HIROSHIMA MATH. J.
6 (1976), 227–255

The Enumeration of Liftings in Fibrations and the Embedding Problem I

Tsutomu YASUI

(Received December 23, 1975)

Introduction

As for the enumeration problem of embeddings of manifolds, many results have been obtained up to the present (e.g. [2], [5], [6], [7], [20] and [21]) but they are small in number compared with those of the existence problem. In this paper, we try one approach to the enumeration problem of embeddings of n-dimensional differentiable manifolds into the real $(2n-1)$-space R^{2n-1}. As an application, we determine the cardinality of the set of isotopy classes of embeddings of the n-dimensional real projective space RP^n into R^{2n-1}.

Our plan is as follows. An embedding $f: M \to R^m$ of a space M into R^m induces a Z_2-equivariant map $F: M \times M - \Delta \to S^{m-1}$ by $F(x, y) = \dfrac{f(x)-f(y)}{\|f(x)-f(y)\|}$ for distinct points x, y of M, where Δ is the diagonal of M and the Z_2-actions on $M \times M - \Delta$ and S^{m-1} are the interchange of the factors and the antipodal action, respectively. Consider the correspondence which associates with an isotopy class of an embedding $f: M \to R^m$ the equivariant homotopy class of the map F made above. Then this correspondence is surjective if $2m \geq 3(n+1)$ and bijective if $2m > 3(n+1)$ for any n-dimensional compact differentiable manifold M by the theorem of A. Haefliger [5, § 1]. On the other hand, there is a one-to-one correspondence between the set of the equivariant homotopy classes of equivariant maps of $M \times M - \Delta$ to S^{m-1} and the set of homotopy classes of cross sections of the sphere bundle $S^{m-1} \to (M \times M - \Delta) \times_{Z_2} S^{m-1} \to (M \times M - \Delta)/Z_2$, where the reduced symmetric product $M^* = (M \times M - \Delta)/Z_2$ of M has the homotopy type of a CW-complex X of dimension less than $2n$ ($n = \dim M$). Therefore, the enumeration problem of embeddings of an n-dimensional manifold M into R^m arrives at the enumeration problem of cross sections of an S^{m-1}-bundle ξ over a CW-complex X of dimension less than $2n$.

Now, consider the case that $m = 2n-1$, and let $p: BO(m-1) \to BO(m)$ be the universal S^{m-1}-bundle. Then the enumeration of cross sections of an S^{m-1}-bundle ξ over X is equivalent to the enumeration of liftings of the classifying map $\xi: X \to BO(m)$ of ξ to $BO(m-1)$. We construct the third stage Postnikov factorization

$$(*)\qquad \begin{tikzcd} BO(m-1) \arrow[r, "q_2"] \arrow[dr, "q_1"] \arrow[ddr, "p"'] & T \arrow[d, "p_2"] \\ & E \arrow[d, "p_1"] \\ & BO(m) \end{tikzcd}$$

of p. Here p_1 is the twisted principal fibration, p_2 is the principal fibration and q_2 is an $(m+1)$-equivalence. Since the dimension of X is less than $m+1$, the enumeration of liftings of ξ to $BO(m-1)$ is equivalent to the enumeration of liftings to T by the theorem of I. M. James and E. Thomas [11, Theorem 3.2].

From the above considerations, this paper is divided into three chapters.

In Chapter I, we study the enumeration problem of liftings of a map into the base space of a certain fibration to the total space. In §1, the twisted principal fibration is defined and the enumeration of liftings for this fibration is treated. Further, we are concerned with the composition of two twisted principal fibrations $T \xrightarrow{q} E \xrightarrow{p} D$ under the assumption that it is stable (see §2). We describe the set of homotopy classes of liftings of a map $u: X \to D$ to the composition $pq: T \to D$ in Theorem A of §2, which is a generalization of the theorem of I. M. James and E. Thomas [12, Theorem 2.2] for principal fibrations. After preparing several propositions for the composition pq in §§3–4 without assuming the stability, Theorem A is proved in §5.

The purpose of Chapter II is to study the enumeration problem of cross sections of sphere bundles. In §6, we notice the cohomology $H^*(X; \underline{Z})$ with coefficients in the local system defined by $\phi: \pi_1(X) \to \mathrm{Aut}(Z)$. In §7, the third stage Postnikov factorization $(*)$ of $p: BO(n-1) \to BO(n)$ is constructed, and we show in §8 that the composition of fibrations $p_1 p_2: T \to BO(n)$ is stable in the sense of §2. From Theorem A and the fact that $q_2: BO(n-1) \to T$ is an $(n+1)$-equivalence, we have the following theorem in §9.

THEOREM B. *Let ξ be a real n-plane bundle over a CW-complex X of dimension less than $n+1$ and let $n \geq 4$. If ξ has a non-zero cross section, then the set cross (ξ) of homotopy classes of non-zero cross sections of ξ is given, as a set, by*

$$cross(\xi) = H^{n-1}(X; Z) \times \mathrm{Coker}\,\Theta,$$

where the homomorphism

$$\Theta: H^{n-2}(X; \underline{Z}) \longrightarrow H^n(X; Z_2)$$

is defined by

$$\Theta(a) = (\rho_2 a) w_2(\xi) + Sq^2 \rho_2 a \qquad \textit{for} \quad a \in H^{n-2}(X; \underline{Z}),$$

ρ_2 is the mod 2 *reduction, $\underline{Z}$ is the local system on X associated with ξ and $w_2(\xi)$ is the second Stiefel-Whitney class of ξ.*

Chapter III is devoted to an application of A. Haefliger's theorem and Theorem B on the enumeration problem of embeddings of n-dimensional manifolds into R^{2n-1}. In § 10, the set $[M \subset R^{2n-1}]$ of isotopy classes of embeddings of n-dimensional closed differentiable manifolds M into R^{2n-1} is described with the cohomology of M^*. As an application for the n-dimensional real projective space RP^n, we calculate the cohomology group $H^{2n-2}((RP^n)^*; \underline{Z})$ and the homomorphism $\Theta: H^{2n-3}((RP^n)^*; Z) \to H^{2n-1}((RP^n)^*; Z_2)$, and we have the following theorem in §§ 11–12.

THEOREM C. *Let $n \neq 2^r$ and $n \geq 6$. Then the n-dimensional real projective space RP^n is embedded in the real $(2n-1)$-space R^{2n-1}, and there are just four and two isotopy classes of embeddings of RP^n into R^{2n-1} for $n \equiv 3(4)$ and $n \not\equiv 3(4)$, respectively.*

Chapter I. Enumeration of liftings in certain fibrations

§ 1. Twisted principal fibrations

Let Z be a given space. By a *Z-space* $X = (X, f)$, we mean a space X together with a (continuous) map $f: X \to Z$. For two Z-spaces $X = (X, f)$ and $Y = (Y, g)$, the pull back

$$X \times_Z Y = \{(x, y) \mid f(x) = g(y)\} \qquad (\subset X \times Y)$$

of f and g is a Z-space with $(f, g): X \times_Z Y \to Z$, $(f, g)(x, y) = f(x) = g(y)$. A map $h: X \to Y$ is called a *Z-map* if $gh = f$, and a homotopy $h_t: X \to Y$ is called a *Z-homotopy* if $gh_t = f$ for all t. In this case, we say that h_0 is *Z-homotopic* to h_1 and denote by $h_0 \simeq_Z h_1$. Further,

$$[X, Y]_Z$$

denotes the set of all Z-homotopy classes of Z-maps of X to Y.

Now, let B be a space (with base point $*$) and π be a discrete group, and assume that π acts on B preserving the base point by a homomorphism $\phi: \pi \to \text{Homeo}(B, *)$. Then, considering the Eilenberg-MacLane space $K = K(\pi, 1)$, the universal covering $\tilde{K} \to K$ and the usual action of π on $\tilde{K}$, we have the fiber bundle

$$B \longrightarrow L_\phi(B) = \tilde{K} \times_\pi B \xrightarrow{q} K = K(\pi, 1) \tag{1.1}$$

with structure group π. Since $\tilde{K} \times_\pi * = K$, we have the canonical cross section

$s: K \to \tilde{K} \times_\pi B$ such that $s(K) = K = \tilde{K} \times_\pi *$.

In this paper, we consider the following situation.

(1.2) *Let π act on an H-group**) *B by ϕ satisfying the following assumptions: The multiplication $\mu: B \times B \to B$ and the homotopy inverse $\nu: B \to B$ of B are π-equivariant and there are π-equivariant homotopies*

$$\mu(1_B, c) \simeq 1_B \simeq \mu(c, 1_B),\ \mu(\mu \times 1_B) \simeq \mu(1_B \times \mu) \text{ and } \mu(\nu, 1_B) \simeq c \simeq \mu(1_B, \nu),$$

where $c: B \to B$ is the constant map to $*$. Also, if B is homotopy abelian, we assume in addition that there is a π-equivariant homotopy $\mu t \simeq \mu$, where $t: B \times B \to B \times B$ is the map defined by $t(x, y) = (y, x)$.

Then, for the K-space $(L_\phi(B), q)$ of (1.1), we can define K-maps

$$(1.3) \qquad \mu_\phi: L_\phi(B) \times_K L_\phi(B) \longrightarrow L_\phi(B), \qquad \nu_\phi: L_\phi(B) \longrightarrow L_\phi(B)$$

by

$$\mu_\phi([\tilde{x}, b], [\tilde{x}, b']) = [\tilde{x}, \mu(b, b')], \qquad \nu_\phi([\tilde{x}, b]) = [\tilde{x}, \nu(b)],$$

and there exist the following relations:

$$\mu_\phi(1 \times sq)\Delta \simeq_K 1 \simeq_K \mu_\phi(sq \times 1)\Delta: L_\phi(B) \longrightarrow L_\phi(B),$$

$$\mu_\phi(\mu_\phi \times 1) \simeq_K \mu_\phi(1 \times \mu_\phi): L_\phi(B) \times_K L_\phi(B) \times_K L_\phi(B) \longrightarrow L_\phi(B),$$

$$\mu_\phi(\nu_\phi \times 1)\Delta \simeq_K sq \simeq_K \mu_\phi(1 \times \nu_\phi)\Delta: L_\phi(B) \longrightarrow L_\phi(B),$$

and

$$\mu_\phi t \simeq_K \mu_\phi: L_\phi(B) \times_K L_\phi(B) \longrightarrow L_\phi(B),$$

if B is homotopy abelian, where Δ is the diagonal map and t is the map defined by $t(x, y) = (y, x)$.

Therefore we have the following

LEMMA 1.4. *Let X be a K-space with a map $u: X \to K$. Then the homotopy set $[X, L_\phi(B)]_K$ of K-maps is a group with unit $[su]$ by the multiplication*

$$[f] \cdot [g] = [\mu_\phi(f \times g)\Delta] \quad \textit{for K-maps } f, g: X \longrightarrow L_\phi(B).$$

If, furthermore, B is homotopy abelian, then this group $[X, L_\phi(B)]_K$ is abelian.

Let $p: E \to A$ be a fibration with fiber $F = p^{-1}(*)$, and assume that p admits a cross section $s: (A, *) \to (E, *)$. Then, we can consider the path spaces

*) The H-group is the homotopy associative H-space with a homotopy inverse.

$$P_AE = \{\lambda: I \longrightarrow E \mid \lambda(0) \in s(A),\ p\lambda(0) = p\lambda(t) \quad \textit{for all} \quad t \in I\},$$

$$\Omega_AE = \{\lambda \in P_AE \mid \lambda(0) = \lambda(1)\},$$

and we have the following well-known lemma.

LEMMA 1.5. *The projection*

$$r: P_AE \longrightarrow E, \qquad r(\lambda) = \lambda(1),$$

is a fibration with fiber ΩF. Furthermore,

$$pr: P_AE \longrightarrow A \quad \textit{and} \quad pr: \Omega_AE \longrightarrow A$$

are fibrations with fibers PF and ΩF, respectively, and they admit the canonical cross sections induced by s, where $PF = \{\lambda: I \to F \mid \lambda(0) = *\}$ and $\Omega F = \{\lambda \in PF \mid \lambda(0) = \lambda(1)\}$ are the ordinary path space and loop space of F.

By applying this lemma to the fibration $q: L_\phi(B) \to K$ of (1.1), we obtain the fibration

$$qr: \Omega_K L_\phi(B) \longrightarrow K, \qquad (qr)^{-1}(*) = \Omega B,$$

admitting the canonical cross section s. On the other hand, the given homomorphism $\phi: \pi \to \mathrm{Homeo}(B, *)$ induces the homomorphism

$$\phi': \pi \longrightarrow \mathrm{Homeo}(\Omega B, *), \qquad \phi'(g)(\lambda)(t) = \phi(g)(\lambda(t)).$$

This determines by (1.1) the fibration

$$q': L_{\phi'}(\Omega B) \longrightarrow K,$$

with fiber ΩB admitting the canonical cross section s', and we have the natural homeomorphism

$$\psi: L_{\phi'}(\Omega B) \xrightarrow{\approx} \Omega_K L_\phi(B), \qquad \psi([\tilde{x}, \lambda])(t) = [\tilde{x}, \lambda(t)],$$

which satisfies $qr\psi = q'$. Also, the loop space ΩB is a homotopy abelian H-group by the join $\vee$ of loops:

$$(\lambda_1 \vee \lambda_2)(t) = \begin{cases} \lambda_1(2t) & 0 \leq 2t \leq 1 \\ \lambda_2(2t-1) & 1 \leq 2t \leq 2, \end{cases}$$

and the action of π on ΩB by ϕ' satisfies (1.2). Therefore, Lemma 1.4 shows that the homotopy set $[X, L_{\phi'}(\Omega B)]_K$ of K-maps is an abelian group by the multiplication induced by $\vee_{\phi'}$. Furthermore, the above natural homeomorphism ψ commutes with $\vee_{\phi'}$ and the K-map

$$\vee : \Omega_K L_\phi(B) \times_K \Omega_K L_\phi(B) \longrightarrow \Omega_K L_\phi(B)$$

given by the join of loops, and we have the following

LEMMA 1.6. *The natural K-homeomorphism* $\psi: L_{\phi'}(\Omega B) \to \Omega_K L_\phi(B)$ *induces an isomorphism*

$$\psi_*: [X, L_{\phi'}(\Omega B)]_K \xrightarrow{\approx} [X, \Omega_K L_\phi(B)]_K$$

for any K-space X, where the domain is the abelian group of Lemma 1.4 *and the multiplication in the range is induced by* $\vee$ *mentioned above.*

Also, applying Lemma 1.5 to $q: L_\phi(B) \to K$ of (1.1), we obtain the fibration

$$r: P_K L_\phi(B) \longrightarrow L_\phi(B) \quad \text{with fiber } \Omega B.$$

Now, let $\theta: D \to L_\phi(B)$ be a given map. Then, from this fibration, θ induces a fibration

$$p: E = D \times_L P_K L_\phi(B) \longrightarrow D \ (L = L_\phi(B)) \quad \text{with fiber } \Omega B,$$

which is called *the twisted principal fibration* with classifying map θ.

Let $u: X \to D$ be a given map and consider the diagram

$$\begin{tikzcd} & E \arrow[d, "p"] & P_K L_\phi(B) \arrow[d, "r"] & \Omega_K L_\phi(B) \arrow[d, "qr"] \\ X \arrow[r, "u"] & D \arrow[r, "\theta"] & L_\phi(B) \arrow[r, "q"] & K. \end{tikzcd}$$

We define a D-map

$$(1.7) \qquad m: \Omega_K L_\phi(B) \times_K E \longrightarrow E$$

by the relation $m(\lambda_1, (x, \lambda_2)) = (x, \lambda_1 \vee \lambda_2)$, where $\vee$ is the join of paths, and the domain is the pull back of K-spaces $(\Omega_K L_\phi(B), qr)$ and $(E, q\theta p)$ and is understood as a D-space $(\Omega_K L_\phi(B) \times_K E, p\pi_2)$ (π_2 is the projection to the second factor in this paper). Hereafter, we often write $\lambda_1 \vee (x, \lambda_2)$ for $m(\lambda_1, (x, \lambda_2))$ simply. By considering a D-space $X = (X, u)$ as a K-space $(X, q\theta u)$, this map m induces a function

$$m_*: [X, \Omega_K L_\phi(B)]_K \times [X, E]_D \longrightarrow [X, E]_D.$$

PROPOSITION 1.8. *The function* m_* *mentioned above is an action of the abelian group* $[X, \Omega_K L_\phi(B)]_K$ *of* Lemma 1.6 *on the homotopy set* $[X, E]_D$. *If* $u: X \to D$ *has a lifting* $v: X \to E$, *that is, if there is a D-map* $v: (X, u) \to (E, p)$, *then the function* $m_*(\ , [v]): [X, \Omega_K L_\phi(B)]_K \to [X, E]_D$ *is a bijection.*

PROOF. This is a straightforward modification of the case that $p: E \to D$

is a usual principal fibration (cf. [12, Lemma 3.1]).

§2. The main result in Chapter I

Let B and C be H-groups with homomorphisms $\phi(B)\colon \pi(B)\to \mathrm{Homeo}(B, *)$ and $\phi(C)\colon \pi(C)\to \mathrm{Homeo}(C, *)$ such that they satisfy the assumption (1.2), and let

$$q_A\colon L(A) = L_{\phi(A)}(A) \longrightarrow K(A) = K(\pi(A), 1) \qquad (A = B, C)$$

be the fiber bundle of (1.1) with the canonical cross section s_A. Consider the following situation:

$$\text{(2.1)} \qquad \begin{tikzcd} & T \arrow[d, "q"'] & & \\ & E \arrow[r, "\rho"] \arrow[d, "p"'] & L(C) \arrow[r, "q_C"] & K(C) \\ X \arrow[r, "u"] & D \arrow[r, "\theta"] \arrow[urr, "\bar{\rho}"] & L(B) \arrow[r, "q_B"] & K(B). \end{tikzcd}$$

Here p is the twisted principal fibration with fiber ΩB induced from $P_{K(B)}L(B) \to L(B)$ by θ, q is the one with fiber ΩC induced from $P_{K(C)}L(C)\to L(C)$ by ρ, and it is assumed that

$$q_C\rho = \bar{\rho}p.^{*)}$$

For a given map $u\colon X\to D$, the homotopy set $[X, T]_D$ of D-maps of the D-space (X, u) to the D-space (T, pq) is the set of homotopy classes of liftings of u to T. The investigation of this set is our main purpose of Chapter I.

From now on, we assume that C is a topological group.**) For the simplicity,

$$n\colon L(C)\times_{K(C)} L(C) \longrightarrow L(C) \quad \text{and} \quad {}^{-1}\colon L(C) \longrightarrow L(C)$$

denote the $K(C)$-maps $\mu_{\phi(C)}$ and $\nu_{\phi(C)}$ of (1.3) induced from the multiplication and the inverse of C.

Let

$$\text{(2.2)} \qquad m_B\colon \Omega_{K(B)}L(B)\times_{K(B)} E \longrightarrow E$$

*) In our applications of the later chapters, we are concerned with the case where $K(C)=*$. For this case, $L(C)=C$ and q is a usual principal fibration and the existence of such a map $\bar{\rho}$ with $q_C\rho=\bar{\rho}p$ is trivial.

**) This assumption gives neat formulas but essentially the same theory carries through in the case that C is an H-group.

be the D-map defined in (1.7), and consider the map

$$\rho_1: \Omega_{K(B)}L(B) \times_{K(B)} E \longrightarrow L(C)$$

defined by

$$\rho_1(\lambda, y) = n(\rho m_B(\lambda, y), [\rho m_B(c_{\lambda(0)}, y)]^{-1}) \qquad \text{for} \quad \lambda \in \Omega_{K(B)}L(B),\ y \in E,$$

where c_x denotes the constant loop at x. Then, ρ_1 maps $E = s_B(K(B)) \times_{K(B)} E$ to $K(C) = s_C(K(C))$, and ρ_1 is a $K(C)$-map, where $\Omega_{K(B)}L(B) \times_{K(B)} E$ is considered as a $K(C)$-space by the composition $\bar{\rho} p \pi_2 = q_C \rho \pi_2$ (π_2 is the projection to the second factor). Therefore, we have $K(C)$-maps ρ_1 and $1 \times p$ in the diagram

$$\text{(2.3)} \qquad \begin{array}{ccc} (\Omega_{K(B)}L(B) \times_{K(B)} E,\ E) & \xrightarrow{\rho_1} & (L(C),\ K(C)) \\ \big\downarrow{\scriptstyle 1\times p} & & \big\| \\ (\Omega_{K(B)}L(B) \times_{K(B)} D,\ D) & \xrightarrow{d} & (L(C),\ K(C)), \end{array}$$

where $\Omega_{K(B)}L(B) \times_{K(B)} D$ is also considered as a $K(C)$-space by the composition $\bar{\rho}\pi_2$.

Now, we say that the composition of fibrations $T \xrightarrow{q} E \xrightarrow{p} D$ in (2.1) is *stable*, if there exists a $K(C)$-map d in (2.3) such that the diagram (2.3) is $K(C)$-homotopy commutative.

Suppose that the composition pq is stable by a $K(C)$-map d. From the fibration $\Omega_{K(B)}L(B) \to K(B)$, we obtain the fibration

$$\Omega^2_{K(B)}L(B) = \Omega_{K(B)}(\Omega_{K(B)}L(B)) \longrightarrow K(B)$$

with the canonical cross section, by Lemma 1.5. Then, the map d induces a $K(C)$-map

$$\text{(2.4)} \qquad d': (\Omega^2_{K(B)}L(B) \times_{K(B)} D,\ D) \longrightarrow (\Omega_{K(C)}L(C),\ K(C))$$

by the equation

$$d'(\lambda, x)(t) = d(\lambda(t), x) \qquad \text{for} \quad \lambda \in \Omega^2_{K(B)}L(B),\ x \in D \quad \text{and} \quad t \in I.$$

For a given D-space $X = (X, u)$, these $K(C)$-maps d and d' induce two functions

$$\text{(2.5)} \qquad \begin{aligned} \Theta_u &: [X, \Omega_{K(B)}L(B)]_{K(B)} \longrightarrow [X, L(C)]_{K(C)}, \\ \Theta'_u &: [X, \Omega^2_{K(B)}L(B)]_{K(B)} \longrightarrow [X, \Omega_{K(C)}L(C)]_{K(C)}, \end{aligned}$$

given by $\Theta_u([a]) = [d(a, u)]$ and $\Theta'_u([b]) = [d'(b, u)]$, where X is considered as a $K(B)$-space $(X, q_B\theta u)$ and $K(C)$-space $(X, \bar{\rho}u)$. Here Θ'_u is a homomorphism of groups by the definition of d' and so $\operatorname{Coker}\Theta'_u$ is defined. Set $\operatorname{Ker}\Theta_u = \Theta_u^{-1}([s_C\bar{\rho}u])$. Then we have the following main theorem in this chapter, which is a generalization of [12, Theorem 2.2].

THEOREM A. *Suppose that the composition of the fibrations*

$$T \xrightarrow{q} E \xrightarrow{p} D$$

in the diagram (2.1) *is stable by the map* d *in* (2.3). *Let* X *be a CW-complex and* $u\colon X \to D$ *admit a lifting* $X \to T$. *Then the set*

$$[X, T]_D$$

of homotopy classes of liftings of u *to* T *is equivalent to the product*

$$\operatorname{Ker} \Theta_u \times \operatorname{Coker} \Theta'_u,$$

where Θ_u *and* Θ'_u *are the functions of* (2.5).

§3. Correlations

Consider the diagram (2.1) and let $v\colon X \to E$ be a lifting of $u\colon X \to D$. We say that two maps $h, h'\colon X \to T$ are *v-related* if (1) $qh = qh' = v$ and (2) h is D-homotopic to h'. The relation "*v-related*" is an equivalence relation, and if v is D-homotopic to v', then the set of v-relation classes is equivalent to the set of v'-relation classes.

For $\eta = [v] \in [X, E]_D$, let $N(\eta)$ denote the set of v-relation classes of D-maps of X to T. Then

$$N(\eta) = q_*^{-1}(\eta) \quad \text{and} \quad [X, T]_D = \cup\{q_*^{-1}(\eta) \mid \eta \in [X, E]_D\},$$

where $q_*\colon [X, T]_D \to [X, E]_D$. Thus we have the following

LEMMA 3.1 [12, Theorem 3.2]. *The set* $[X, T]_D$ *is equivalent to the disjoint union of the set* $N(\eta)$, *where* η *runs through the elements of* $[X, E]_D$.

Since the set $[X, E]_D$ is equivalent to the group $[X, \Omega_{K(B)}L(B)]_{K(B)}$ by Proposition 1.8, we study the set $N(\eta)$ for each $\eta \in [X, E]_D$ in the rest of this section.

As is constructed in (1.7), there is a D-map

$$m_C\colon \Omega_{K(C)}L(C) \times_{K(C)} T \longrightarrow T.$$

This D-map m_C induces an action of the group $[X, \Omega_{K(C)}L(C)]_{K(C)}$ on $[X, T]_D$ by the same way as Proposition 1.8. It is easily seen that (1) if $h\colon X \to T$ is a D-map and if $k, k'\colon X \to \Omega_{K(C)}L(C)$ are $K(C)$-homotopic, then $m_C(k, h)$ and $m_C(k', h)$ are v-related, where $v = qh$, and (2) if $k\colon X \to \Omega_{K(C)}L(C)$ is a $K(C)$-map and if h, $h'\colon X \to T$ are v-related, then $m_C(k, h)$ and $m_C(k, h')$ are v-related. Hence, using

Proposition 1.8, we see that the above action of $[X, \Omega_{K(C)}L(C)]_{K(C)}$ is transmitted to a transitive action on $N(\eta)$. We, therefore, have the following

LEMMA 3.2. *Let η be the element in the image of $q_*\colon [X, T]_D \to [X, E]_D$. The set $N(\eta)$ is equivalent to the quotient of $[X, \Omega_{K(C)}L(C)]_{K(C)}$ by the stabilizer of an element of $N(\eta)$.*

Let $p\colon E \to A$ be a fibration with fiber F and let

$$\Omega_A^* E = \{\lambda\colon I \longrightarrow E \mid p\lambda(t) = p\lambda(0) \quad \textit{for all} \quad t \in I,\ \lambda(0) = \lambda(1)\},$$

$$\Omega^* F = \{\lambda\colon I \longrightarrow F \mid \lambda(0) = \lambda(1)\}.$$

Then the following results are known and will be used later on.

LEMMA 3.3. *Let $r\colon \Omega_A^* E \to E$ be a map defined by $r(\lambda) = \lambda(1)$. Then $r\colon \Omega_A^* E \to E$ is a fibration with fiber ΩF and $pr\colon \Omega_A^* E \to A$ is also a fibration with fiber $\Omega^* F$.*

The map $\rho\colon E \to L(C)$ in (2.1) induces a map

$$\rho'\colon \Omega_D^* E \longrightarrow \Omega_{K(C)}^* L(C),$$

which is given by $\rho'(\lambda)(t) = \rho(\lambda(t))$, and there follows a commutative diagram below,

$$\begin{array}{ccccc} \Omega_D^* E & \xrightarrow{r} & E & \xrightarrow{p} & D \\ \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \bar\rho} \\ \Omega_{K(C)}^* L(C) & \xrightarrow{r} & L(C) & \longrightarrow & K(C). \end{array}$$

Therefore we have a commutative diagram

$$\begin{array}{ccccc} & & [X, \Omega_D^* E]_D & \xrightarrow{r_*} & [X, E]_D \\ & & \downarrow{\scriptstyle \rho'_*} & & \downarrow{\scriptstyle \rho_*} \\ [X, \Omega_{K(C)}L(C)]_{K(C)} & \xrightarrow{i_*} & [X, \Omega_{K(C)}^* L(C)]_{K(C)} & \xrightarrow{r_*} & [X, L(C)]_{K(C)}, \end{array}$$

where $i\colon \Omega_{K(C)}L(C) \to \Omega_{K(C)}^* L(C)$ is the natural inclusion. We say that an element $\gamma \in [X, \Omega_{K(C)}L(C)]_{K(C)}$ is *ρ-correlated to* $\eta \in [X, E]_D$ if there is an element $\chi \in [X, \Omega_D^* E]_D$ such that $r_*(\chi) = \eta$ and $\rho'_*(\chi) = i_*(\gamma)$.

LEMMA 3.4. *Let $h\colon X \to T$ be a D-map and let $v = qh$. Suppose that $k \vee h = m_C(k, h)$ is v-related to h for a $K(C)$-map $k\colon X \to \Omega_{K(C)}L(C)$. Then the class of k in $[X, \Omega_{K(C)}L(C)]_{K(C)}$ is ρ-correlated to the D-homotopy class of $v\colon X \to E$.*

LEMMA 3.5. *For a $K(C)$-map $k\colon X \to \Omega_{K(C)}L(C)$, suppose that the class of* *k in* $[X, \Omega_{K(C)}L(C)]_{K(C)}$ *is* ρ*-correlated to the D-homotopy class of* $v: X\to E$. *Then* $k\vee h$ *is* v*-related to* h *for any lifting* $h: X\to T$ *of* v.

Combining Lemma 3.2 and Lemmas 3.4–5, we have the following

PROPOSITION 3.6. *If* $\eta\in[X, E]_D$ *lies in the image of* $q_*: [X, T]_D\to[X, E]_D$, *then the set* $N(\eta)=q_*^{-1}(\eta)$ *is equivalent to the factor group of* $[X, \Omega_{K(C)}L(C)]_{K(C)}$ *by the subgroup of elements which are* ρ*-correlated to* η.

PROOF OF LEMMA 3.4. Let $g_t: X\to T$ be a D-homotopy such that $g_0=h$ and $g_1=k\vee h$ and let $g: X\to\Omega_D^*E$ be a D-map given by $g(x)(t)=qg_t(x)$ for any $x\in X$ and $t\in I$. Then $rg(x)=g(x)(1)=qg_1(x)=v(x)$. Hence it is sufficient to show that $i_*([k])=\rho'_*([g])$ in $[X, \Omega_{K(C)}^*L(C)]_{K(C)}$. Let $\tilde\rho: T\to P_{K(C)}L(C)$ be the map induced by ρ, which makes the following diagram commutative:

$$\begin{array}{ccc} T & \xrightarrow{\tilde\rho} & P_{K(C)}L(C) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ E & \xrightarrow{\rho} & L(C)\,. \end{array}$$

Then there is a homotopy $l_s: X\to\Omega_{K(C)}^*L(C)$ ($s\in I$) given by

$$l_s(x)(t)=\begin{cases}\tilde\rho g_{1+2st-2s}(x)(t/2) & 0\le 2s\le 1\\ \tilde\rho g_t(x)(2s+t-st-1) & 1\le 2s\le 2\end{cases}$$

which is a $K(C)$-homotopy between ik and $\rho'g$. q.e.d.

PROOF OF LEMMA 3.5. Let $g: X\to\Omega_D^*E$ be a D-map such that $rg\simeq_D v$ and $\rho'g\simeq_{K(C)}ik$. Since $\Omega_D^*E\to E$ is a fibration by Lemma 3.3, we may assume that $rg=v$. Let $\tau: \Omega_{K(C)}L(C)\to\Omega_{K(C)}L(C)$ be a $K(C)$-map given by $\tau(\lambda)(t)=\lambda(1-t)$ for all $t\in I$. Let $k': X\to\Omega_{K(C)}L(C)$ be a $K(C)$-map defined by $k'=\tilde\rho h\vee\rho'g\vee\tau(\tilde\rho h)$. Then ik' is $K(C)$-homotopic to $l_0: X\to\Omega_{K(C)}^*L(C)$ defined by

$$l_0(x)(t)=\begin{cases}\tilde\rho h(x)(3t) & 0\le 3t\le 1\\ \rho'g(x)(3t-1) & 1\le 3t\le 2\\ \tilde\rho h(x)(3-3t) & 2\le 3t\le 3.\end{cases}$$

Let $l_s: X\to\Omega_{K(C)}^*L(C)$ be a $K(C)$-homotopy which is defined by

$$l_s(x)(t)=\begin{cases}l_0(x)(t+s/3) & 0\le 3t\le 1-s\\ l_0(x)((t+s)/(1+2s)) & 1-s\le 3t\le 2+s\\ l_0(x)(t-s/3) & 2+s\le 3t\le 3.\end{cases}$$

Then $l_1(x)(t)=l_0(x)((1+t)/3)=\rho' g(x)(t)$ and so $i_*([k'])=\rho'_*([g])$. Therefore, there follows $ik\simeq_{K(C)} ik'$ because $i_*([k])=\rho'_*([g])$ by the assumption. Let $f_t\colon X\to\Omega^*_{K(C)}L(C)$ be a $K(C)$-homotopy between ik' and ik, and let $f\colon X\to \Omega_{K(C)}L(C)$ be a $K(C)$-map given by $f(x)(t)=f_t(x)(0)$. Then it is easily seen that $k'\vee f\simeq_{K(C)} f\vee k$, i.e., $[k'\vee f]=[f\vee k]$ in $[X, \Omega_{K(C)}L(C)]_{K(C)}$. Because $[X, \Omega_{K(C)}L(C)]_{K(C)}$ is an abelian group by Lemma 1.6, it follows that $[k]=[k']$. Therefore, we have

$$k\vee\tilde{\rho}h\simeq_{K(C)} k'\vee\tilde{\rho}h\simeq_{K(C)}(\tilde{\rho}h\vee\rho' g\vee\tau(\tilde{\rho}h))\vee\tilde{\rho}h\simeq_{K(C)}\tilde{\rho}h\vee\rho' g.$$

Let $w\colon X\to T$ be the map defined by $w(x)=(v(x), (\tilde{\rho}h\vee\rho' g)(x))$. Then w is a lifting of v and w is D-homotopic to $(v, k\vee\tilde{\rho}h)=k\vee h$, i.e., w is v-related to $k\vee h$. On the other hand, let $w_s\colon X\to T$ be a homotopy which is given by

$$w_s(x)=(g(x)(1-s),\ l'_s(x)),$$

$$l'_s(x)(t)=\begin{cases}\tilde{\rho}h(x)(2t/(1+s)) & 0\le 2t\le 1+s\\ \rho' g(x)(2t-1-s) & 1+s\le 2t\le 2.\end{cases}$$

Then w_s is a D-homotopy between w and h. Therefore, w is v-related to h and so $k\vee h$ is v-related to h. q.e.d.

§4. Compositions of twisted principal fibrations

Let $p\colon E\to D$ be the twisted principal fibration with fiber F $(=\Omega B)$ in the diagram (2.1) and let

$$m_B\colon (\Omega_{K(B)}L(B)\times_{K(B)}E,\ \Omega_{K(B)}L(B)\times_{K(B)}F)\longrightarrow (E, F)$$

be the map of (2.2). Obviously, $\Omega_{K(B)}L(B)\times_{K(B)}F=F\times F$ and $m_B\colon F\times F\to F$ is the ordinary multiplication of $F=\Omega B$. Consider the map

$$m'_B\colon (\Omega^2_{K(B)}L(B)\times_{K(B)}E,\ \Omega^2_{K(B)}L(B)\times_{K(B)}F)\longrightarrow(\Omega^*_D E,\ \Omega^* F),$$

which is given by

$$m'_B(\lambda, x)(t)=m_B(\lambda(t), x)\quad\text{for}\quad \lambda\in\Omega^2_{K(B)}L(B),\quad x\in E\quad\text{and}\quad t\in I.$$

It is easily seen that $\Omega^2_{K(B)}L(B)\times_{K(B)}F=\Omega F\times F$ and $m'_B\colon \Omega F\times F\to\Omega^*F$ coincides with the map defined in [10, Theorem 2.7]. Now, $pr\colon \Omega^*_D E\to D$ is a fibration with fiber Ω^*F by Lemma 3.3 on the one hand and on the other hand $p\pi_2\colon \Omega^2_{K(B)}L(B)\times_{K(B)}E\to D$ (π_2 is the projection to the second factor) is a fibration with fiber $\Omega F\times F$, and m'_B makes the following diagram of fibrations commutative:

$$\begin{array}{ccccc} \Omega F \times F & \xrightarrow{\subset} & \Omega^2_{K(B)} L(B) \times_{K(B)} E & \xrightarrow{p\pi_2} & D \\ \downarrow{\scriptstyle m'_B} & & \downarrow{\scriptstyle m'_B} & & \downarrow{\scriptstyle 1} \\ \Omega^* F & \xrightarrow{\subset} & \Omega^*_D E & \xrightarrow{pr} & D. \end{array}$$

The map m'_B: $\Omega F \times F \to \Omega^* F$ is a weak homotopy equivalence by [10, Theorem 2.7] and so is the map m'_B: $\Omega^2_{K(B)} L(B) \times_{K(B)} E \to \Omega^*_D E$, which is seen immediately by using the homotopy exact sequences of fibrations and the five lemma. Therefore the function

$$m'_{B*}\colon [X, \Omega^2_{K(B)} L(B)]_{K(B)} \times [X, E]_D \longrightarrow [X, \Omega^*_D E]_D$$

is a bijection for all CW-complex X, by [11, Theorem 3.2].

The $K(C)$-map ρ_1 in (2.3) induces a $K(C)$-map

$$\rho'_1\colon (\Omega^2_{K(B)} L(B) \times_{K(B)} E,\ E) \longrightarrow (\Omega_{K(C)} L(C),\ K(C)),$$

which is defined by

$$\rho'_1(\lambda, x)(t) = \rho_1(\lambda(t), x).$$

If v: $X \to E$ is a D-map and a, b: $X \to \Omega^2_{K(B)} L(B)$ are $K(B)$-maps, then the relation

$$\rho'_1(a \vee b, v) = \rho'_1(a, v) \vee \rho'_1(b, v)$$

holds. Therefore the function

$$\Delta(\rho, [v])\colon [X, \Omega^2_{K(B)} L(B)]_{K(B)} \longrightarrow [X, \Omega_{K(C)} L(C)]_{K(C)}, \tag{4.1}$$

defined by

$$\Delta(\rho, [v])([a]) = [\rho'_1(a, v)],$$

is a homomorphism of groups. We consider also a $K(C)$-map

$$n'\colon \Omega_{K(C)} L(C) \times_{K(C)} L(C) \longrightarrow \Omega^*_{K(C)} L(C),$$

defined by the relation

$$n'(\lambda, x)(t) = n(\lambda(t), x) \quad \text{for} \quad \lambda \in \Omega_{K(C)} L(C), \quad x \in L(C) \quad \text{and} \quad t \in I,$$

where $n = \mu_{\phi(C)}$: $L(C) \times_{K(C)} L(C) \to L(C)$ is the induced multiplication of (1.3). Because C is a topological group, the map n' is a $K(C)$-homeomorphism. Therefore the induced function

$$n'_*\colon [X, \Omega_{K(C)} L(C)]_{K(C)} \times [X, L(C)]_{K(C)} \longrightarrow [X, \Omega^*_{K(C)} L(C)]_{K(C)}$$

is a bijection for any space X. By the direct calculations, we obtain

$$n'(\rho'_1, \rho r m_B)\Delta = \rho' m'_B\colon \Omega^2_{K(B)}L(B)\times_{K(B)}E \longrightarrow \Omega^*_{K(C)}L(C),$$

where Δ is the diagonal map. This implies the following lemma.

LEMMA 4.2. *There are the following relations*:

(1) $r_* m'_{B*}(\beta, \eta) = \eta$,

(2) $\rho'_* m'_{B*}(\beta, \eta) = n'_*(\Delta(\rho, \eta)(\beta), \rho_*\eta)$,

(3) $n'_*(\gamma, [s_C\bar{\rho}u]) = i_*(\gamma)$.

Using the above lemma, we can prove the following

PROPOSITION 4.3. *Under the situation of* (2.1), *the conditions* (i) *and* (ii) *are equivalent.*
(i) *The element* $\eta\in[X, E]_D$ *is contained in the image of* $q_*\colon [X, T]_D\to[X, E]_D$ *and* $\gamma\in[X, \Omega_{K(C)}L(C)]_{K(C)}$ *is* ρ-*correlated to* η.
(ii) *The element* $\eta\in[X, E]_D$ *is contained in* $\rho_*^{-1}([s_C\bar{\rho}u])$ *and* γ *lies in the image of* $\Delta(\rho, \eta)\colon [X, \Omega^2_{K(B)}L(B)]_{K(B)}\to[X, \Omega_{K(C)}L(C)]_{K(C)}$.

From Lemma 3.1, Proposition 3.6 and Proposition 4.3, we have the following

THEOREM 4.4. *Under the situation of* (2.1), *the set* $[X, T]_D$ *is equivalent to the disjoint union of* Coker $\Delta(\rho, \eta)$ *of the homomorphism* $\Delta(\rho, \eta)$ *of* (4.1), *as* η *runs through* $\rho_*^{-1}([s_C\bar{\rho}u])$, *where* $\rho_*\colon [X, E]_D\to[X, L(C)]_{K(C)}$.

§5. Proof of Theorem A in §2

Assume that the composition of fibrations $T\xrightarrow{q}E\xrightarrow{p}D$ in the diagram (2.1) is stable by a $K(C)$-map $d\colon (\Omega_{K(B)}L(B)\times_{K(B)}D, D)\to(L(C), K(C))$, i.e., the following diagram is $K(C)$-homotopy commutative:

$$\begin{array}{ccc}
(\Omega_{K(B)}L(B)\times_{K(B)}E, E) & \xrightarrow{\rho_1} & (L(C), K(C)) \\
\downarrow{\scriptstyle 1\times p} & & \| \\
(\Omega_{K(B)}L(B)\times_{K(B)}D, D) & \xrightarrow{d} & (L(C), K(C)),
\end{array}$$

where ρ_1 is the map defined in (2.3). Let

$$d'\colon (\Omega^2_{K(B)}L(B)\times_{K(B)}D, D) \longrightarrow (\Omega_{K(C)}L(C), K(C))$$

be the map induced from the map d by $d'(\lambda, x)(t) = d(\lambda(t), x)$. Then the diagram below is $K(C)$-homotopy commutative:

$$\begin{array}{ccc}(\Omega^2_{K(B)}L(B)\times_{K(B)}E,\ E) & \xrightarrow{\rho'_1} & (\Omega_{K(C)}L(C),\ K(C)) \\ \downarrow{\scriptstyle 1\times p} & & \| \\ (\Omega^2_{K(B)}L(B)\times_{K(B)}D,\ D) & \xrightarrow{d'} & (\Omega_{K(C)}L(C),\ K(C)).\end{array}$$

For any map $u\colon X\to D$, there are two functions

$$\Theta_u\colon [X,\ \Omega_{K(B)}L(B)]_{K(B)} \longrightarrow [X,\ L(C)]_{K(C)},$$

$$\Theta'_u\colon [X,\ \Omega^2_{K(B)}L(B)]_{K(B)} \longrightarrow [X,\ \Omega_{K(C)}L(C)]_{K(C)},$$

which are defined by

$$\Theta_u([a]) = [d(a,\ u)], \qquad \Theta'_u([b]) = [d'(b,\ u)].$$

If $u\colon X\to D$ has a lifting to E, then the homomorphism Θ'_u is equal to the homomorphism $\Delta(\rho,\ \eta)$ of (4.1) for any $\eta\in[X,\ E]_D$ by the definition of $\Delta(\rho,\ \eta)$ and the above commutative diagram. Therefore

$$\operatorname{Coker}\Theta'_u = \operatorname{Coker}\Delta(\rho,\ \eta) \qquad \text{for any} \quad \eta\in[X,\ E]_D.$$

Let $\eta=[v]\in[X,\ E]_D$. Then

$$\Theta_u([a]) = [d(a,\ u)] = [\rho_1(a,\ v)] = [n(\rho m_B(a,\ v),\ \rho m_B(c_{a(0)},\ v)^{-1})]$$

by definition. If $v\colon X\to E$ has a lifting to T, then $[\rho m_B(c_{a(0)},\ v)]$ is equal to the unit $[s_C\bar{\rho}u]$. Thus the function

$$\rho_* m_{B*}(\ ,\ \eta)\colon [X,\ \Omega_{K(B)}L(B)]_{K(B)} \longrightarrow [X,\ E]_D \longrightarrow [X,\ L(C)]_{K(C)}$$

is equal to Θ_u, if u has a lifting to T. Since $m_{B*}(\ ,\ \eta)$ is a bijection by Proposition 1.8, we see that $\rho_*^{-1}([s_C\bar{\rho}u])$ is equivalent to $\operatorname{Ker}\Theta_u=\Theta_u^{-1}([s_C\bar{\rho}u])$.

The above argument and Theorem 4.4 complete the proof of Theorem A.

REMARK. We see easily that the function Θ_u is also a homomorphism.

Chapter II. Enumeration of cross sections of sphere bundles

§6. Some remarks on the cohomology with local coefficients

The non-trivial homomorphism $\phi\colon Z_2\to\operatorname{Aut}(Z)$, where $\operatorname{Aut}(Z)$ is the group of automorphisms of the infinite cyclic group Z, induces a homomorphism $\phi\colon Z_2\to\operatorname{Homeo}(K(Z,\ n))$ $(n>1)$. As indicated in (1.1), there is a fibration

$$K(Z,\ n) \xrightarrow{i} L_\phi(Z,\ n) \xrightarrow{q} K = K(Z_2,\ 1), \quad L_\phi(Z,\ n) = L_\phi(K(Z,\ n)),$$

with a canonical cross section s. A map $u\colon X\to K$ determines a local system on

X which is given by $\phi u_*\colon \pi_1(X)\to\pi_1(K)=Z_2\to\mathrm{Aut}(Z)$. We denote the cohomology with coefficients in the above local system by $H^*(X; Z_{u^*\phi})$ or $H^*(X; \underline{Z})$ simply. Notice that the following results.

PROPOSITION 6.1 [13, § 1 and § 3]. *There is a unique element $\lambda\in H^n(L_\phi(Z, n), K; Z_{q^*\phi})$ such that $\bar{i}^*\lambda=\iota_n\in H^n(K(Z, n); Z)$, the fundamental class of $K(Z, n)$, where $\bar{i}\colon K(Z, n)\to(L_\phi(Z, n), K)$ is the natural inclusion, and there is a natural isomorphism*

$$\Phi\colon [X, A; L_\phi(Z, n), K]_K \xrightarrow{\approx} H^n(X, A; Z_{u^*\phi})$$

for any pair of regular cell complex (X, A) and for any map $u\colon X\to K$ which is defined by

$$\Phi([a]) = a^*(\lambda).$$

If A is empty, this is the isomorphism

$$\Phi\colon [X, L_\phi(Z, n)]_K \longrightarrow H^n(X; Z_{u^*\phi}), \qquad \Phi([a]) = a^*j^*\lambda,$$

where $j\colon L_\phi(Z, n)\to(L_\phi(Z, n), K)$ is the natural inclusion.

We say that the elements λ and $j^*\lambda$ are the fundamental classes of the fibration $q\colon L_\phi(Z, n)\to K$ and we denote λ, $j^*\lambda$ and their mod 2 reductions by the same symbol λ, whenever no confusion can arise.

For a map $u\colon X\to K$, consider the pull back of $q\colon L_\phi(Z, n)\to K$ by u,

$$\begin{array}{ccccc} K(Z, n) & \xrightarrow{i} & L_\phi(Z, n)\times_K X & \xrightarrow{\pi_1} & L_\phi(Z, n) \\ & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle q} \\ & & X & \xrightarrow{\quad u \quad} & K, \end{array}$$

(π_i is the projection to the i-th factor). Then $i^*\pi_1^*\lambda=\iota_n$ follows immediately from the relation $\bar{i}^*\lambda=\iota_n$. Therefore, we see easily the following

LEMMA 6.2. *Let $\nu\colon H^*(K(Z, n); Z_2)\to H^*(L_\phi(Z, n)\times_K X; Z_2)$ be the homomorphism of Z_2-algebras given by $\nu(Sq^I\iota_n)=Sq^I\lambda_X$, where ι_n is the image of the* mod 2 *reduction of the fundamental class ι_n of $K(Z, n)$ and $\lambda_X=\pi_1^*\lambda\in H^n(L_\phi(Z, n)\times_K X; Z_2)$. Then*

$$\nu\otimes\pi_2^*\colon H^*(K(Z, n); Z_2)\otimes H^*(X; Z_2) \longrightarrow H^*(L_\phi(Z, n)\times_K X; Z_2)$$

is an isomorphism of Z_2-algebras and so any element x in $H^*(L_\phi(Z, n)\times_K X; Z_2)$ is described uniquely in the form

$$x = \sum_i Sq^{I_i}\lambda_X\pi_2^*a_i, \qquad a_i\in H^*(X; Z_2).$$

§7. The third stage Postnikov factorization of $BO(n-1)\to BO(n)$

Let $p\colon BO(n-1)\to BO(n)$ be the universal S^{n-1}-bundle ($n\geq 4$). Our purpose in this section is the construction of the third stage Postnikov factorization of this bundle using the methods of J. F. McClendon [13] and E. Thomas [19].

Let $\phi\colon \pi_1(BO(n))=Z_2\to \mathrm{Aut}(\pi_{n-1}(S^{n-1}))=\mathrm{Aut}(Z)$ be the local system on $BO(n)$ associated with $p\colon BO(n-1)\to BO(n)$, and let s_{n-1} be the generator of $H^{n-1}(S^{n-1}; Z)=Z$. Then, by [13, Theorem 4.1 and §§2-3], there is a map $W\colon BO(n)\to L_\phi(Z, n)$ such that $[W]\in[BO(n), L_\phi(Z, n)]_K=H^n(BO(n); \underline{Z})$ is the transgression image of s_{n-1}, and we have a commutative diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "s_{n-1}"'] & BO(n-1) \arrow[d, "q_1"] & & \\ \Omega K(Z, n) \arrow[r] & E \arrow[r] \arrow[d, "p_1"] & P_K L_\phi(Z, n) \arrow[d] & \\ & BO(n) \arrow[r, "W"] & L_\phi(Z, n) \arrow[r, "q"] & K, \end{tikzcd}$$

where $p_1 q_1=p$ and p_1 is the twisted principal fibration induced by W. By using the homotopy exact sequences of fibrations, we see easily that both maps s_{n-1} and q_1 are homotopically equivalent to the fibrations $F\xrightarrow{\subset} S^{n-1}\xrightarrow{s_{n-1}}\Omega K(Z, n)$ and $F\xrightarrow{\subset} BO(n-1)\xrightarrow{q_1} E$ (cf. [19, §1]) and

$$\pi_i(F)=\begin{cases} 0 & \text{for } i\leq n-1 \\ \pi_i(S^{n-1}) & \text{for } i\geq n. \end{cases}$$

Therefore $q_1\colon BO(n-1)\to E$ is an n-equivalence.*) Since the generator of $H^n(F; Z_2)=Z_2$ is transgressive for the fibration $q_1\colon BO(n-1)\to E$, its transgression image is a non-zero element ρ in $H^{n+1}(E; Z_2)$ and there is a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & BO(n-1) \arrow[d, "q_2"] & \\ K(Z_2, n) \arrow[r] & T \arrow[d, "p_2"] & \\ & E \arrow[r, "\rho"] & K(Z_2, n+1). \end{tikzcd}$$

Here $p_2 q_2=q_1$, p_2 is the principal fibration with the classifying map ρ and it is easily seen that q_2 is an $(n+1)$-equivalence and $q_2|F$ represents the generator of

*) A map $g\colon X\to Y$ (X, Y are connected) is called an n-equivalence if $g_*\colon \pi_i(X)\to\pi_i(Y)$ is isomorphic for $i<n$ and epimorphic for $i=n$.

$H^n(F; Z_2)$.

In the rest of this section, we concentrate ourselves on the characterization of the map $\rho: E \to K(Z_2, n+1)$. Let

$$m: \Omega_K L_\phi(Z, n) \times_K E \longrightarrow E$$

be the action defined in (1.7) and set

$$\mu = m(1 \times q_1): \Omega_K L_\phi(Z, n) \times_K BO(n-1) \longrightarrow E. \tag{7.1}$$

The map μ makes the following diagram commutative:

$$\begin{array}{ccc} \Omega_K L_\phi(Z, n) \times_K BO(n-1) & \xrightarrow{\mu} & E \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle p_1} \\ BO(n-1) & \xrightarrow{p} & BO(n). \end{array}$$

The projection π_2 to the second factor admits a cross section s defined by $s(x) = (c_{qWp(x)}, x)$, where c_y is the constant loop at y, and the relation

$$\mu s \simeq_{BO(n)} q_1 \tag{7.2}$$

holds obviously. The local system $\pi_1(BO(n)) = Z_2 \to \mathrm{Aut}(H^i(K(Z, n-1); Z_2))$ on $BO(n)$, which is associated with $p_1: E \to BO(n)$, is trivial for $i = n-1$ and hence so for all i. Also $H^i(K(Z, n-1); Z_2) = 0$ for $0 < i < n-1$ and $H^i(BO(n), BO(n-1); Z_2) = 0$ for $i < n$. Therefore, by the similar proof to [19, Property 4], we see that the sequence

$$\begin{aligned} \cdots \longrightarrow H^i(\Omega_K L_\phi(Z, n) \times_K BO(n-1); Z_2) &\xrightarrow{\tau_0} H^{i+1}(BO(n), BO(n-1); Z_2) \\ \xrightarrow{p_1^* j^*} H^{i+1}(E; Z_2) \xrightarrow{\mu^*} H^{i+1}(\Omega_K L_\phi(Z, n) &\times_K BO(n-1); Z_2) \longrightarrow \\ \cdots \longrightarrow H^{2n-2}(E; Z_2) & \end{aligned}$$

is exact, where $j: BO(n) \to (BO(n), BO(n-1))$ is the natural inclusion, and τ_0 is the relative transgression. On the other hand, $p^*: H^i(BO(n); Z_2) \to H^i(BO(n-1); Z_2)$ is epimorphic for all i. Also $\mathrm{Ker}\, p^*$ is the ideal generated by the universal n-th Stiefel-Whitney class w_n. Since w_n is the transgression image of s_{n-1} of $p: BO(n-1) \to BO(n)$, we have $w_n = \tau(\iota_{n-1}) \in \mathrm{Ker}\, p_1^*$, where τ is the transgression of $K(Z, n-1) \xrightarrow{\subset} E \xrightarrow{p_1} BO(n)$. Thus we see that $\mathrm{Ker}\, p^* = \mathrm{Ker}\, p_1^*$. Therefore, the same argument as in [19, Property 5] provides the exact sequence

$$\begin{aligned} 0 \longrightarrow H^i(E; Z_2) \xrightarrow{\mu^*} H^i(\Omega_K L_\phi(Z, n) \times_K BO(n-1); Z_2) \\ \xrightarrow{\tau_1} H^{i+1}(BO(n); Z_2) \end{aligned} \tag{7.3}$$

for $t<2n-2$, where $\tau_1=j^*\tau_0$. (7.2) and (7.3) imply that

$$\text{(7.4)} \qquad \mu^*\colon \operatorname{Ker} q_1^* \longrightarrow \operatorname{Ker} s^* \cap \operatorname{Ker} \tau_1$$

is isomorphic in dimension less than $2n-2$.

By considering $\Omega_K L_\phi(Z, n)=L_\phi(Z, n-1)$ by the natural K-homeomorphism ψ of Lemma 1.6, there is an element $\lambda_{BO(n-1)}$ in $H^{n-1}(\Omega_K L_\phi(Z, n)\times_K BO(n-1); Z_2)$ by Lemma 6.2 for the fibration $\Omega_K L_\phi(Z, n)\times_K BO(n-1)\to BO(n-1)$ such that $i^*\lambda_{BO(n-1)}=\iota_{n-1}$, the mod 2 reduction of the fundamental class of $K(Z, n-1)$. Here the diagram

$$\begin{tikzcd}
\Omega K(Z, n) \arrow[r, "i"] \arrow[d] & \Omega_K L_\phi(Z, n)\times_K BO(n-1) \arrow[r, "\pi_2"] \arrow[d, "\mu"] & BO(n-1) \arrow[d, "p"] \\
\Omega K(Z, n) \arrow[r] & E \arrow[r, "p_1"] & BO(n)
\end{tikzcd}$$

implies that $\tau_1(\lambda_{BO(n-1)})=j^*\tau_0(\lambda_{BO(n-1)})=\tau i^*(\lambda_{BO(n-1)})=\tau(\iota_{n-1})=w_n$. Any element x in $H^{n+1}(\Omega_K L_\phi(Z, n)\times_K BO(n-1); Z_2)$ is described in the form

$$x = \pi_2^* b+\varepsilon_1\lambda_{BO(n-1)}\pi_2^* w_1^2+\varepsilon_2\lambda_{BO(n-1)}\pi_2^* w_2+\varepsilon_3 Sq^2\lambda_{BO(n-1)},$$

where $\varepsilon_i=0$ or 1 for $i=1, 2, 3$ by Lemma 6.2. If $x\in \operatorname{Ker} s^*\cap\operatorname{Ker}\tau_1$, then $0=s^*x=b$. Because τ_1 is an $H^*(BO(n); Z_2)$-homomorphism and $\tau_1 Sq^i=Sq^i\tau_1$ by [19, § 3], it follows that

$$\tau_1(\lambda_{BO(n-1)}\pi_2^* w_1^2) = w_n w_1^2, \qquad \tau_1(\lambda_{BO(n-1)}\pi_2^* w_2) = w_n w_2,$$

$$\tau_1(Sq^2\lambda_{BO(n-1)}) = Sq^2 w_n = w_n w_2.$$

Hence $\operatorname{Ker} s^*\cap\operatorname{Ker}\tau_1=Z_2$ generated by $\lambda_{BO(n-1)}\pi_2^* w_2+Sq^2\lambda_{BO(n-1)}$ and so the map $\rho\colon E\to K(Z_2, n+1)$ is characterized by the relation

$$\text{(7.5)} \qquad \mu^*\rho = \lambda_{BO(n-1)}\pi_2^* w_2+Sq^2\lambda_{BO(n-1)}.$$

Summing up the above arguments, we have

THEOREM 7.6. *The third stage Postnikov factorization of $p\colon BO(n-1)\to BO(n)$ is given as follows*:

$$\text{(7.7)} \qquad \begin{tikzcd}
BO(n-1) \arrow[r, "q_2"] \arrow[dr, "q_1"] \arrow[ddr, "p"'] & T \arrow[d, "p_2"] & \\
& E \arrow[r, "\rho"] \arrow[d, "p_1"] & K(Z_2, n+1) \\
& BO(n) \arrow[r, "W"] & L_\phi(Z, n),
\end{tikzcd}$$

where $\phi\colon \pi_1(K(Z_2, 1))=Z_2\to\mathrm{Aut}(Z)$ is the non-trivial local system on $K(Z_2, 1)$, $p_1\colon E\to BO(n)$ is the twisted principal fibration induced by the map W, $p_2\colon T\to E$ is the principal fibration with classifying map ρ, $q_1\colon BO(n-1)\to E$ is an n-equivalence, $q_2\colon BO(n-1)\to T$ is an $(n+1)$-equivalence and the map ρ is characterized by the relation (7.5).

§8. The stability of the third stage Postnikov factorization of p: $BO(n-1)\to BO(n)$

There is a map

$$(8.1)\qquad d\colon (\Omega_K L_\phi(Z, n)\times_K BO(n), BO(n)) \longrightarrow (K(Z_2, n+1), *),$$

which represents the element $\lambda_{BO(n)}\pi_2^* w_2+Sq^2\lambda_{BO(n)}$ in $H^{n+1}(\Omega_K L_\phi(Z, n)\times_K BO(n), BO(n); Z_2)$, i.e., $d^*(\iota)=\lambda_{BO(n)}\pi_2^* w_2+Sq^2\lambda_{BO(n)}$, where ι is the fundamental class of $K(Z_2, n+1)$. The relation

$$(8.2)\qquad (1\times p_1)^* d^*(\iota)=\lambda_E\pi_2^* p_1^* w_2+Sq^2\lambda_E\in H^{n+1}(\Omega_K L_\phi(Z, n)\times_K E, E; Z_2)$$

follows easily. Let

$$\rho_1\colon (\Omega_K L_\phi(Z, n)\times_K E, E)\longrightarrow (K(Z_2, n+1), *)$$

be the map given by the relation $\rho_1(k, y)=\rho m(k, y)\cdot[\rho m(c_{k(0)}, y)]^{-1}$ (cf. (2.3)). Then the following relation holds:

$$(8.3)\qquad \rho_1^*(\iota)=m^*\rho^*(\iota)-\pi_2^*\rho^*(\iota)\in H^{n+1}(\Omega_K L_\phi(Z, n)\times_K E, E; Z_2).$$

To see that the composition of fibrations $T\xrightarrow{p_2}E\xrightarrow{p_1}BO(n)$ in the diagram (7.7) is stable by the map d in the sense of §2, it is sufficient to show that

$$(8.4)\qquad (m^*-\pi_2^*)\rho^*(\iota)=\lambda_E\pi_2^* p_1^* w_2+Sq^2\lambda_E,$$

by (8.2) and (8.3). Now, consider the map μ of (7.1). Then the diagram

$$\begin{array}{ccc} H^{n+1}(E; Z_2) & \xrightarrow{m^*-\pi_2^*} & H^{n+1}(\Omega_K L_\phi(Z, n)\times_K E; Z_2)\\ \cup & & \big\downarrow{\scriptstyle (1\times q_1)^*}\\ \mathrm{Ker}\, q_1^* & \xrightarrow{\mu^*} & H^{n+1}(\Omega_K L_\phi(Z, n)\times_K BO(n-1); Z_2)\end{array}$$

is commutative because $(1\times q_1)^*(m^*-\pi_2^*)(x)=(1\times q_1)^* m^*(x)-(1\times q_1)^*\pi_2^*(x)=\mu^*(x)$ for any x in $\mathrm{Ker}\, q_1^*$. Therefore we have

$$\begin{aligned}(1\times q_1)^*(m^*-\pi_2^*)\rho^*(\iota) &= \mu^*\rho^*(\iota) \qquad \text{by}\quad \rho^*(\iota)\in\mathrm{Ker}\, q_1^*\\ &= \lambda_{BO(n-1)}\pi_2^* p^* w_2+Sq^2\lambda_{BO(n-1)} \quad \text{by (7.5)}\end{aligned}$$

$$= (1 \times q_1)^*(\lambda_E \pi_2^* p_1^* w_2 + Sq^2 \lambda_E).$$

Consider the following commutative diagram:

$$\begin{array}{ccc}
H^{n+1}(\Omega_K L_\phi(Z, n) \times_K E; Z_2) & \xleftarrow{\nu \otimes \pi_2^*} & \sum_{i=0}^{2} H^{n-i}(K(Z, n-1); Z_2) \otimes H^i(E; Z_2) \\
\downarrow{\scriptstyle (1 \times q_1)^*} & & \downarrow{\scriptstyle 1 \otimes q_1^*} \\
H^{n+1}(\Omega_K L_\phi(Z, n) \times_K BO(n-1); Z_2) & \xleftarrow{\nu \otimes \pi_2^*} & \sum_{i=0}^{2} H^{n-i}(K(Z, n-1); Z_2) \otimes H^i(BO(n-1); Z_2).
\end{array}$$

The horizontal maps are monomorphisms by Lemma 6.2. Further $q_1^*: H^i(E; Z_2) \to H^i(BO(n-1); Z_2)$ is monomorphic for $i \leq 2$ because q_1 is an n-equivalence, and so the vertical map in the right hand side is a monomorphism. This result and the above equality imply (8.4), and we have the following

PROPOSITION 8.5. *The composition of the fibrations* $T \xrightarrow{p_2} E \xrightarrow{p_1} BO(n)$ *in the diagram* (7.7) *is stable by the map* d *in* (8.1).

§9. Enumeration of cross sections of sphere bundles

Let ξ be a real n-plane bundle over a CW-complex X. If ξ has a non-zero cross section, $cross(\xi)$ denotes the set of (free) homotopy classes of non-zero cross sections of ξ. The space X is a $BO(n)$-space with the classifying map $\xi: X \to BO(n)$ of ξ. Then the relation

$$cross(\xi) = [X, BO(n-1)]_{BO(n)}$$

follows from [11, Lemma 2.2]. If the dimension of X is less than $n+1$ and $n \geq 4$, then

$$[X, BO(n-1)]_{BO(n)} = [X, T]_{BO(n)}$$

follows from [11, Theorem 3.2], because $q_2: BO(n-1) \to T$ is an $(n+1)$-equivalence. On the other hand, it follows from Theorem A of §2 that

$$[X, T]_{BO(n)} = \operatorname{Ker} \Theta_\xi \times \operatorname{Coker} \Theta'_\xi.$$

Here

$$\Theta_\xi: [X, \Omega_K L_\phi(Z, n)]_K \longrightarrow [X, K(Z_2, n+1)] = H^{n+1}(X; Z_2) = 0,$$

$$\Theta'_\xi: [X, \Omega_K^2 L_\phi(Z, n)]_K \longrightarrow [X, \Omega K(Z_2, n+1)] = H^n(X; Z_2),$$

and $\Theta'_\xi([a]) = [d'(a, \xi)]$, where $d': (\Omega_K^2 L_\phi(Z, n) \times_K BO(n), BO(n)) \to (\Omega K(Z_2, n+1), *)$ is the map given by $d'(a, x)(t) = d(a(t), x)$ (cf. (2.4)). Also,

$$[X, \Omega_K L_\phi(Z, n)]_K = H^{n-1}(X; \underline{Z}), \qquad [X, \Omega_K^2 L_\phi(Z, n)]_K = H^{n-2}(X; \underline{Z})$$

by Proposition 6.1, where $\underline{Z}$ is the local system on X associated with ξ given by the composition

$$\pi_1(X) \xrightarrow{\xi_*} \pi_1(BO(n)) \xrightarrow{q_* W_*} \pi_1(K) = Z_2 \xrightarrow{\phi} \mathrm{Aut}(Z), \qquad (K = K(Z_2, 1)).$$

Now, we show that the homomorphism $\Theta'_\xi\colon H^{n-2}(X; \underline{Z}) \to H^n(X; Z_2)$ is given by

$$(9.1) \qquad \Theta'_\xi(a) = (\rho_2 a) w_2(\xi) + Sq^2 \rho_2 a, \qquad \text{for any,} \quad a \in H^{n-2}(X; \underline{Z}),$$

where ρ_2 is the mod 2 reduction and $w_2(\xi)$ is the second Stiefel-Whitney class of ξ.

Let $\iota' \in H^n(K(Z_2, n); Z_2)$ be the fundamental class of $K(Z_2, n)$. Then

$$(9.2) \qquad \Theta'_\xi([a]) = (a, \xi)^* d'^*(\iota')$$

for any K-map $a\colon X \to \Omega_K^2 L_\phi(Z, n)$. Consider the two commutative diagrams of the mod 2 cohomology groups

$$\begin{array}{ccccc}
H^i(K', *) & \xrightarrow[\approx]{r^*} & H^i(PK', \Omega K') & \xleftarrow[\approx]{\delta} & H^{i-1}(\Omega K', *) \\
\downarrow{\scriptstyle d^*} & & \downarrow{\scriptstyle d'^*} & & \downarrow{\scriptstyle d'^*} \\
H^i(\Omega' \times_K B, B) & \xrightarrow{r^*} & H^i(P_K \Omega' \times_K B, \Omega_K \Omega' \times_K B) & \xleftarrow[\approx]{\delta} & H^{i-1}(\Omega_K \Omega' \times_K B, B),
\end{array}$$

$$\begin{array}{ccccc}
H^{n-1}(\Omega', K) & \xrightarrow{r^*} & H^{n-1}(P_K \Omega', \Omega_K \Omega') & \xleftarrow[\approx]{\delta} & H^{n-2}(\Omega_K \Omega', K) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\
H^{n-1}(K'') & \xrightarrow[\approx]{r^*} & H^{n-1}(PK'', \Omega K'') & \xleftarrow[\approx]{\delta} & H^{n-2}(\Omega K''),
\end{array}$$

where $K' = K(Z_2, n+1)$, $\Omega' = \Omega_K L_\phi(Z, n)$, $B = BO(n)$, $K'' = \Omega K(Z, n)$ and d': $P_K \Omega' \times_K B \to PK'$ is the map defined by the same equation $d'(b, x)(t) = d(b(t), x)$ as (2.4). Since $\delta^{-1} r^*(\iota_{n-1}) = \iota_{n-2}$, we have

$$\delta^{-1} r^* \lambda = \lambda', \qquad \delta^{-1} r^* \lambda_B = \lambda'_B,$$

where $\lambda \in H^{n-1}(\Omega', K)$ and $\lambda' \in H^{n-2}(\Omega_K \Omega', K)$ are the fundamental classes of the fibrations $\Omega' \to K$ and $\Omega_K \Omega' \to K$ of Proposition 6.1 and $\lambda_B = \pi_1^* \lambda \in H^{n-1}(\Omega' \times_K B, B)$, $\lambda'_B = \pi_1^* \lambda' \in H^{n-2}(\Omega_K \Omega' \times_K B, B)$. Therefore, by the equation $d^*(\iota) = \lambda_B \pi_2^* w_2 + Sq^2 \lambda_B$ by (8.1) and $\delta^{-1} r^*(\iota) = \iota'$, we have $d'^*(\iota') = \delta^{-1} r^* d^*(\iota) = \lambda'_B \pi_2^* w_2 + Sq^2 \lambda'_B = (\pi_1^* \lambda')(\pi_2^* w_2) + Sq^2 \pi_1^* \lambda'$. This equality and (9.2) yield

$$\begin{aligned}
\Theta'_\xi([a]) &= (a, \xi)^*((\pi_1^* \lambda') \cdot (\pi_2^* w_2) + Sq^2 \pi_1^* \lambda') \\
&= (a^* \lambda')(\xi^* w_2) + Sq^2 a^* \lambda'.
\end{aligned}$$

Therefore, the homomorphism $\Theta'_\xi\colon H^{n-2}(X;\ \underline{Z})\to H^n(X;\ Z_2)$ is given by

$$\Theta'_\xi(a)=(\rho_2 a)w_2(\xi)+Sq^2\rho_2 a$$

by Proposition 6.1, where $w_2(\xi)$ is the second Stiefel-Whitney class of ξ and ρ_2 is the mod 2 reduction.

From the consideration made above, we obtain the following

THEOREM B. *Let ξ be a real n-plane bundle over a CW-complex X of dimension less than $n+1$ and let $n\geq 4$. If ξ admits a non-zero cross section, then the set cross(ξ) of homotopy classes of non-zero cross sections of ξ is, as a set, given by*

$$cross(\xi)=H^{n-1}(X;\ \underline{Z})\times \operatorname{Coker}\Theta,$$

where $\Theta\colon H^{n-2}(X;\ \underline{Z})\to H^n(X;\ Z_2)$ is defined by

$$\Theta(a)=(\rho_2 a)w_2(\xi)+Sq^2\rho_2 a,\qquad for\quad a\in H^{n-2}(X;\ \underline{Z}),$$

ρ_2 is the mod 2 *reduction and $\underline{Z}$ is the local system on X associated with ξ.*

Chapter III. Enumeration of embeddings

§10. Enumeration of embeddings of manifolds

Let M be an n-dimensional differentiable closed manifold. Let M^* be the reduced symmetric product of M obtained from $M\times M-\varDelta$ ($\varDelta$ is the diagonal of M) by identifying (x, y) and (y, x) and let η be the real line bundle over M^* associated with the double covering $M\times M-\varDelta\to M^*$. Then the set $[M\subset R^{2n-1}]$ of isotopy classes of embeddings of M into the real $(2n-1)$-space R^{2n-1} for $n\geq 6$ is equivalent to the set of homotopy classes of cross sections of the S^{2n-2}-bundle $(M\times M-\varDelta)\times_{Z_2}S^{2n-2}\to M^*$ by the theorem of A. Haefliger [5, §1]. Because this bundle is the associated S^{2n-2}-bundle of $(2n-1)\eta$, we have

$$[M\subset R^{2n-1}]=cross((2n-1)\eta).$$

Since M^* is an open $2n$-dimensional manifold, there is a proper Morse function on M^* with no critical point of index $2n$ by [15, Lemma 1.1], and so M^* has the homotopy type of a CW-complex of dimension less than $2n$ by [14, Theorem 3.5]. Therefore we have the following proposition from Theorem B of §9 and the fact

$$w_2((2n-1)\eta)=\binom{2n-1}{2}w_1(\eta)^2.$$

PROPOSITION 10.1. *Let $n \geq 6$ and let M be an n-dimensional differentiable closed manifold which is embedded in R^{2n-1}. Then the set $[M \subset R^{2n-1}]$ of isotopy classes of embeddings of M into R^{2n-1} is, as a set, given by*

$$[M \subset R^{2n-1}] = H^{2n-2}(M^*; \underline{Z}) \times \operatorname{Coker} \Theta,$$

where the homomorphism

$$\Theta: H^{2n-3}(M^*; \underline{Z}) \longrightarrow H^{2n-1}(M^*; Z_2)$$

is given by

$$\Theta(a) = \binom{2n-1}{2} w_1(\eta)^2 \rho_2 a + Sq^2 \rho_2 a,$$

$w_1(\eta)$ is the first Stiefel-Whitney class of the double covering $M \times M - \Delta \to M^*$ and $\underline{Z}$ is the local system on M^* defined from this double covering.

COROLLARY 10.2. *In addition to the conditions of the above proposition, we assume that $H_1(M; Z_2) = 0$. Then we have*

$$[M \subset R^{2n-1}] = H^{2n-2}(M^*; \underline{Z}).$$

PROOF. Because $H_1(M; Z_2) = 0$, we have $H_1(M \times M, \Delta; Z_2) = 0$ by the exact sequence of the pair $(M \times M, \Delta)$. The Thom-Gysin exact sequence

$$\longrightarrow H^{2n-1}(M \times M - \Delta; Z_2) \longrightarrow H^{2n-1}(M^*; Z_2) \longrightarrow H^{2n}(M^*; Z_2) \quad (=0)$$

and the Poincaré duality $H^{2n-1}(M \times M - \Delta; Z_2) = H_1(M \times M, \Delta; Z_2)$ $(=0)$ yield $H^{2n-1}(M^*; Z_2) = 0$, which implies that $\operatorname{Coker} \Theta = 0$.

REMARK. There is a description in [6, 1.3, e, Théorème] that

$$[M \subset R^{2n-1}] = H^{2n-2}(M^*; \underline{Z}) = \begin{cases} H^{n-2}(M; Z) & \text{if } n-1 \text{ is odd} \\ H^{n-2}(M; Z_2) & \text{if } n-1 \text{ is even,} \end{cases}$$

under the assumption $H_1(M; Z) = 0$.

§11. Enumeration of embeddings of real projective spaces RP^n

Our purpose in this section is to prove the following

THEOREM C. *Let $n \neq 2^r$ and let $n \geq 6$. Then the n-dimensional real projective space RP^n is embedded into the real $(2n-1)$-space R^{2n-1}. Furthermore, the cardinality $\#[RP^n \subset R^{2n-1}]$ of the set $[RP^n \subset R^{2n-1}]$ of isotopy classes of embeddings of RP^n into R^{2n-1} is given by*

$$\sharp[RP^n \subset R^{2n-1}] = \begin{cases} 4 & n \equiv 3(4) \\ 2 & otherwise. \end{cases}$$

The first half of this theorem is shown in [1, Theorem 1] for even n and in [9, Theorem 1.1] for odd n. Thus we concentrate ourselves on the study of the set $[RP^n \subset R^{2n-1}]$. Let η be the real line bundle associated with the double covering $RP^n \times RP^n - \Delta \to (RP^n)^*$. Then the set $[RP^n \subset R^{2n-1}]$ is equivalent to the set $cross((2n-1)\eta)$ (cf. § 10).

In [8, (2.5–6)],

(11.1) *there is a commutative diagram of the double coverings*

$$\begin{array}{ccc} V_{n+1,2}/(Z_2+Z_2) = Z_{n+1,2} & \xrightarrow{f'} & RP^n \times RP^n - \Delta \\ \downarrow & & \downarrow \\ V_{n+1,2}/D_4 = SZ_{n+1,2} & \xrightarrow{f} & (RP^n)^*, \end{array}$$

where $V_{n+1,2}$ is the Stiefel manifold of 2-frames in R^{n+1}, D_4 is the dihedral group of order 8, both maps f and f' are homotopy equivalences and both spaces $Z_{n+1,2}$ and $SZ_{n+1,2}$ are $(2n-1)$-dimensional manifolds.

The mod 2 cohomology of $(RP^n)^*$ (and so $SZ_{n+1,2}$) is calculated by S. Feder [2], [3] and D. Handel [8] and is given as follows:

(11.2) *Let $G_{n+1,2}$ be the Grassmann manifold of 2-planes in the real $(n+1)$-space R^{n+1}. Then the* mod 2 *cohomology of $G_{n+1,2}$ is given by*

$$H^*(G_{n+1,2}; Z_2) = Z_2[x, y]/(a_n, a_{n+1}),$$

where $\deg x = 1$, $\deg y = 2$ *and* $a_r = \sum_i \binom{r-i}{i} x^{r-2i} y^i$ $(r = n, n+1)$, *and there is a relation*

$$x^{2i} y^{n-i-1} \neq 0 \quad \text{if and only if} \quad i = 2^t - 1 \quad \text{for some} \quad t.$$

$H^*((RP^n)^*; Z_2)$ has $\{1, v\}$ as a basis of an $H^*(G_{n+1,2}; Z_2)$-module, where $v \in H^1((RP^n)^*; Z_2)$ is the first Stiefel-Whitney class of the double covering $RP^n \times RP^n - \Delta \to (RP^n)^*$ and there are the relations

$$v^2 = vx, \; Sq^1 y = xy \quad \text{and} \quad x^{2^{r+1}-1} = 0 \quad \text{for} \quad n = 2^r + s, \; 0 \leq s < 2^r.$$

By the Poincaré duality and (11.1–2),

(11.3) *$H^t((RP^n)^*; Z_2)$ $(n = 2^r + s, \; 0 < s < 2^r)$ for $2n-3 \leq t \leq 2n-1$ are given as follows* [20], [21]:

t	$H^t((RP^n)^*; Z_2)$	basis
$2n-1$	Z_2	$vx^{2^{r+1}-2}y^s$
$2n-2$	Z_2+Z_2	$vx^{2^{r+1}-3}y^s,\ x^{2^{r+1}-2}y^s$
$2n-3$	$Z_2+Z_2+Z_2$	$vx^{2^{r+1}-4}y^s,\ x^{2^{r+1}-3}y^s,\ vx^{2^{r+1}-2}y^{s-1}$

To apply Proposition 10.1, we must study the cohomology groups $H^i((RP^n)^*; \underline{Z})$ $(i=2n-2, 2n-3)$ with coefficients in the local system associated with the double covering $RP^n\times RP^n-\Delta\to(RP^n)^*$.

Let $\rho_2\colon H^i((RP^n)^*; \underline{Z})\to H^i((RP^n)^*; Z_2)$ be the mod 2 reduction.

LEMMA 11.4. *Let* $n\equiv 0(2)$. *Then* $H^{2n-2}((RP^n)^*; \underline{Z})=Z_2$ *and* $\rho_2 H^{2n-3}((RP^n)^*; \underline{Z})=Z_2+Z_2$ *generated by* $\{vx^{2^{r+1}-4}y^s,\ vx^{2^{r+1}-2}y^{s-1}\}$.

LEMMA 11.5. *Let* $n\equiv 1(2)$. *Then* $H^{2n-2}((RP^n)^*; \underline{Z})=Z_2$ *and* $\rho_2 H^{2n-3}((RP^n)^*; \underline{Z})=Z_2+Z_2$ *generated by* $\{vx^{2^{r+1}-4}y^s+x^{2^{r+1}-3}y^s,\ vx^{2^{r+1}-2}y^{s-1}\}$.

The proofs of Lemmas 11.4–5 will be made in the next section and we go on proving Theorem C. By Proposition 10.1,

$$[RP^n\subset R^{2n-1}] = H^{2n-2}((RP^n)^*; \underline{Z})\times \operatorname{Coker}\Theta,$$

where

$$\Theta\colon H^{2n-3}((RP^n)^*; \underline{Z})\longrightarrow H^{2n-1}((RP^n)^*; Z_2),\quad \Theta(a) = Sq^2\rho_2 a+\binom{2n-1}{2}v^2\rho_2 a.$$

Now, there are relations

$$Sq^2(vx^{2^{r+1}-2}y^{s-1}) = (s-1)vx^{2^{r+1}-2}y^s,$$

$$Sq^2(vx^{2^{r+1}-4}y^s) = \left(s+\binom{s}{2}\right)vx^{2^{r+1}-2}y^s,$$

$$Sq^2(x^{2^{r+1}-2}y^s) = 0,$$

which are easily seen by using (11.2) and the fact $Sq^2(y^t)=ty^{t+1}+\binom{t}{2}x^2y^t$. Therefore we have

$$\left(Sq^2+\binom{2n-1}{2}v^2\right)(vx^{2^{r+1}-2}y^{s-1}) = \begin{cases} vx^{2^{r+1}-2}y^s & n\equiv 0(2) \\ 0 & n\equiv 1(2), \end{cases}$$

$$\left(Sq^2+\binom{2n-1}{2}v^2\right)(vx^{2^{r+1}-4}y^s+x^{2^{r+1}-3}y^s) = \begin{cases} vx^{2^{r+1}-2}y^s & n\equiv 1(4) \\ 0 & n\equiv 3(4). \end{cases}$$

From Lemmas 11.4–5 and (11.3), these relations show that

$$\operatorname{Coker}\Theta = \begin{cases} Z_2 & n \equiv 3(4) \\ 0 & \text{elsewhere.} \end{cases}$$

Since $H^{2n-2}((RP^n)^*; \underline{Z}) = Z_2$ by Lemmas 11.4–5, we have Theorem C.

§ 12. Proofs of Lemmas 11.4–5

There are two exact sequences of cohomology groups associated with the double covering $RP^n \times RP^n - \Delta \to (RP^n)^*$ (cf. [17, pp. 282–283]), which is called the Thom-Gysin exact sequence:

$$(12.1) \quad \cdots \to H^{i-1}(M^*; Z) \to H^i(M^*; \underline{Z}) \to H^i(M \times M - \Delta; Z) \to H^i(M^*; \underline{Z}) \to \cdots,$$

$$\cdots \to H^{i-1}(M^*; \underline{Z}) \to H^i(M^*; Z) \to H^i(M \times M - \Delta; Z) \to H^i(M^*; Z) \to \cdots,$$

where $M = RP^n$. Moreover, there is the Bockstein exact sequence [18]

$$(12.2) \qquad \cdots \longrightarrow H^{i-1}(M^*; Z_2) \xrightarrow{\tilde{\beta}_2} H^i(M^*; \underline{Z}) \xrightarrow{\times 2} H^i(M^*; \underline{Z})$$

$$\xrightarrow{\rho_2} H^i(M^*; Z_2) \xrightarrow{\tilde{\beta}_2} \cdots, \qquad (M = RP^n),$$

associated with the short exact sequence $0 \longrightarrow \underline{Z} \xrightarrow{\times 2} \underline{Z} \xrightarrow{\rho_2} Z_2 \longrightarrow 0$. The homomorphism $\tilde{\beta}_2$ is called the twisted Bockstein operator, and by [4] and [16], the homomorphism $\rho_2\tilde{\beta}_2$: $H^{i-1}((RP^n)^*; Z_2) \to H^i((RP^n)^*; Z_2)$ is given by

$$(12.3) \qquad \rho_2\tilde{\beta}_2(a) = Sq^1 a + va \quad \textit{for} \quad a \in H^{i-1}((RP^n)^*; Z_2),$$

where v is the first Stiefel-Whitney class of the double covering $RP^n \times RP^n - \Delta \to (RP^n)^*$.

From now on, set $n = 2^r + s$, $0 < s < 2^r$.

PROOF OF LEMMA 11.4. Since n is even, the space $SZ_{n+1,2}$ is an orientable $(2n-1)$-dimensional manifold by [2, § 3] and so it follows that

$$H^{2n-1}(SZ_{n+1,2}; Z) = Z,$$

$$H^{2n-2}(SZ_{n+1,2}; Z) = H_1(SZ_{n+1,2}; Z) = D_4/[D_4, D_4] = Z_2 + Z_2.$$

Since the total space $Z_{n+1,2}$ is also orientable and $\pi_1(Z_{n+1,2}) = Z_2 + Z_2$, the following relations hold:

$$H^{2n-1}(Z_{n+1,2}; Z) = Z, \quad H^{2n-2}(Z_{n+1,2}; Z) = Z_2 + Z_2.$$

Hence (11.1) and the Thom-Gysin exact sequence (12.1) give rise to the two exact sequences

$$Z_2+Z_2\to H^{2n-1}((RP^n)^*;\ \underline{Z})\to Z\to Z\to 0,$$

$$Z_2+Z_2\to H^{2n-2}((RP^n)^*;\ \underline{Z})\to Z\to Z\to H^{2n-1}((RP^n)^*;\ \underline{Z})\to 0.$$

A simple calculation yields

$$H^{2n-2}((RP^n)^*;\ \underline{Z})=Z_2 \quad \text{or} \quad Z_2+Z_2 \quad \text{or} \quad 0. \tag{12.4}$$

On the other hand, there are relations

$$\rho_2\tilde{\beta}_2(x^{2^{r+1}-2}y^s) = vx^{2^{r+1}-2}y^s,$$

$$\rho_2\tilde{\beta}_2(x^{2^{r+1}-3}y^s) = x^{2^{r+1}-2}y^s+vx^{2^{r+1}-3}y^s,$$

$$\rho_2\tilde{\beta}_2(x^{2^{r+1}-4}y^s) = vx^{2^{r+1}-4}y^s,\quad \rho_2\tilde{\beta}_2(x^{2^{r+1}-2}y^{s-1}) = vx^{2^{r+1}-2}y^{s-1},$$

by (11.2) and (12.3) since n is even. Consider the Bockstein exact sequence (12.2)

$$\cdots\longrightarrow H^{2n-3}((RP^n)^*;\ \underline{Z}) \xrightarrow{\rho_2} H^{2n-3}((RP^n)^*;\ Z_2) \xrightarrow{\tilde{\beta}_2} H^{2n-2}((RP^n)^*;\ \underline{Z})$$
$$\xrightarrow{\times 2} H^{2n-2}((RP^n)^*;\ \underline{Z}) \xrightarrow{\rho_2} H^{2n-2}((RP^n)^*;\ Z_2) \longrightarrow \cdots.$$

The last three relations of the above and (11.3) show the last half of Lemma 11.4. Also, the first two relations of the above show that the image $\rho_2 H^{2n-2}((RP^n)^*;\ \underline{Z})=Z_2$ generated by $x^{2^{r+1}-3}y^s+vx^{2^{r+1}-3}y^s$. Therefore we have the first half of Lemma 11.4 by the above Bockstein exact sequence, (11.3) and (12.4).

PROOF OF LEMMA 11.5. Consider the Bockstein exact sequence (12.2)

$$H^{2n-3}((RP^n)^*;\ \underline{Z}) \xrightarrow{\rho_2} H^{2n-3}((RP^n)^*;\ Z_2) \xrightarrow{\tilde{\beta}_2} H^{2n-2}((RP^n)^*;\ \underline{Z})$$
$$\xrightarrow{\times 2} H^{2n-2}((RP^n)^*;\ \underline{Z}) \xrightarrow{\rho_2} H^{2n-2}((RP^n)^*;\ Z_2).$$

Since n is odd, there are relations

$$\rho_2\tilde{\beta}_2(x^{2^{r+1}-2}y^s) = vx^{2^{r+1}-2}y^s,$$

$$\rho_2\tilde{\beta}_2(x^{2^{r+1}-3}y^s) = vx^{2^{r+1}-3}y^s,$$

$$\rho_2\tilde{\beta}_2(vx^{2^{r+1}-3}y^{s-1}) = vx^{2^{r+1}-2}y^{s-1},$$

$$\rho_2\tilde{\beta}_2(x^{2^{r+1}-4}y^s) = vx^{2^{r+1}-4}y^s+x^{2^{r+1}-3}y^s,$$

by (11.2) and (12.3). Therefore, the lemma can be proved in the same way as the proof of Lemma 11.4, by using the Bockstein exact sequence (12.2) and (11.3).

References

[1] D. B. A. Epstein and R. L. E. Schwarzenberger, *Imbeddings of real projective spaces*, Ann. of Math. **76** (1962), 180–184.

[2] S. Feder, *The reduced symmetric product of a projective space and the embedding problem*, Bol. Soc. Mat. Mexicana **12** (1967), 76–80.

[3] ———, *The reduced symmetric product of projective spaces and the generalized Whitney theorem*, Illinois J. Math. **16** (1972), 323–329.

[4] R. Greenblatt, *The twisted Bockstein coboundary*, Proc. Camb. Phil. Soc. **61** (1965), 295–297.

[5] A. Haefliger, *Plongements différentiables dans le domaine stable*, Comment. Math. Helv. **37** (1962), 155–176.

[6] ———, *Plongement de variétés dans le domaine stable*, Séminaire Bourbaki **150** (1962/63), n°. 245.

[7] A. Haefliger and M. W. Hirsch, *On the existence and classification of differentiable embeddings*, Topology **2** (1963), 129–135.

[8] D. Handel, *An embedding theorem for real projective spaces*, Topology **7** (1968), 125–130.

[9] I. M. James, *Some embeddings of projective spaces*, Proc. Camb. Phil. Soc. **55** (1959), 294–298.

[10] I. M. James and E. Thomas, *An approach to the enumeration problem for non-stable vector bundles*, J. Math. Mech. **14** (1965), 489–506.

[11] ———————, *Note on the classification of cross-sections*, Topology **4** (1966), 351–359.

[12] ———————, *On the enumeration of cross-sections*, Topology **5** (1966), 95–114.

[13] J. F. McClendon, *Obstruction theory in fiber spaces*, Math. Z. **120** (1971), 1–17.

[14] J. Milnor, Morse Theory, Princeton Univ. Press, 1963.

[15] A. Phillips, *Submersions of open manifolds*, Topology **6** (1967), 171–206.

[16] H. Samelson, *A note on the Bockstein operator*, Proc. Amer. Math. Soc. **15** (1964), 450–453.

[17] E. H. Spanier, Algebraic Topology, McGraw-Hill, 1966.

[18] N. Steenrod, The Topology of Fibre Bundles, Princeton Univ. Press, 1951.

[19] E. Thomas, Seminar on Fiber Spaces, Lecture Notes in Math. **13**, Springer-Verlag, 1966.

[20] T. Yasui, *Note on the enumeration of embeddings of real projective spaces*, Hiroshima Math. J. **3** (1973), 409–418.

[21] ———, ———————, II, Hiroshima Math. J. **6** (1976), 221–225.

Department of Mathematics,
Faculty of Education,
Yamagata University