Design of estimators using covariance information with uncertain observations in linear discrete-time distributed parameter systems

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Abstract

This paper presents recursive least-squares (RLS) estimation algorithms using the covariance information in linear discrete-time distributed parameter systems. The signal is estimated with the observations containing some uncertain observations. In the uncertain observations, there are cases where the observed value does not contain the signal and consists of observation noise only. The probability that the signal exists in the observed value is used in the estimation algorithms. The algorithms are derived based on the invariant imbedding method.

Keywords. Image restoration; uncertain observations; distributed parameter systems.
Running head: Estimators with uncertain observations

1. Introduction

In Nakamori [1], optimal filtering algorithm using the covariance information is presented in linear discrete-time distributed parameter systems. Here, the autocovariance function of the signal is expressed in the semi-degenerate kernel form. Also, in [2], linear RLS algorithms with uncertain observations for the filtering estimate is proposed by using the state-space model. In [3],[4], linear discrete-time RLS algorithms with uncertain observations for

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the prediction, filtering and smoothing estimates are proposed in the case where state and measurement noises are correlated. In [5], RLS filter and predictor based on the state-space model are proposed with uncertain observations in linear discrete-time distributed parameter systems. Here, in the uncertain observations, there are cases where the observed value does not contain the signal and consists of observation noise only. The probability that the signal exists in the observed value is used in the estimation algorithms. The uncertain observations occur based on an intermittent failure in the observation mechanisms.

By combining the previous works [1],[2], with the uncertain observations, this paper newly derives the RLS algorithms using the covariance information for the filtering and fixed-point smoothing estimates in linear discrete-time distributed parameter systems. It is assumed that observation noise is white. The proposed algorithms use the following information. (1) The autocovariance function of the signal in the semi-degenerate kernel form. (2) The variance of the observation noise. (3) The probability. It is advantageous that the proposed filtering algorithm can be applied to the case where a difference equation, which generates the signal process, is unknown in linear discrete-time distributed parameter systems.

A numerical simulation example is shown for the restoration of image from a degraded observed image that contains signal plus white noise with the probability for the existence of the signal.

2. Linear least-squares estimation problem

Let D be a connected bounded open domain of a p-dimensional Euclidean space R^p. The spacial coordinate vector is denoted by $\eta=(\eta_1,\eta_2,\dots,\eta_p)\in D$. Let $u(k,\eta)$ be an n-dimensional zero-mean signal vector in linear discrete-time stochastic system:

$$u(k,\eta)=Col[u_1(k,\eta),\ \cdots,\ u_n(k,\eta)],\ k\geq 0. \tag{1}$$

Let us assume that the measurement data are taken at fixed m points $\eta^1,\ \cdots,\ \eta^m$. Furthermore, let us define an mn-dimensional column vector

$$u_m(k)=Col[u(k,\eta^1),\ \cdots,\ u(k,\eta^m)]=\begin{bmatrix} u(k,\eta^1) \\ \vdots \\ u(k,\eta^m) \end{bmatrix}. \tag{2}$$

Assume that the uncertain observation equation is described by

$$z(k) = U(k)H(k)u_m(k) + v(k), \tag{3}$$

where $z(k)$ is an r-dimensional measurement vector at the points $\eta^1, \cdots, \eta^m$. $\{U(k);\ k\geq 0\}$ is a sequence of independent Bernoulli random variables with $P[U(k)=1]=p(k)$. $U(k)$ has the following stochastic properties.

$$\begin{aligned} &E[U(k)]=p(k) \\ &E[U(k)U(j)]=p(k)p(j),\ k \neq j \\ &E[U^2(k)]=p(k) \end{aligned} \tag{4}$$

$H(k)$ is a known $r\times mn$ matrix function. $v(k)$, $k\geq 0$, is a vector-valued white process that is uncorrelated with $u_m(\cdot)$. The mean and covariance of $v(\cdot)$ are given by

$$E[v(k)]=0, \quad E[v(k)v^T(s)]=R(k)\delta_K(k-s),\ R(k)>0, \tag{5}$$

where $\delta_K(k-s)$ represents the Kronecker delta function. It is assumed that the processes $\{U(k);k\geq 0\}$, $\{v(k);k\geq 0\}$ and $\{u(k,\eta);k\geq 0,\eta\in D\}$ are uncorrelated with each other.

Let $\hat{u}(k,L,\eta)$ denote a fixed-point smoothing estimate of $u(k,\eta)$,

$$\hat{u}(k,L,\eta) = \sum_{l'=1}^{L} h(k,l',L,\eta)z(l'), \tag{6}$$

where $h(k,l',L,\eta)$ is called an impulse response function. The fixed-point smoothing estimate, which minimizes the mean-square value of the fixed-point smoothing error
$u(k,\eta) - \hat{u}(k,L,\eta)$,

$$E\left[\left\|u(k,\eta) - \hat{u}(k,L,\eta)\right\|^2\right], \tag{7}$$

is optimal in a sense of least-squares estimation, where $\| \ \|$ denotes the Euclidean norm. Minimizing (7) leads to the Wiener-Hopf equation

$$E\left[u(k,\eta)z^T(l)\right] = \sum_{l'=1}^{L} h(k,l',L,\eta)E\left[z(l')z^T(l)\right]. \tag{8}$$

From (3) and (4), $E[u(k,\eta)z^{r}(l)]$ is calculated as

$$
\begin{aligned}
E\left[u(k,\eta)z^T(l)\right] &= E\left[u(k,\eta)u_m^T(l)H^T(l)U(l)\right] \\
&= p(l)E\left[u(k,\eta)u_m^T(l)\right]H^T(l).
\end{aligned}
\tag{9}
$$

Let us define the function $K_m(k,\eta,l) = E\left[u(k,\eta)u_m^T(l)\right]$ and $F_m(k,l) = E\left[u_m(k)u_m^T(l)\right]$.
Also, $E\left[z(l')z^T(l)\right]$ is calculated as

$$
\begin{aligned}
E\left[z(l')z^T(l)\right] &= E\left[U(l')H(l')u_m(l')u_m^T(l)H^T(l)U(l)\right] + R(l')\delta_K(l'-l), \\
&= \begin{cases} p(l')H(l')F_m(l',l)H^T(l)p(l), l' \neq l, \\ p(l)H(l)F_m(l,l)H^T(l) + R(l), l' = l. \end{cases}
\end{aligned}
\tag{10}
$$

By substituting (9) and (10) into (8), we obtain

$$
\begin{aligned}
&h(k,l,L,\eta)(R(l) + p(l)H(l)F_m(l,l)H^T(l) - p(l)H(l)F_m(l,l)H^T(l)p(l)) \\
&= p(l)K_m(k,\eta,l)H^T(l) - \sum_{l'=1}^{L} h(k,l',L,\eta)p(l')H(l')F_m(l',l)H^T(l)p(l).
\end{aligned}
\tag{11}
$$

Let us assume that the autocovariance function of the signal $u(k,\eta)$ is expressed by [1]

$$
K(k,\eta,l,\xi) = E\left[u(k,\eta)u^T(l,\xi)\right] = \begin{cases} \alpha(k,\eta,\xi)\beta^T(l,\eta,\xi), & 0 \le l \le k, \\ \gamma(k,\eta,\xi)\zeta^T(l,\eta,\xi), & 0 \le k \le l, \end{cases}
\tag{12}
$$

in the form of the semi-degenerate kernel. Here, $\alpha(k,\eta,\xi)$, $\beta(l,\eta,\xi)$, $\gamma(k,\eta,\xi)$ and $\zeta(l,\eta,\xi)$ are $n \times m'$ bounded matrices. The covariance function as (12) is appropriate for expressing a separable autocovariance function (see Section 5) which is often adopted based on the two-dimensional image model [1]. Then $K_m(k,\eta,l)$ is written as

$$
\begin{aligned}
K_m(k,\eta,l) &= E\left[u(k,\eta)u_m^T(l)\right] \\
&= \begin{cases} \tilde{\alpha}(k,\eta)\tilde{\beta}^T(l,\eta), & 0 \le l \le k, \\ \tilde{\gamma}(k,\eta)\tilde{\zeta}^T(l,\eta), & 0 \le k \le l, \end{cases}
\end{aligned}
\tag{13}
$$

where $\tilde{\alpha}(k,\eta)$, $\tilde{\beta}(l,\eta)$, $\tilde{\gamma}(k,\eta)$ and $\tilde{\zeta}(l,\eta)$ are bounded matrices of

$$\tilde{\alpha}(k,\eta)=\left[\alpha(k,\eta,\eta^{1}) \ \cdots \ \alpha(k,\eta,\eta^{m})\right],$$
$$\tilde{\beta}(l,\eta)=Diag\left[\beta(l,\eta,\eta^{1}) \ \cdots \ \beta(l,\eta,\eta^{m})\right],$$
$$\tilde{\gamma}(k,\eta)=\left[\gamma(k,\eta,\eta^{1}) \ \cdots \ \gamma(k,\eta,\eta^{m})\right],$$
$$\tilde{\zeta}(l,\eta)=Diag\left[\zeta(l,\eta,\eta^{1}) \ \cdots \ \zeta(l,\eta,\eta^{m})\right].$$

Also $F_m(k,l)$ is written as

$$F_m(k,l)=E\left[u_m(k)u_m^T(l)\right]$$
$$=\begin{cases}\alpha(k)\beta^T(l), & 0\le l\le k.\\ \gamma(k)\zeta^T(l), & 0\le k\le l,\end{cases} \qquad (15)$$

where $\alpha(k)$, $\beta(l)$, $\gamma(k)$ and $\zeta(l)$ are bounded matrices of

$$\alpha(k)=Diag\left[\tilde{\alpha}(k,\eta^{1}) \ \cdots \ \tilde{\alpha}(k,\eta^{m})\right],$$
$$\beta(l)=\left[\tilde{\beta}(l,\eta^{1}) \ \cdots \ \tilde{\beta}(l,\eta^{m})\right],$$
$$\gamma(k)=Diag\left[\tilde{\gamma}(k,\eta^{1}) \ \cdots \ \tilde{\gamma}(k,\eta^{m})\right],$$
$$\zeta(l)=\left[\tilde{\zeta}(l,\eta^{1}) \ \cdots \ \tilde{\zeta}(l,\eta^{m})\right]. \qquad (16)$$

It is quite difficult to obtain an analytical solution of $h(k,l,L,\eta)$ in (11). The algorithms for calculating the linear least-squares filtering and fixed-point smoothing estimates of $u(k,\eta)$ are shown in Theorem 1.

3. Linear least-squares algorithms for the filtering and fixed-point smoothing estimates

Theorem 1 presents the linear least-squares filtering and fixed-point smoothing algorithms using covariance information.

THEOREM 1. Let the autocovariance function of the signal $u(k,\eta)$ be given by (12) in the semi-degenerate kernel form, let the uncertain probability, that the observed value includes the signal, be $p(k)$ and let the variance of white observation noise be $R(k)$, then the linear RLS algorithms for the filtering and fixed-point smoothing estimates consist of (17)-(26).

Fixed-point smoothing estimate of the signal $u(k,\eta)$: $\hat{u}(k,L,\eta)$

$$\hat{u}(k,L,\eta)=\hat{u}(k,L-1,\eta)+h(k,L,L,\eta)(z(L)-p(L)H(L)\alpha(L)O(L-1)) \tag{17}$$

Smoother gain: $h(k,L,L,\eta)$

$$\begin{aligned} h(k,L,L,\eta)=&(p(L)\tilde{\gamma}(k,\eta)\tilde{\zeta}^T(L,\eta)H^T(L)-p(L)S(k,L-1,\eta)\zeta^T(L)H^T(L)) \\ &\left[R(L)+p(L)H(L)(F_m(L,L)-p(L)\alpha(L)r(L-1)\zeta^T(L))H^T(L)\right]^{-1} \end{aligned} \tag{18}$$

$$\begin{aligned} &S(k,L,\eta)=S(k,L-1,\eta)+h(k,L,L,\eta)p(L)H(L)(\gamma(L)-\alpha(L)r(L-1)), \\ &S(L,L,\eta)=\tilde{\alpha}(L,\eta)\tilde{r}(L,\eta) \end{aligned} \tag{19}$$

Filtering estimate of the signal $u(L,\eta)$: $\hat{u}(L,L,\eta)$

$$\hat{u}(L,L,\eta)=\tilde{\alpha}(L,\eta)\tilde{O}(L,\eta) \tag{20}$$

$$\tilde{r}(L,\eta)=\tilde{r}(L-1,\eta)+\tilde{J}(L,L,\eta)p(L)H(L)(\gamma(L)-\alpha(L)r(L-1)),\tilde{r}(0,\eta)=0 \tag{21}$$

$$r(L)=r(L-1)+J(L,L)p(L)H(L)(\gamma(L)-\alpha(L)r(L-1)),r(0)=0 \tag{22}$$

$$\begin{aligned} J(L,L)=&\left[(\beta^T(L)-r(L-1)\zeta^T(L))H^T(L)p(L)\right] \\ &\left[R(L)+p(L)H(L)(F_m(L,L)-p(L)\alpha(L)r(L-1)\zeta^T(L))H^T(L)\right]^{-1} \end{aligned} \tag{23}$$

$$\tilde{O}(L,\eta)=\tilde{O}(L-1,\eta)+\tilde{J}(L,L,\eta)(z(L)-p(L)H(L)\alpha(L)O(L-1)),\tilde{O}(0,\eta)=0 \tag{24}$$

$$\begin{aligned} \tilde{J}(L,L,\eta)=&\left[\tilde{\beta}^T(L,\eta)H^T(L)p(L)-\tilde{r}(L-1,\eta)\zeta^T(L)H^T(L)p(L)\right] \\ &\left[R(L)+p(L)H(L)(F_m(L,L)-p(L)\alpha(L)r(L-1)\zeta^T(L))H^T(L)\right]^{-1} \end{aligned} \tag{25}$$

$$O(L)=O(L-1)+J(L,L)(z(L)-p(L)H(L)\alpha(L)O(L-1)),\quad O(0)=0 \tag{26}$$

Also, the recursive algorithm for the optimal impulse response function $h(k,l,L,\eta)$ in the fixed-point smoothing problem consists of (27), (28) with (18), (19), (21), (22), (23) and (25).

$$h(k,l,L,\eta) = h(k,l,L-1,\eta) - h(k,L,L,\eta)p(L)H(L)\alpha(L)J(l,L-1) \tag{27}$$

Initial condition of $h(k,l,L,\eta)$ at $l=L$ is $h(k,L,L,\eta)$. $h(k,L,L,\eta)$ is calculated by (18) together with (19), (21), (22), (23) and (25).

$$J(l,L) = J(l,L-1) - J(L,L)p(L)H(L)\alpha(L)J(l,L-1) \tag{28}$$

Initial condition of $J(l,L)$ at $l=L$ is $J(L,L)$. $J(L,L)$ is calculated by (23) together with (22).

The recursive algorithm for the optimal impulse response function $h(L,l,L,\eta)$ in the filtering problem consists of (29), (30), (28) with (21), (22), (23) and (25).

$$h(L,l,L,\eta) = \tilde{\alpha}(L,\eta)\tilde{J}(L,l,\eta) \tag{29}$$

$$\tilde{J}(L,l,\eta) = \tilde{J}(L-1,l,\eta) - \tilde{J}(L,L,\eta)p(L)H(L)\alpha(L)J(l,L-1) \tag{30}$$

Initial condition of $\tilde{J}(L,l,\eta)$ at $l=L$ is $\tilde{J}(L,L,\eta)$. $\tilde{J}(L,L,\eta)$ is calculated by (25) together with (21), (22) and (23).

The proof of Theorem 1 is given in Appendix A.

4. Filtering and fixed-point smoothing error covariance functions

Let us derive equations for filtering and fixed-point smoothing error covariance functions. The filtering error covariance function is defined by

$$\tilde{P}(k,\eta,l,\xi) = E\left[(u(k,\eta) - \hat{u}(k,k,\eta))(u(l,\xi) - \hat{u}(l,l,\xi))^T\right]. \tag{31}$$

From an orthogonal projection lemma that the filtering error $u(k,\eta) - \hat{u}(k,k,\eta)$ is orthogonal to $\hat{u}(l,l,\xi)$, $l \le k$, and (12), we have

$$\tilde{P}(k,\eta,l,\xi)=K(k,\eta,l,\xi)-E\left[\hat{u}(k,k,\eta)u^{T}(l,\xi)\right]. \tag{32}$$

Using (3), (6) and $K_m(k,\eta,l)=E\left[u(k,\eta)u_m^T(l)\right]$, we can rewrite (32) as

$$\tilde{P}(k,\eta,l,\xi)=K(k,\eta,l,\xi)-\sum_{l'=1}^{k}p(l')h(k,l',k,\eta)H(l')K_m^T(l,\xi,l'). \tag{33}$$

Substituting (29) into (33) and introducing a function

$$\Psi(k,\eta,l,\xi)=\sum_{l'=1}^{k}p(l')\tilde{J}(k,l',\eta)H(l')K_m^T(l,\xi,l'), \tag{34}$$

we rewrite (33) as

$$\tilde{P}(k,\eta,l,\xi)=K(k,\eta,l,\xi)-\tilde{\alpha}(k,\eta)\Psi(k,\eta,l,\xi). \tag{35}$$

Subtracting $\Psi(k,\eta,l,\xi)$ from the equation obtained by putting $k\to k+1$ in (34), we have (36).

$$\Psi(k+1,\eta,l,\xi)-\Psi(k,\eta,l,\xi)=p(k+1)\tilde{J}(k+1,k+1,\eta)H(k+1)K_m^T(l,\xi,k+1)+\\ \sum_{l'=1}^{k}p(l')(\tilde{J}(k+1,l',\eta)-\tilde{J}(k,l',\eta))H(l')K_m^T(l,\xi,l') \tag{36}$$

Substituting (30) into (36) and introducing a function

$$T(k,l,\xi)=\sum_{l'=1}^{k}p(l')J(l',k)H(l')K_m^T(l,\xi,l'), \tag{37}$$

we rewrite (36) as (38).

$$\Psi(k+1,\eta,l,\xi)=\Psi(k,\eta,l,\xi)+p(k+1)\tilde{J}(k+1,k+1,\eta)H(k+1)(K_m^T(l,\xi,k+1)\\ -\alpha(k+1)T(k,l,\xi)) \tag{38}$$

Initial condition of $\Psi(k,\eta,l,\xi)$ at k=0 is $\Psi(0,\eta,l,\xi)=0$ from (34).

Subtracting $T(k,l,\xi)$ from the equation obtained by putting $k\to k+1$ in (37), we have (39).

$$T(k+1,l,\xi)-T(k,l,\xi)=p(k+1)J(k+1,k+1)H(k+1)K_m^T(l,\xi,k+1)+$$
$$\sum_{l'=1}^{k}p(l')(J(l',k+1)-J(l',k))H(l')K_m^T(l,\xi,l') \tag{39}$$

Substituting (28) into (39) and using (37), we obtain (40).

$$T(k+1,l,\xi)=T(k,l,\xi)+p(k+1)J(k+1,k+1)H(k+1)(K_m^T(l,\xi,k+1)-\alpha(k+1)T(k,l,\xi)) \tag{40}$$

Initial condition of $T(k,l,\xi)$ at k=0 is $T(0,l,\xi)=0$ from (37).

Hence, the filtering error covariance function $\tilde{P}(k,\eta,l,\xi)$ is calculated by (35), (38) and (40) with (21), (22), (23),(25).

Now, the fixed-point smoothing error covariance function is defined by

$$\tilde{P}_F(k,\eta,l,\xi,L)=E[(u(k,\eta)-\hat{u}(k,L,\eta))(u(l,\xi)-\hat{u}(l,L,\xi))^T]. \tag{41}$$

From an orthogonal projection lemma that the filtering error $u(k,\eta)-\hat{u}(k,L,\eta)$ is orthogonal to $\hat{u}(l,L,\xi)$ and (12), we have

$$\tilde{P}_F(k,\eta,l,\xi,L)=K(k,\eta,l,\xi)-E[\hat{u}(k,L,\eta)u^T(l,\xi)]. \tag{42}$$

Using (3), (6) and $K_m(k,\eta,l)=E[u(k,\eta)u_m^T(l)]$, we can rewrite (42) as

$$\tilde{P}_F(k,\eta,l,\xi,L)=K(k,\eta,l,\xi)-\sum_{l'=1}^{k}p(l')h(k,l',L,\eta)H(l')K_m^T(l,\xi,l'). \tag{43}$$

Introducing

$$V(k,\eta,l,\xi,L)=\sum_{l'=1}^{L}p(l')h(k,l',L,\eta)H(l')K_m^T(l,\xi,l'), \tag{44}$$

we rewrite (43) as

$$\tilde{P}_F(k,\eta,l,\xi,L)=K(k,\eta,l,\xi)-V(k,\eta,l,\xi,L). \tag{45}$$

Subtracting $V(k,\eta,l,\xi,L)$ from the equation obtained by putting $L\rightarrow L+1$ in (44), we have (46).

$$V(k,\eta,l,\xi,L+1)-V(k,\eta,l,\xi,L)=p(L+1)h(k,L+1,L+1,\eta)H(L+1)K_m^T(l,\xi,L+1)$$
$$+\sum_{l'=1}^{L}p(l')(h(k,l',L+1,\eta)-h(k,l',L,\eta))H(l')K_m^T(l,\xi,l') \tag{46}$$

Substituting (27) into (46) and introducing a function

$$Q(l,\xi,L)=\sum_{l'=1}^{L}p(l')J(l',L)H(l')K_m^T(l,\xi,l'), \tag{48}$$

we obtain (49).

$$\begin{aligned}&V(k,\eta,l,\xi,L+1)=V(k,\eta,l,\xi,L)+\\&p(L+1)h(k,L+1,L+1,\eta)H(L+1)(K_m^T(l,\xi,L+1)-\alpha(L+1)Q(l,\xi,L))\end{aligned} \tag{49}$$

Initial condition of $V(k,\eta,l,\xi,L)$ at L=0 is $V(k,\eta,l,\xi,0)=0$ from (44).

Subtracting $Q(l,\xi,L)$ from the equation obtained by putting $L\to L+1$ in (48), we have (50).

$$Q(l,\xi,L+1)-Q(l,\xi,L)=p(L+1)J(L+1,L+1)H(L+1)K_m^T(l,\xi,L+1)$$
$$+\sum_{l'=1}^{L}p(l')(J(l',L+1)-J(l',L))H(l')K_m^T(l,\xi,l') \tag{50}$$

Substituting (28) into (50) and using (48), we obtain (51).

$$\begin{aligned}&Q(l,\xi,L+1)=Q(l,\xi,L)+p(L+1)J(L+1,L+1)H(L+1)(K_m^T(l,\xi,L+1)\\&-\alpha(L+1)Q(l,\xi,L))\end{aligned} \tag{51}$$

Initial condition of $Q(l,\xi,L)$ at L=0, $Q(l,\xi,0)=0$, is from (48).

The fixed-point smoothing error covariance function $\tilde{P}_F(k,\eta,l,\xi,L)$ is calculated by (45), (49) and (51) with (18), (19), (21), (22), (23), (25).

From Theorem 1, the condition that the filtering and fixed-point smoothing estimates exist is given by

$$R(L)+p(L)H(L)(F_m(L,L)-p(L)\alpha(L)r(L-1)\zeta^T(L))H^T(L)>0. \tag{52}$$

It might be seen that (52) expresses the autovariance function of the innovations process $z(L)$-$p(L)H(L)\alpha(L)O(L$-1). From this, it is deduced that $p(L)H(L)(F_m(L,L)$-$p(L)\alpha(L)r(L$-

$1)\zeta^T(L))H^T(L)$ is always a positive semi-definite matrix. Hence, the condition (52) is assured if $R(L)$ is a positive definite matrix.

5. A numerical simulation example

Let the autocovariance function for the signal $u(k,\eta)$ be given by

$$K(k,\eta,l,\xi)=\sigma^2 e^{-\theta_1|k-l|-\theta_2|\eta-\xi|}. \tag{53}$$

According to Habibi [6], the image field with the separable autocovariance function can be modeled by

$$\begin{aligned} &u(k+1,\eta+1)=A_1u(k,\eta+1)+A_2u(k+1,\eta)-A_1A_2u(k,\eta)+\sqrt{(1-A_1^2)(1-A_2^2)}w(k,\eta),\\ &A_1=e^{-\theta_1}, A_2=e^{-\theta_2}, E[w(k,\eta)w(l,\xi)]=\sigma^2\delta_K(k-l)\delta_K(\eta-\xi). \end{aligned} \tag{54}$$

From (12), we have

$$\begin{aligned} &\alpha(k,\eta,\xi)=\sigma^2 e^{-\theta_1 k}, \beta(l,\eta,\xi)=e^{\theta_1 l-\theta_2|\eta-\xi|},\\ &\gamma(k,\eta,\xi)=\sigma^2 e^{\theta_1 k}, \zeta(l,\eta,\xi)=e^{-\theta_1 l-\theta_2|\eta-\xi|}. \end{aligned} \tag{55}$$

The signal is observed at an observation point η^1.

$$z(k)=U(k)u(k,\eta^1)+v(k) \tag{56}$$

We find that $\tilde{\alpha}(L,\eta)$, $\alpha(L)$, $\tilde{\gamma}(k,\eta)$ and $\gamma(L)$ are not affected by η. $\tilde{\beta}(L,\eta)$, $\beta(L)$, $\tilde{\zeta}(L,\eta)$ and are $\zeta(L)$ are not affected by η for the case of $\eta=\eta^1$, where the observation point $\eta=\eta^1$ varies according to η. Hence, $\tilde{\alpha}(L,\eta^1)=\sigma^2 e^{-\theta_1 L}(=\alpha(L))$, $\tilde{\gamma}(k,\eta^1)=\sigma^2 e^{\theta_1 k}$, $\gamma(L)=\sigma^2 e^{\theta_1 L}$, $\tilde{\beta}(L,\eta^1)=e^{\theta_1 L}$, $\beta(L)=e^{\theta_1 L}$, $\tilde{\zeta}(L,\eta^1)=e^{-\theta_1 L}$ and $\zeta(L)=e^{-\theta_1 L}$. Taking into account these relationships, we calculate the linear least-squares filtering and fixed-point smoothing estimates of $u(k,\eta)$ by substituting the covariance information of signal and noise with the uncertain probability into Theorem 1. Figure 1 displays the original image 'Lena.tif'. The picture has 512×512 pixels and it has 256 gray levels. Parameters in the semi-causal model of (54) are calculated as A_1= 0.9984, A_2= 0.9979, σ^2= 19798, θ_1= 0.0016 and θ_2= 0.0021. Also, the uncertain probability is set to $p(k)$ = 0.96. Figure 2 displays the degraded observed image for white Gaussian observation noise $N(0,30^2)$ (S/N ratio is 26.8474 [dB]), where noise variance is 30^2 with the mean 0. In Fig.2 the probability that the observed value includes the signal is set to $p(k)(=0.96)$. Fig. 3 illustrates the restored image by the filter in Theorem 1. Table 1 shows the mean-square values (MSVs) of the filtering and fixed-point smoothing errors by the estimators in Theorem 1. The MSVs of the filtering and fixed-point smoothing errors are calculated by $\sum_{i=1}^{511}\sum_{j=1}^{490}(u(i,j)-\hat{u}(i,i,j))^2/250390$ and $\sum_{i=1}^{511}\sum_{j=1}^{490}\sum_{k=1}^{20}(u(i,j)-\hat{u}(i,i+k,j))^2/5007800$. The MSV becomes large with an increase in the variance of white Gaussian observation noise both for the filter and the smoother. The estimation accuracy of the filter is better than the fixed-point smoother.

In the simulation example, MATLAB [7],[8] with its image processing toolbox was used.

6. Conclusions

In this paper, new algorithms for the filtering and fixed-point smoothing estimates are proposed with the uncertain observations contaminated with white observation noise in linear discrete-time distributed parameter systems. The proposed filter and fixed-point smoother use the covariance information of the signal and observation noise and the uncertain probability without requiring the information of the state-space model for the signal.

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APPENDIX A. Proof of Theorem 1.

A Cauchy system for the RLS filtering and fixed-point smoothing estimates are obtained by using an invariant imbedding method [9].

In (11) let us put $W(l)$ as

$$W(l)=R(l)+p(l)H(l)F_m(l,l)H^T(l)-p(l)H(l)F_m(l,l)H^T(l)p(l). \tag{A.1}$$

Then (11) is written as

$$h(k,l,L,\eta)W(l)=p(l)K_m(k,\eta,l)H^T(l)$$
$$-\sum_{l'=1}^{L}h(k,l',L,\eta)p(l')H(l')F_m(l',l)H^T(l)p(l). \tag{A.2}$$

Subtracting the equation obtained by putting $L\rightarrow L-1$ from (A.2), we have (A.3).

$$(h(k,l,L,\eta)-h(k,l,L-1,\eta))W(l)=-h(k,L,L,\eta)p(L)H(L)F_m(L,l)H^T(l)p(l)$$
$$-\sum_{l'=1}^{L-1}(h(k,l',L,\eta)-h(k,l',L-1,\eta))p(l')H(l')F_m(l',l)H^T(l)p(l) \tag{A.3}$$

Let us introduce a function $\Lambda(l,L-1)$ which satisfies

$$\Lambda(l,L-1)W(l)=F_m(L,l)H^T(l)p(l)-\sum_{l'=1}^{L-1}\Lambda(l',L-1)p(l')H(l')F_m(l',l)H^T(l)p(l). \tag{A.4}$$

From (A.2) and (A.3), we have

$$h(k,l,L,\eta)-h(k,l,L-1,\eta)=-h(k,L,L,\eta)p(L)H(L)\Lambda(l,L-1). \tag{A.5}$$

By introducing an auxiliary function $J(l,L-1)$, which satisfies

$$J(l,L-1)W(l)=\beta^T(l)H^T(l)p(l)-\sum_{l'=1}^{L-1}J(l',L-1)p(l')H(l')F_m(l',l)H^T(l)p(l), \tag{A.6}$$

and by noting the semi-degenerate property of (15) for $F_m(L,l)$ in (A.4), we obtain

$$\Lambda(l, L-1) = \alpha(L)J(l,L-1). \tag{A.7}$$

(27) is clear from (A.5) and (A.7).

Subtracting the equation obtained by putting $L \to L-1$ in (A.6) from (A.6), we have (A.8).

$$\begin{aligned}&(J(l,L-1)-J(l,L-2))W(l)=\\&-J(L-1,L-1)p(L-1)H(L-1)F_m(L-1,l)H^T(l)p(l)\\&-\sum_{l'=1}^{L-2}(J(l',L-1)-J(l',L-2))p(l')H(l')F_m(l',l)H^T(l)p(l)\end{aligned} \tag{A.8}$$

From (A.4) and (A.8), we obtain (A.9).

$$J(l,L-1)-J(l,L-2) = -J(L-1,L-1)p(L-1)H(L-1)\Lambda(l,L-2) \tag{A.9}$$

(28) is clear from (A.7) and (A.9).

If we put $l \to L-1$ in (A.6), we have (A.10).

$$\begin{aligned}&J(L-1,L-1)W(L-1)=\beta^T(L-1)H^T(L-1)p(L-1)-\\&\sum_{l'=1}^{L-1}J(l',L-1)p(l')H(l')F_m(l',L-1)H^T(L-1)p(L-1)\end{aligned} \tag{A.10}$$

Using the semi-degenerate kernel property of (15) and introducing a function

$$\gamma(L-1)=\sum_{l'=1}^{L-1}J(l',L-1)p(l')H(l')\gamma(l'), \tag{A.11}$$

we obtain (A.12).

$$\begin{aligned}&J(L-1,L-1)W(L-1)=\beta^T(L-1)H^T(L-1)p(L-1)-\\&r(L-1)\zeta^T(L-1)H^T(L-1)p(L-1)\end{aligned} \tag{A.12}$$

Subtracting the equation obtained by putting $L \to L-1$ in (A.11) from (A.11), we have (A.13).

$$\begin{aligned}&r(L-1)-r(L-2)=J(L-1,L-1)p(L-1)H(L-1)\gamma(L-1)+\\&\sum_{l'=1}^{L-2}(J(l',L-1)-J(l',L-2))p(l')H(l')\gamma(l')\end{aligned} \tag{A.13}$$

Substituting (A.7) and (A.9) into (A.13), and using (A11), we obtain (22). The initial condition $r(0)=0$ for updating $r(L)$ by (22) is clear from (A.11). From (A.12) and (22), we obtain (23) after some manipulations.

Putting $l=L$ in (A.2), we have (A.14).

$$h(k,L,L,\eta)W(L)=p(L)K_m(k,\eta,L)H^T(L) - \sum_{l'=1}^{L} h(k,l',L,\eta)p(l')H(l')F_m(l',L)H^T(L)p(L) \tag{A.14}$$

Using the semi-degenerate kernel properties of (13) and (15) and introducing a function

$$S(k,L,\eta)=\sum_{l'=1}^{L} h(k,l',L,\eta)p(l')H(l')\gamma(l'), \tag{A.15}$$

we obtain (A.16).

$$h(k,L,L,\eta)W(L)=p(L)\tilde{\gamma}(k,\eta)\tilde{\zeta}^T(L,\eta)H^T(L)-S(k,L,\eta)\zeta^T(L)H^T(L)p(L) \tag{A.16}$$

Subtracting the equation obtained by putting $L\to L-1$ in (A.15) from (A.15), we have (A.17).

$$S(k,L,\eta)-S(k,L-1,\eta)=h(k,L,L,\eta)p(L)H(L)\gamma(L) + \sum_{l'=1}^{L-1}(h(k,l',L,\eta)-h(k,l',L-1,\eta))p(l')H(l')\gamma(l') \tag{A.17}$$

Substituting (A.5) with (A.7) into (A.17) and using (A.11), we obtain (19). From (A.16) and (19), we obtain (18) after some manipulations.

If we put $k=L$ in (A.2), we have (A.18).

$$h(L,l,L,\eta)W(l)=p(l)K_m(L,\eta,l)H^T(l) - \sum_{l'=1}^{L} h(L,l',L,\eta)p(l')H(l')F_m(l',l)H^T(l)p(l) \tag{A.18}$$

Substituting $K_m(L,\eta,l)=\tilde{\alpha}(L,\eta)\tilde{\beta}^T(l,\eta),\ 0\le l\le L$, from (13) into (A.18) and introducing an auxiliary function $\tilde{J}(L,l,\eta)$, which satisfies

$$\tilde{J}(L,l,\eta)W(l) = p(l)\tilde{\beta}^T(l,\eta)H^T(l) - \sum_{l'=1}^{L} \tilde{J}(L,l',\eta)p(l')H(l')F_m(l',l)H^T(l)p(l), \tag{A.19}$$

we obtain (A.20).

$$h(L,l,L,\eta) = \tilde{\alpha}(L,\eta)\tilde{J}(L,l,\eta) \tag{A.20}$$

Subtracting the equation obtained by putting $L \to L-1$ in (A.19) from (A.19), we have (A.21).

$$\begin{aligned}(\tilde{J}(L,l,\eta) - \tilde{J}(L-1,l,\eta))W(l) &= -\tilde{J}(L,L,\eta)p(L)H(L)F_m(L,l)H^T(l)p(l) \\ &-\sum_{l'=1}^{L-1} (\tilde{J}(L,l',\eta) - \tilde{J}(L-1,l',\eta))p(l')H(l')F_m(l',l)H^T(l)p(l)\end{aligned} \tag{A.21}$$

Substituting $F_m(L,l)=\alpha(L)\beta^T(l)$, $0 \le l \le L$, from (15) into (A.21) and using (A.6), we obtain (30).

Putting $l = L$ in (A.19), we have (A.22).

$$\begin{aligned}\tilde{J}(L,L,\eta)W(l) &= p(L)\tilde{\beta}^T(L,\eta)H^T(L) - \\ &\sum_{l'=1}^{L} \tilde{J}(L,l',\eta)p(l')H(l')F_m(l',L)H^T(L)p(L)\end{aligned} \tag{A.22}$$

Substituting $F_m(l',L)=\gamma(l')\zeta^T(L)$, $0 \le l' \le L$, from (15) into (A.22) and introducing a function

$$\tilde{r}(L,\eta) = \sum_{l'=1}^{L} \tilde{J}(L,l',\eta)p(l')H(l')\gamma(l'), \tag{A.23}$$

we obtain (A.24).

$$\tilde{J}(L,L,\eta)W(L) = \tilde{\beta}^T(L,\eta)H^T(L)p(L) - \tilde{r}(L,\eta)\zeta^T(L)H^T(L)p(L) \tag{A.24}$$

Subtracting the equation obtained by putting $L \to L-1$ in (A.23) from (A.23), we have (A.25).

$$\tilde{r}(L,\eta)-\tilde{r}(L-1,\eta)=\tilde{J}(L,L,\eta)p(L)H(L)\gamma(L)+\sum_{l'=1}^{L-1}(\tilde{J}(L,l',\eta)-\tilde{J}(L-1,l',\eta))p(l')H(l')\gamma(l') \tag{A.25}$$

Substituting (30) previously derived into (A.25) and using (A.11), we obtain (21). From (A.24) and (21), we obtain (25) after some manipulations. The initial condition $\tilde{r}(0,\eta)=0$ for updating $\tilde{r}(L,\eta)$ by (21) is clear from (A.23).

Substituting (19) into (A.16), we obtain (18) after some manipulations.

If we put $k=L$ in (A.15), we have (A.26).

$$S(L,L,\eta)=\sum_{l'=1}^{L}h(L,l',L,\eta)p(l')H(l')\gamma(l') \tag{A.26}$$

Substituting (A.20) into (A.26), we have (A.27).

$$S(L,L,\eta)=\sum_{l'=1}^{L}\tilde{\alpha}(L,\eta)\tilde{J}(L,l',\eta)p(l')H(l')\gamma(l') \tag{A.27}$$

From (A.27) with (A.23), we obtain the initial condition in (19) for $S(k,L,\eta)$.

From (6) the filtering estimate $\hat{u}(L,L,\eta)$ of the signal $u(L,\eta)$ is written as (A.28).

$$\hat{u}(L,L,\eta)=\sum_{l'=1}^{L}h(L,l',L,\eta)z(l') \tag{A.28}$$

Substituting (A.20) into (A.28), we have (A.29).

$$\hat{u}(L,L,\eta)=\sum_{l'=1}^{L}\tilde{\alpha}(L,\eta)\tilde{J}(L,l',\eta)z(l') \tag{A.29}$$

Introducing a function

$$\tilde{O}(L,\eta)=\sum_{l'=1}^{L}\tilde{J}(L,l',\eta)z(l'), \tag{A.30}$$

we obtain (20) for the filtering estimate $\hat{u}(L,L,\eta)$.

Subtracting the equation obtained by putting $L\to L-1$ in (A.29) from (A.29), we have (A.31).

$$\tilde{O}(L,\eta)-\tilde{O}(L-1,\eta)=\tilde{J}(L,L,\eta)z(L)+\sum_{l'=1}^{L-1}(\tilde{J}(L,l',\eta)-\tilde{J}(L-1,l',\eta))z(l') \tag{A.31}$$

Substituting (30) previously obtained into (A.31) and using (A.30), we have (A.32).

$$\tilde{O}(L,\eta)-\tilde{O}(L-1,\eta)=\tilde{J}(L,L,\eta)z(L)-\sum_{l'=1}^{L-1}\tilde{J}(L,L,\eta)p(L)H(L)\alpha(L)J(l',L-1)z(l') \tag{A.32}$$

Introducing a function

$$O(L)=\sum_{l'=1}^{L}J(l',L)z(l'), \tag{A.33}$$

we obtain (24). The initial condition $\tilde{O}(0,\eta)=0$ for updating $\tilde{O}(L,\eta)$ by (24) is clear from (A.30).

Subtracting the equation obtained by putting $L \to L-1$ in (A.33) from (A.33), we have (A.34).

$$O(L)-O(l-1)=J(L,L)z(L)+\sum_{l'=1}^{L-1}(J(l',L)-J(l',L-1))z(l') \tag{A.34}$$

Substituting (28) into (A.34) and using (A.33), we obtain (26). The initial condition $O(0)=0$ for updating $O(L)$ by (26) is clear from (A.33).

The fixed-point smoothing estimate $\hat{u}(k,L,\eta)$ of $u(k,L)$ is formulated by (6). Subtracting the equation obtained by putting $L \to L-1$ in (6) from (6), we have (A.35).

$$\hat{u}(k,L,\eta)-\hat{u}(k,L-1,\eta)=h(k,L,L,\eta)z(L)+\sum_{l'=1}^{L-1}(h(k,l',L,\eta)-h(k,l',L-1,\eta))z(l') \tag{A.35}$$

Substituting (27) into (A.35) and using (A.33), we obtain (17) (Q.E.D.).

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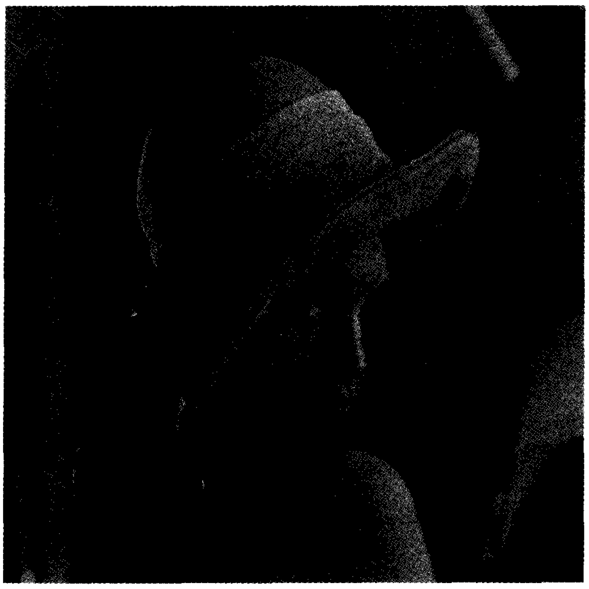

Fig.1 Original image, "Lena.tif".

Fig.2 Observed degraded image for the probability $p(k)$(=0.96) and S/N=26.8474 [dB].

Fig.3 Restored image by the proposed filter for S/N=26.8474 [dB].

Table 1 MSVs of the filtering and fixed-point smoothing errors.

S/N[dB]	MSV of filtering error	MSV of fixed-point smoothing error
45.9322	311.0684	440.1181
26.8474	447.3113	836.6552
21.8498	537.9950	1127.80
17.9734	631.0103	1452.0