve with the genus $g(C_0) \geq 1$. We put $X := C_1 \times C_2$. Taking a point $q_0 \in C_2$, we put $D_X := C_1 \times q_0$ and $D_S := C_0$. We denote by $p_1 : X \to D_X$ the projection to the first factor, and by $\nu_X : D_X \longrightarrow X$ the inclusion map. We denote the same letter g the map from D_X to D_S induced by $g : C_1 \longrightarrow C_0$. Then the *push-out*, or *fibered sum*

$$S := X \coprod_{D_X} D_S \tag{5.3.1}$$

of X and D_S over D_X in the category of complex spaces exists (cf. [17]), and it is a compact complex surface with ordinary singularities. By this construction, we have the following commutative diagram

$$\begin{array}{ccc} D_X & \xrightarrow{\nu_X} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ D_S & \xrightarrow[\nu_S]{} & S. \end{array} \tag{5.3.2}$$

5.3.1 Proposition. *The surface S in* (5.3.1) *is projective.*

Proof:. By a criterion for a line bundle over a comact complex space to be *positive* due to H. Grauert ([3]), it suffices to show that there exists a line bundle $\mathcal{L}$ over S satisfying the following conditions:

(5.3.3)

(1) For any irreducible curve C on S, there exist a natural number k and a cross-section s of $\mathcal{L}^{\otimes k}$ over C which has at least one zero point on C and is not identically zero on C.
(2) There exist a natural number k' and a cross-section s' of $\mathcal{L}^{\otimes k'}$ over S which has at least one zero point on S and is not identicall zero on S.

For this purpose, we take a point $p_0 \in D_S$ with $p_0 \notin \Sigma_{C_S}$ where Σ_{C_S} denotes the branch locus of the double covering map $g : D_X \longrightarrow D_S$. We put

$$D_1' := p_1^{-1}(g^{-1}(p_0)),$$

which is a divisor on X consisting of two reduced, irreducible curves. Then the *push-out* $D_1 := D_1' \coprod_{\nu_X(g^{-1}(p_0))} p_0$ of D_1' and p_0 over $\nu_X(g^{-1}(p_0))$ is realized as a curve on S. Further we take a point $q_0' \in C_2$ with $q_0' \neq q_0$, and define $D_2 := f(p_2^{-1}(q_0'))$, which is a reduced, irreducible curve on S where p_2 denotes the projection $X = C_1 \times C_2 \longrightarrow C_2$. Note that, since $p_2^{-1}(q_0') \hookrightarrow X - D_X$ and f gives rise to a biholomorphic map between $X - D_X$ and $S - D_S$, D_2 is biholomorphic to $p_2^{-1}(q_0')$. Now we define

$$\mathcal{L} := [D_1 + D_2].$$

We claim that the line bundle $\mathcal{L}$ satisfies that the condition (1) and (2) in (5.3.3). Indeed, it is obvious that $\mathcal{L}$ satisfies the condition (2). Now we are going to check the condition (1). Let C be an irreducible curve on X. In what follows, for each $i = 1, 2$, we denote by s_i the cross-section of the line bundle $[D_i]$ whose zero locus is $[D_i]$.

i) In the case where C is neither an irreducible component of D_1 nor D_2: $s_1 \otimes s_2$ is a cross-section of $\mathcal{L}$ which does not vanish identically on C and has at least one zero point on C.

ii) In the case where C is one of irreducible components of D_1: We take a sufficiently large natural number k so that there is an effective divisor $d_1 = \Sigma_{i=1}^k p_i$ on D_0, consisting of finite points $p_1, \cdots, p_k$ included in $D_S - \Sigma c - p_0$, and being linearly equivalent to kp_0. This is always possible,because $[p_0]$ is a positive line bundle on D_S. We put $E_1' := \Sigma_{i=1}^k p_1^{-1}(g^{-1}(p_i))$, which is a divisor on X. Then the *push-out*

$$E_1 := E_1' \coprod_{\Sigma_{i=1}^k \nu_X(g^{-1}(p_i))} \Sigma_{i=1}^k p_i$$

of E_1' and $\Sigma_{i=1}^k p_i$ over $\Sigma_{i=1}^k \nu_X(g^{-1}(p_i))$ is realized as a curve on S which is linearly equivalent to kD_1, since $\Sigma_{i=1}^k p_i$ is linearly equivalent to kp_0. Let s_1' be the cross-section of the line bundle $[kD_1]$ whose zero locus is E_1. Then the cross-section $s_1' \otimes s_2^k$ of the line bundle $\mathcal{L}^k = [kD_1 + kD_2]$ satisfies the condition (1) in (5.3.3) for the curve C.

iii) In the case where $C = D_2$: We take a sufficiently large number ℓ so that there is an effective divisor $d_2 = \Sigma_{i=1}^\ell q_i$ on C_2, consisting of finite distinct points $q_1, \cdots, q_\ell$ included in $C_2 - \{q_0, q_0'\}$, and being linerly equivalent to $\ell q_0'$. This is always possible because $[q_0']$ is a positive line bundle on C_2. We put $E_2 := \Sigma_{i=1}^\ell f(p_2^{-1}(q_i))$, which is a curve on S and is linearly equivalent to ℓD_2 since $d_2 = \Sigma_{i=1}^\ell q_i$ is linearly equivalent to $\ell q_0'$ on C_2. Let s_2' be the cross-section of the line bundle $[\ell D_2]$ whose zero locus is E_2. Then the cross-section $s_1^\ell \otimes s_2'$ of the line bundle $\mathcal{L}^{\otimes \ell} = [\ell D_1 + \ell D_2]$ satisfies the condition (1) in (5.3.3) for the curve C.

Q.E.D.

First we shall compute the dimension of the cohomology groups $H^1(X, \Theta_X(-log\, D_X))$ and $H^1(S, \Theta_S)$ which are the infinitesimal locally trivial deformation space of the pair (X, D_X) and S, respectively. We denote by $p_i : X := C_1 \times C_2 \to C_i$ $(i = 1, 2)$ the projection to C_i. Since $D_X := C_1 \times q$, $q \in C_2$, we have

$$\Theta_X(-log\, D_X) = p_1^*\Theta_{C_1} \oplus p_2^*\Theta_{C_2}(-q), \tag{5.3.4}$$

where $\Theta_{C_2}(-q)$ denotes the sheaf of germs of holomorphic vector fields on C_2 which vanish at q. Since

(5.3.5)

$$R^s p_{1*} p_1^* \Theta_{C_1} = \Theta_{C_1} \otimes R^s p_{1*} \mathcal{O}_X = \begin{cases} \Theta_{C_1} \otimes H^s(C_2, \mathcal{O}_{C_2}) & s = 0, 1 \\ 0 & \text{otherwise,} \end{cases}$$

there arises an exact sequence

(5.3.6) $$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(X, p_1^*\Theta_{C_1}) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow H^2(X, p_1^*\Theta_{C_1})$$

from the Leray spectral sequence

$$E_2^{r,s} = H^r(C_1, R^s p_{1*} p_1^* \Theta_{C_1}) \Longrightarrow E_\infty^{r+s} = H^{r+s}(C_1, p_1^* \Theta_{C_1})$$

(cf. [R. Godement: Topologie algébrique et théorie des faisceaux, Chapitre I, Théorème 4.5.1]). Since $H^0(C_1, \Theta_{C_1}) = 0$, it follows from (5.3.6) that

(5.3.7) $$H^1(C_1, \Theta_{C_1}) \simeq H^1(X, p_1^*\Theta_{C_1}).$$

Since

$$\begin{aligned} & R^s p_{2*} p_2^* \Theta_{C_2}(-q_0) \\ &= \Theta_{C_2}(-q_0) \otimes R^s p_{2*} \mathcal{O}_X = \begin{cases} \Theta_{C_2}(-q_0) \otimes H^s(C_1, \mathcal{O}_{C_1}) & s = 0, 1 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

by the same reasoning as above, we have

(5.3.8) $$H^1(C_2, \Theta_{C_2}(-q_0)) \simeq H^1(X, p_2^*\Theta_{C_2}(-q_0)).$$

Now the exact sequence of cohomology

(5.3.9) $$0 \longrightarrow H^0(q_0, N_{q_0/C_2}) \longrightarrow H^1(C_2, \Theta_{C_2}(-q_0)) \longrightarrow H^1(C_2, \Theta_{C_2}) \longrightarrow 0.$$

follows from the folloing eaxact sequence of $\mathcal{O}_{C_2}$-modules

$$0 \longrightarrow \Theta_{C_2}(-q_0) \longrightarrow \Theta_{C_2} \longrightarrow N_{q_0/C_2} \longrightarrow 0.$$

By (5.3.4), (5.3.7) and (5.3.8) and (5.3.9), we have the following proposition

5.3.2 Proposition.

$$\begin{aligned} & dim\, H^1(X, \Theta_X(-log\, D_X)) = dim\, H^1(C_1, \Theta_{C_1}) + dim\, H^1(C_2, \Theta_{C_2}(-q_0)) \\ &= dim\, H^1(C_1, \Theta_{C_1}) + dim\, H^1(C_2, \Theta_{C_2}) + dim\, H^0(q_0, N_{q_0/C_2}) \\ &= dim\, H^1(X, \Theta_X) + dim\, H^0(D_X, N_{D_X/X}), \end{aligned}$$

where $N_{D_X/X}$ denotes the sheaf of germs of holomorphic sections of the normal bundle of D_X in X.

Note that the third equality in Proposition 5.3.2 follows from the fact that

$$\Theta_X = p_1^*\Theta_{C_1} \oplus p_2^*\Theta_{C_2}, \quad \text{and} N_{D_X/X} = p_2^* N_{q_0/C_2}$$

5.3.3 Corollary.

$$
\begin{aligned}
&\mathit{dim}\, H^1(S,\Theta_S)\\
&= \mathit{dim}\, H^1(C_0,\Theta_{C_0}(-\Sigma\mathfrak{c}_0)) + \mathit{dim}\, H^1(C_2,\Theta_{C_2}) + \mathit{dim}\, H^0(q_0, N_{q_0/C_2})
\end{aligned}
$$

where $\Sigma\mathfrak{c}_0$ *denotes the branch locus of the double covering map* $g: C_1 \longrightarrow C_0$.

Proof. Note that we may consider that $D_X = C_1$ and $D_S = C_0$ in the diagram (5.3.2). Since $H^0(D_X,\Theta_{D_X}) = 0$ by the assumption $g(D_X) \geq 2$, since S is triple point free, and since the map $H^1(X,\Theta_X(-\log D_X)) \longrightarrow H^1(D_X,\Theta_{D_X})$ is surjective by Proposition 5.3.2, the short exact sequence in Theorem 3.19 in Part I implies the exact sequence

$$
\begin{aligned}
&0 \longrightarrow H^1(S,\Theta_S) \longrightarrow H^1(X,\Theta_X(-\log D_X)) \oplus H^1(D_S,\Theta_{D_S}(-\Sigma\mathfrak{c}_S))\\
&\qquad\qquad \longrightarrow H^1(D_X,\Theta_{D_X}) \longrightarrow 0.
\end{aligned} \tag{5.3.10}
$$

Therefore, by (5.3.10) and Proposition 5.3.2, we obtain the equality in the corollary.

Q.E.D.

Here we shoud notice that $H^1(C_0,\Theta_{C_0}(-\Sigma\mathfrak{c}_0))$ is nothing but the infinitesimal deformation space of the map $g: C_1 \longrightarrow C_0$. In fact, the following proposition holds:

5.3.4 Proposition. *There exists an analytic family* $(\mathcal{C}_1, G, \mathcal{C}_0, \pi_1, \pi_2, M_1, o_1, \phi_1, \psi_1)$ *of deformations of the holomorphic map* $g: C_1 \longrightarrow C_0$ *(for the notation see* [12]*), parametrized by a pointed complex manifold* (M_1, o_1) *which enjoys the following properties:*

(1) *For any point* $t \in M$, *the characteristic map at* t

$$\sigma_t : T_tM \longrightarrow H^1(C_{0t},\Theta_{C_{0t}}(-\Sigma\mathfrak{c}_{0t}))$$

is defined, where T_tM *denotes the tangent space of* M *at* t, *and* $C_{0t} := \pi_1^{-1}(t)$, *and* σ_t *is injective.*

(2) *The family is complete (versal) at any point* $t \in M$ *with respect to deformations of a holomorphic map.*

Proof. By (4.6) the double branched covering map $g: C_1 \longrightarrow C_0$ is a stable holomorphic map in the sense of J. N. Mather (cf. [7]). Therefore the results in [11] is valid for deformations of g. The obstruction class of this deformation sits in $H^2(C_0,\Theta_{C_0}(-\Sigma\mathfrak{c}_0))$, which is zero. Hence, by Theorem 2.2 in [11], there exists an analytic family $(\mathcal{C}_0, G, \mathcal{C}_1, \pi_1, \pi_2, M_1, o_1, \phi_1, \psi_1)$ of deformations of the holomorphic map $g : C_1 \to C_0$, parametrized by a complex manifold M_1, which enjoys the property that the characteristic map

$$\sigma_{o_1} : T_{o_1}M_1 \to H^1(C_o, \Theta_{C_o}(-\Sigma\mathfrak{c}_o))$$

at $o_1 \in M_1$ is bijective, and also enjoys the property (2) in the proposition. Since g is locally stable, after schrinking M sufficiently small around o_1 if necessary, we may assume that $g_t : C_{1t} \to C_{0t}$ is a double branched covering for any $t \in M_1$. Since

$$dim\, H^1(C_{0t}, \Theta_{C_{ot}}(-\Sigma\mathfrak{c}_{ot})) = 3g(C_{ot}) - 3 + \#\Sigma\mathfrak{c}_{ot}$$

under the assumption $g(C_{ot}) \geq 1$, where $g(C_{ot})$ denotes the genus of C_{ot} and $\#\Sigma\mathfrak{c}_{ot}$ denotes the cardinal number of the set $\Sigma\mathfrak{c}_{ot}$, it is independent on $t \in M_1$. Cosequently, we conclude that $\sigma_t : T_tM \to H^1(C_{ot}, \Theta_{C_ot}(-\Sigma\mathfrak{c}_{ot}))$ is bijective at any point $t \in M$.

Q.E.D.

We denote by $(\mathcal{C}_2, \varpi_2, M_2, o_2, \phi_2)$ the universal family of deformations of C_2. Note that M_2 is a complex manifold of dimension $3g(C_2) - 3$, and the characteristic map

$$\sigma_t : T_tM_2 \to H^1(C_{2t}, \Theta_{C_{2t}})$$

is bijective at any point $t \in M_2$, where $C_{2t} := \varpi_2^{-1}(t)$ for $t \in M$. We take an open neighborhood U of q_0 in C_2 and define a map

$$\widetilde{\nu} : \mathcal{C}_1 \times M_2 \times U \to \mathcal{C}_1 \times \mathcal{C}_2 \times U$$

by $(x_1, t_2, q') \to (x_1, (t_2, \phi_2(q')), q')$ for $(x_1, t_1, q') \in \mathcal{C}_1 \times M_2 \times U$, where ϕ_2 is the isomorphism from C_2 to $\varpi_2^{-1}(o_2)$. Then we obtain the following diagram:

$$\begin{array}{ccc} \mathcal{C}_1 \times M_2 \times U & \xrightarrow{\widetilde{\nu}} & \mathcal{C}_1 \times \mathcal{C}_2 \times U \\ {\scriptstyle G\times id_{M_2}\times id_U}\downarrow & & \\ \mathcal{C}_0 \times M_2 \times U. & & \end{array}$$

Since $\widetilde{\nu}$ gives rises to a closed embedding after schrinking M_2 sufficiently small around o_2 if necessary, and since $G \times id_{M_2} \times id_U$ is a finite map, there exists a *push-out*, or *fibered sum*

$$\mathcal{S} := (\mathcal{C}_1 \times \mathcal{C}_2 \times U) \coprod_{\mathcal{C}_1 \times M_2 \times U} (\mathcal{C}_o \times M_2 \times U)$$

of $\mathcal{C}_1 \times \mathcal{C}_2 \times U$ and $\mathcal{C}_o \times M_2 \times U$ over $\mathcal{C}_1 \times M_2 \times U$, and there exists naturally a surjective holomorphic map φ from $\mathcal{S}$ to $M_1 \times M_2 \times U$ such that:

(1) for $(t_1, t_2, q') \in M_1 \times M_2 \times U$,

$$\varpi^{-1} := (C_{1t_1} \times C_{2t_2} \times q') \coprod_{C_{1t_1} \times t_2 \times q'} (C_{ot_1} \times t_2 \times q')$$

is a complex projective surface with ordinary singularities whose double curve is $C_{ot_1} \times t_2 \times q'$,

(2) $\varpi^{-1}(o_1, o_2, q_0)$ is isomorphic to $S := X \coprod_{D_X} D_S$, where $X := C_1 \times C_2$, $D_X := C_1 \times q_0$, $D_S := C_2 \times q_0$.

We denote by ϕ the isomorphism from S to $\varphi^{-1}(o_1, o_2, q)$. We put $M := M_1 \times M_2 \times U$, and $o := (o_1, o_2, q_0)$.

5.3.5 Proposition. *The analytic family $(\mathcal{S}, \varpi, M, o, \phi)$ constructed above is the one of locally trivial deformations of S, parametrized by a complex manifold M such that:*

(1) *For any point $t \in M$, $S_t := \varpi^{-1}(t)$ is projective.*

(2) *For any point $t \in M$, the characteristic map at $t \in M$*

$$\sigma_t : T_t M \longrightarrow H^1(S_t, \Theta_{S_t})$$

is bijective.

(3) *It is complete at any point $t \in M$ with respect to locally trivial deformation of S_t.*

Proof. The assertion (1) follows from Proposition 5.3.1. Now we are going to prove the assertion (2). By the assumption $g(C_i) \geq 2 (i = 1, 2)$, we have

$$H^0(X, \Theta_X) = H^0(C_t, \Theta_{C_t}) \oplus H^0(C_2, \Theta_{C_2}) = 0.$$

Hence the exact sequence of cohomology groups

$$(5.3.11) \qquad 0 \longrightarrow H^0(D_X, N_{D_X/X}) \longrightarrow H^1(X, \Theta_X(-log\, D_X)) \longrightarrow H^1(X, \Theta_X)$$

follows from the short exact sequence of $\mathcal{O}$-modules in (4.24). By the arguments used to derive Proposition 5.3.2, we have

$$H^0(D_X, N_{D_X/X}) \simeq H^0(C_2, N_{q_0/C_2}),$$

$$H^1(X, \Theta_X(-log\, D_X)) \simeq H^1(C_1, \Theta_{C_1}) \oplus H^1(C_2, \Theta_{C_2}(-q_0)),$$

$$H^1(X, \Theta_X) \simeq H^1(C_1, \Theta_{C_1}) \oplus H^1(C_2, \Theta_{C_2}).$$

Furthermore, the homomorphism $H^1(C_2, \Theta_{C_2}(-q_0)) \longrightarrow H^1(C_2, \Theta_{C_2})$ is surjective (cf. (5.3.9)). Therefore, the long exact sequence of cohomology groups in (5.3.11) reduces to

$$\begin{aligned} &0 \longrightarrow H^0(q_0, N_{q_0/C_2}) \longrightarrow H^1(X, \Theta_X(-log\, D_X)) \\ &\qquad\qquad \longrightarrow H^1(C_1, \Theta_{C_1}) \oplus H^1(C_2, \Theta_{C_2}) \longrightarrow 0 \quad \text{(exact)}. \end{aligned} \tag{5.3.12}$$

On the other hand, since $H^0(D_X, \Theta_{D_X}) = 0$ $(D_X = C_1 \times q_0)$ by the assumption $g(C_1) \geq 2$. Then, by Proposition 3.6 and (5.3.12), we have

$$\begin{aligned} &0 \longrightarrow H^1(S, \Theta_S) \longrightarrow H^1(X, \Theta_X(-log\, D_X) \oplus H^1(D_S, \Theta_{D_S}(-\Sigma\mathfrak{c}_S)) \\ &\qquad\qquad \longrightarrow H^1(D_X, \Theta_{D_X}) \longrightarrow 0. \quad \text{(exact)} \end{aligned} \tag{5.3.13}$$

Here note that $D_X^* = D_X$, $\Sigma t_X^* = \emptyset$. By Proposition 4.1 and Corollary 4.5, the homomorphism $H^1(D_S, \Theta_{D_S}(-\Sigma\mathfrak{c}_S)) \longrightarrow H^1(D_X, \Theta_{D_X})$ and $H^1(S, \Theta_S) \longrightarrow H^1(X, \Theta_X(-log\, D_X))$ are injective. Hence we can regard $H^1(D_S, \Theta_{D_S}(-\Sigma\mathfrak{c}_S))$ (resp. $H^1(S, \Theta_S)$ as a subspace of $H^1(D_X, \Theta_{D_X})$ (resp. $H^1(X, \Theta_X(-log\, D_X))$). Then, since the homomorphism

$$H^1(X, \Theta_X(-log\, D_X)) \longrightarrow H^1(D_X, \Theta_{D_X})$$

in (5.3.13) factorizes through the homomorphism

$$H^1(X, \Theta_X(-log\, D_X)) \longrightarrow H^1(D_X, \Theta_{D_X}) \oplus H^1(C_2, \Theta_{C_2}) \qquad (D_X = C_1),$$

in (5.3.12), and since the image of $H^0(q_0, N_{q_0/C_2})$ in $H^1(X, \Theta_X(-log\, D_X))$ is included in $H^1(S, \Theta_S)$, which is because of (5.3.12) and (5.3.13), we have the following exact sequence

$$\begin{aligned} &0 \longrightarrow H^0(q_0, N_{q_0/C_2}) \longrightarrow H^1(S, \Theta_S) \\ &\qquad\qquad \longrightarrow H^1(C_0, \Theta_{C_0}(-\Sigma\mathfrak{c}_0)) \oplus H^1(C_2, \Theta_{C_2}) \longrightarrow 0 \end{aligned} \tag{5.3.14}$$

$(C_0 = C_S)$. We should now recall how we have constructed the family $\mathcal{S}$. Then we conclude that the characteristic map $\sigma_o : T_o M \longrightarrow H^1(S, \Theta_S)$ maps the subspaces $T_{o_1} M_1$, $T_{o_2} M_2$ and $T_q U$ of $T_o M$ isomorphically to

$$H^1(C_0, \Theta_{C_0}(-\Sigma\mathfrak{c}_o)),\ H^1(C_2, \Theta_{C_2}) \text{ and } H^0(q_0, N_{q_0/C_2}),$$

respectively. Therefore, it follows from (5.3.14) that the characteristic map $\sigma_o : T_o M \longrightarrow H^1(S, \Theta_S)$ is bijective. Note that the arguments above is also valid for $\sigma_t : T_t M \longrightarrow H^1(S_t, \Theta_{S_t})$ for any point $t \in M$. Therefore we also conclude that the characteristiv map σ_t is bijective at any point $t \in M$. The assertion (3) follows from the assertion (2) and from the way by which the family $\mathcal{S}$ has been constructed.

Q.E.D.

We are now going to consider the cohomological infinitesimal mixed Torelli problem for the surface S. We will check the conditions in Theorem 4.12. First, note that in the case we are now consideing, we have

$$C_o \simeq D_S = D_S^*,\ \ C_1 \simeq D_X \equiv D_X^*,\ \ \text{and}\ \ \Sigma t_S = \Sigma t_X = \Sigma t_X^* = \emptyset.$$

In what follows we identify D_S and D_X with C_0 and C_1, respectively. Since $H^0(D_X, \Theta_{D_X}) = 0$ by the assumption $g(C_1) \geq 2$, the condition (i) is fullfiled. Hence any cohomology class of $H^1(S, \Theta_S)$ is represented by a pair of cohomology classes

$$(\theta_X, \theta_{D_X}) \in H^1(X, \Theta_X) \oplus H^1(D_S, \Theta_{D_S}(-\Sigma\, \mathfrak{c}_S))$$

which is subject to the condition $\widehat{\omega\nu_X}(\theta_X) = \widehat{\omega g}(\theta_{D_S})$ in $H^1(D_X, \Theta_{D_X})$. Since $\Theta_X = p_1^*\Theta_{C_1} \oplus p_2^*\Theta_{C_2}$, it follows that

$$\begin{aligned} H^1(X, \Theta_X) &\simeq H^1(X, p_1^*\Theta_{C_1}) \oplus H^1(X, p_2^*\Theta_{C_2}) \\ &\simeq H^1(C_1, \Theta_{C_1}) \oplus H^1(C_2, \Theta_{C_2}). \end{aligned}$$

Using this isomorphism, we represent $\theta_X \in H^1(X, \Theta_X)$ as $\theta_X = (p_1^*\theta_{C_1}, p_2^*\theta_{C_2})$ where $\theta_{C_i} \in H^1(C_i, \Theta_{C_i})$ for $i = 1, 2$. Then the condition $\widehat{\omega\nu_X}(\theta_X) = \widehat{\omega g}(\theta_{D_S})$ is equivalent to $\theta_{C_1} = \widehat{\omega g}(\theta_{C_2})$. Since $\Omega_X^2 = p_1^*\Omega_{C_1}^1 \otimes p_2^*\Omega_{C_2}^1$ and $\Omega_X^1 = p_1^*\Omega_{C_1}^1 \oplus p_2^*\Omega_{C_2}^1$, by (5.3.6), we have

$$R^s p_{j*} p_i^* \Omega_{C_i}^1 = \Omega_{C_i}^1 \otimes R^s p_{i*} \mathcal{O}_X \simeq \begin{cases} \Omega_{C_i}^1 \otimes H^s(C_j, \mathcal{O}_{C_j}) & s = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$
$$(i \neq j,\ 1 \leq i, j \leq 2)$$

and

$$R^s p_{i*} \Omega_X^2 = \Omega_{C_i}^1 \otimes R^s p_{j*} \Omega_{C_j}^1 = \begin{cases} \Omega_{C_i}^1 \otimes H^s(C_j, \Omega_{C_j}^1) & s = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$
$$(i \neq j,\ 1 \leq i, j \leq 2).$$

From these it follows that

(5.3.15) $$H^0(X, \Omega_X^2) \simeq H^0(C_1, \Omega_{C_1}^1) \otimes H^0(C_2, \Omega_{C_2}^1),$$

(5.3.16) $$H^0(X, \Omega_X^1) \simeq H^0(C_1, \Omega_{C_1}^1) \oplus H^0(C_2, \Omega_{C_2}^1),$$

and that there injections

(5.3.17) $$0 \longrightarrow H^1(C_i, \Omega_{C_i}^1) \longrightarrow H^1(X, p_i^*\Omega_{C_i}^1) \qquad i = 1, 2.$$

By (5.3.15), if $\omega_1^{(1)}, \cdots \omega_{g(C_1)}^{(1)}$ (resp. $\omega_1^{(2)}, \cdots \omega_{g(C_2)}^{(2)}$) form a basis for $H^0(C_1, \Omega^1_{C_1})$ (resp. $H^0(C_2, \Omega^1_{C_2})$), then $p_1^*\omega_i^{(1)} \otimes p_2^*\omega_j^{(2)}$, $1 \leq i \leq g(C_i)$, $1 \leq j \leq g(C_2)$ form a basis of $H^0(X, \Omega^2_X)$. Therefore, for $\theta_X = (p_1^*\theta_{C_1}, p_2^*\theta_{C_1})$ ($\theta_{C_i} \in H^1(C_i, \Theta_{C_i})$, $i = 1, 2$), we have

$$
\begin{aligned}
&\theta \lfloor (p_1^*\omega_1^{(1)} \otimes p_2^*\omega_j^{(2)}) \\
&= (p_1^*(\theta_{C_1} \lfloor \omega_i^{(1)}), p_2^*(\theta_{C_2} \lfloor \omega_j^{(2)})) \in H^1(X, p_1^*\Omega^1_{C_1}) \oplus H^1(X, p_2^*\Omega^1_{C_2}) \simeq H^1(X, \Omega^1_X)
\end{aligned}
$$

By (5.3.17), if $(\theta_X \lfloor (p_1^*\omega_i^{(1)} \otimes p_2^*\omega_j^{(2)}) = 0$ for any i, j with $1 \leq i \leq g(C_1)$, $1 \leq g \leq g(C_2)$, then we have

$$
\begin{aligned}
&\theta_{C_1} \lfloor \omega_i^{(1)} = 0 \qquad \text{for } 1 \leq i \leq g(C_1), \\
&\theta_{C_2} \lfloor \omega_j^{(2)} = 0 \qquad \text{for } 1 \leq j \leq g(C_2).
\end{aligned}
\tag{5.3.18}
$$

Here we remember the fact that the homomorphism

$$
H^1(C_i, \Theta_{C_i}) \to Hom_{\mathbb{C}}(H^0(C_i, \Omega^1_{C_i}), H^1(C_i, \mathcal{O}_{C_i})) \qquad (i = 1, 2)
$$

is injective if C_i is non-hyperelliptic. From now on we assume that both C_i, $i = 1, 2$, are non-hyperelliptic. Then it follows that $\theta_{C_1} = \theta_{C_2} = 0$, and so $\theta_X = 0$ from (5.3.18). Namely, the condition (ii) in Theorem 4.12 is fullfilled.

Finally, we check the condition (iii) in Theorem 4.12. In the case we are considering, the condition (iii) in Theorem 4.12 is reduced to that the homomorphism

$$
\begin{aligned}
\overline{\mu}^{(1)} : Ker\{ H^0(D_X, N_{D_X/X}) &\to H^1(D_X, \Theta_{D_X}) \} \\
&\to Hom_{\mathbb{C}}(\{Ker\{H^0(\Omega^1_X) \oplus H^0(\Omega^1_{D_S}) \to H^0(\Omega^1_{D_X})\} \\
&\qquad \to H^0(\mathcal{O}_{D_X})/Im\{H^0(\mathcal{O}_X) \oplus H^0(\mathcal{O}_{D_S})\}\})
\end{aligned}
$$

defined by *contraction* and the *pull-back* is injective. By (5.3.16) we have

$$
\begin{aligned}
&Ker\{ H^0(\Omega^1_X) \oplus H^0(\Omega^1_{D_S}) \to H^0(\Omega^1_{D_X}) \} \\
&\simeq H^0(\Omega^1_{D_S^*}) \oplus H^0(\Omega^1_{C_2}).
\end{aligned}
$$

Since $N_{D_X/X} = p_2^* N_{q/C_2}$, we have

$$
H^0(D_X, N_{D_X/X}) \simeq H^0(C_2, N_{q/C_2}).
$$

Hence any $s_X \in H^0(D_X, N_{D_X/X})$ is represented as

$$
s_X = p_2^* s_{C_2}, \qquad s_{C_2} \in H^0(C_2, N_{q/C_2}).
$$

Therefore, if $\overline{\mu}^{(1)}(s_X) = 0$ for $s_X \in Ker\{ H^0(D_X, N_{D_X/X}) \to H^1(D_X, \Theta_{D_X}) \}$, then we have

$$\begin{aligned}(5.3.19) \qquad & \overline{\mu}^{(1)}(s_X)(p_2^*\omega_{C_2}^{(i)}, 0) \\ & = s_{C_2} \lfloor \omega_{C_2|q}^{(i)} = s_{C_2}^{(i)}(\omega_{C_2}^{(i)}(q)) = 0\end{aligned}$$

for any $\omega_{C_2}^{(i)}$, $1 \le i \le g(C_2)$, where $\omega_{C_2}^{(i)}(q)$ denotes the value of $\omega_{C_2}^{(i)}$ at q. Since $g(C_2) \ge 2$, there is at least one i_0, $1 \le i_0 \le g(C_2)$ with $\omega_{C_2}^{(i_0)}(q) \ne 0$, i.e., the canonical linear system $|\mathfrak{K}_{C_2}|$ on C_2 has no base point. Conseqently, by (5.3.19) we conclude that $s_{C_2} = 0$ which implies $s_X = 0$, that is, the homomorphism $\overline{\mu}^{(1)}$ is injective as requied.

Summarizing the arguments above, we obtain the following theorem.

5.3.6 Theorem. *Assume $g(C_i) \ge 2$ and C_i is non-hyperelliptic for $i = 1, 2$. Then, for the surface S in* (5.3.1), *the homomorphism*

$$\oplus_{i=1}^2 \tau^{(i)} : H^1(S, \Theta_S) \to \oplus_{p=1}^2 Hom_{\mathbb{C}}(\mathbb{H}^0(\Omega_{X_\bullet}^p[1], \mathbb{H}^1(\Omega_{X_\bullet}^{p-1}[1]))$$

is injective. That is, infinitesimal mixed Torelli problem is affirmatively solved for the surface S.

REFERENCES

1. Enriques F., *Le Superficie Algebriche*, Nicola Zanichelli Editore, Bologna (1949).
2. Flenner H., *Über Deformation holomorpher Abbildungen*, Osnabrcher Schriften zur Mathematik, Universität Osnabrück (1979).
3. Grauert H., *Über Modifikationen und exzeptionelle analytisch Mengen*, Math. Ann. **146** (1962), 331-368.
4. Hartshorne R., *Algebraic Geometry*, GTM 52, Springer-Verlag (1977).
5. Iitaka S., *Max Nöther's theorem on a regular projective algebraic variety*, J. Fac. Sci. Univ. Tokyo, Sec. I **8** (1966), 129-137.
6. Kodaira K., *On characteristic systems of families of surfaces with ordinary singularities in a projective space*, Amer. J. Math. **87** (1965), 227-256.
7. Mather J. N., *Stability of C^∞ Mappings*, II: *Infinitesimal stability implies stability*, Ann. of Math. **89, No.2** (1969), 254-291.
8. Mather J. N., *Stable map-germs and algebraic geometry*, Lecture Notes in Math., Manifolds-Amsterdam **197** (1970), 176-193.
9. Roberts J., *Hypersurfaces with non-singular normalization and Their double loci*, J. Algebra **53, No.1** (1978), 253-267.
10. Tsuboi S., *Deformations of locally stable holomorphic maps and locally trivial displacements of analytic subvarieties with ordinary singularities*, Sci. Rep. Kagoshima Univ. **35** (1986), 9-90.
11. S. Tsuboi, *Deformations of complex analytic subspaces with locally stable parametrizations of compact complex manifolds*, Proc. Japan Acad. **66A** (1990), 157-160.
12. Tsuboi S., *On deformations of locally stable holomorphic maps*, Japan. J. Math. **19, No.2** (1993), 325-342.

13. Tsuboi S., *Infinitesimal parameter spaces of locally trivial deformations of compact complex surfaces with ordinary singularities,*, Finite or infinite dimensional complex analysis, Lecture notes of pure and applied mathematics (Dekker) **214**, (2000), 523-532.
14. Tsuboi S., *Infinitesimal mixed Torelli problem for algebraic surfaces with ordinary singularities,* I, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) **37** (2004), 1-71.
15. Tsuboi S. and M. Ôkawa, *Some results on the local moduli of non-singular normalizations of surfaces with ordinary singularities*, Tohoku Mathematical Journal **40/2** (1988), 269-291.
16. Usui S., *Deformations and local Torelli theorem for certain surfaces of general type*, Lecture Note in Math. **732**, Algebraic Geometry, Proceedings, Copenhagen (1978).
17. Van Straten D., *Weakly normal singularities and their improvements*, Doctor Thesis, Leiden University (1987).
18. Wavrik J.J., *Deformations of Banach coverings of complex manifolds*, Amer. J. Math. **90** (1968), 926-960.

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