Rational integrals of the second kind on a complex projective manifold and its primitive cohomology

Rational integrals of the second kind on a complex projective manifold and its primitive cohomology *†

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ABSTRACT: Let X be a complex algebraic manifold of dimension $n+1$ embedded in a sufficiently higher dimensional complex projective space $\mathbb{P}^N(\mathbb{C})$, and Y a generic hyperplane section of X. By sheaf cohomological method, we prove the well-known facts that the primitive cohomology group $H^p(X,\mathbb{C})_0$ $(1 \leq p \leq n+1)$ is isomorphic to the De Rham cohomology group $I^p(X,(p+1)Y)_0$ of closed rational p-forms of the 2nd kind on X, having poles of order $p+1$ (at most) along Y only, and that the *Hodge filtration* of $H^p(X,\mathbb{C})_0$ is isomorphic to the one of $I^p(X,(p+1)Y)_0$ defined by the order of poles along Y. On the other hand, we have a long exact sequence of cohomology

$$\to H^p(X,\mathbb{C}) \xrightarrow{r^p} H^p(X-Y,\mathbb{C}) \xrightarrow{R^p} H^{p-1}(Y,\mathbb{C}) \xrightarrow{G^{p-1}} H^{p+1}(X,\mathbb{C}) \to \cdots,$$

which is dual to

$$\to H_p(X,\mathbb{C}) \xleftarrow{\iota_p} H^c_p(X-Y,\mathbb{C}) \xleftarrow{\tau_{p-1}} H_{p-1}(Y,\mathbb{C}) \xleftarrow{G_{p+1}} H_{p+1}(X,\mathbb{C}) \to \cdots,$$

where H^c_* denotes compact support homology group (cf. (1.2)). Using these exact sequences, we describe the mixed Hodge structure on $H^p(X-Y,\mathbb{C})$ and the Hodge filtration of the middle primitive cohomology group $H^n(Y,\mathbb{C})_0$ of Y in terms of rational integrals on X.

Key words: Primitive cohomology, Rational integral of the 2nd kind, Generalized Poincaré résidue map, Hodge filtration, Mixed Hodge structure

Contents

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Summary

Let X be a non-singular irreducible algebraic variety of dimension $n+1$ embedded in a sufficiently higher dimensional complex projective space $\mathbb{P}^N(\mathbb{C})$, and Y a generic hyperplane section of X. We shall use the following notation:

Ω_X^q : the sheaf of germs of holomorphic q-forms on X,
$\Omega_X^q(kY)$: the sheaf of germs of meromorphic q-forms having poles of order k (at most) along Y as their only singularities on X,
$\Omega_X^q(*Y)$: the sheaf of germs of meromorphic q-forms having poles of arbitrary order along Y as their only singularities on X,
$\Omega_X^q(\log Y)$: the sheaf of germs of meromorphic q-forms having logarithmic poles (at most) along Y as their only singularities on X.

We denote by Φ_X^q, $\Phi_X^q(kY)$, e.t.c., the subsheaves consisting of closed forms of each ones. On the complex $\Omega_X^{\cdot}$ we define a decreasing filtration $F = \{F^k\}_{0 \leq k \leq n+1}$ (the Hodge filtration) by the subcomplexes

$$F^k(\Omega_X^{\cdot})^q = \begin{cases} 0 & q < k \\ \Omega_X^q & k \leq q. \end{cases}$$

On the complex $\Omega_X^{\cdot}(\log Y)$ we define the Hogde filtartion similarly, and another increasing filtration $W = \{W_0 \subset W_1\}$ (the weight filtration) by

$$W_0(\Omega_X^{\cdot}(\log Y)) = \Omega_X^{\cdot}, \quad W_1(\Omega_X^{\cdot}(\log Y)) = \Omega_X^{\cdot}(\log Y).$$

Then $(\Omega_X^{\cdot}, F)$ becomes the *cohomological Hodge complex*, and $(\Omega_X^{\cdot}(\log Y), W, F)$ the *cohomological mixed Hodge complex* (cf. §3). They induce the Hodge structure on the cohomology $H^p(X, \mathbb{C})$, and the mixed Hodge structure on the cohomology $H^p(X - Y, \mathbb{C})$. We define

$$I_k^p(X, (p+1)Y) := \frac{\Gamma(X, \Phi_X^p((p-k+1))Y)}{d\Gamma(X, \Omega_X^{p-1}((p-k))Y)} \quad (0 \leq k \leq p)$$

and denote by $I_k^p(X, (p+1)Y)_0$ the subspace of $I_k^p(X, (p+1)Y)$ generated by closed moromorphic p-forms of the *second kind* (cf. Definition 2.2). Assume that

$$H^p(X, \Omega_X^q(kY)) = 0 \quad \text{for} \quad p \geq 1, q \geq 0 \quad \text{and} \quad k \geq 1.$$

Then we have

$$\begin{aligned}
&F^k H^p(X - Y, \mathbb{C}) \simeq I_k^p(X, (p+1)Y) \quad 0 \leq k \leq p, \\
&F^k H^p(X, \mathbb{C})_0 \simeq I_k^p(X, (p+1)Y)_0 \quad 0 \leq k \leq p, \\
&Gr_q^{W[q]} H^q(X - Y, \mathbb{C}) = W[q]_q H^q(X - Y, \mathbb{C}) = I^q(X, *Y)_0, \\
&Gr_{q+1}^{W[q]} H^q(X - Y, \mathbb{C}) = I^q(X, *Y)/I^q(X, *Y)_0, \\
&F^k Gr_q^{W[q]} H^q(X - Y, \mathbb{C}) \simeq F^k H^q(X, \mathbb{C})_0, \quad \text{and} \\
&F^k Gr_{q+1}^{W[q]} H^q(X - Y, \mathbb{C}) \simeq \operatorname{Ker}\{F[-1]^k H^{q-1}(Y, \mathbb{C})_0 \xrightarrow{G} F^k H^{q+2}(Y, \mathbb{C})\},
\end{aligned}$$

where $H^p(X, \mathbb{C})_0$ denotes the p-th primitive coholomology of X, F^k the k-th Hodge filtration of cohomology, and $W[q]$ the shift to the right on the degree of W by q. (Theorem 3.1, Theorem 3.3 and Proposition 2.3). Furthermore, let Y' be a generic hypersurface of $\mathbb{P}^N(\mathbb{C})$ of sufficiently higher degree so that

$$H^p(Y, \Omega_Y^q(kZ)) = 0 \quad \text{for} \quad p \geq 1, q \geq 0 \quad \text{and} \quad k \geq 1,$$

where $Z = Y \cdot Y'$. Then we can define the *generalized Poincaré résidue map*

$$\textit{Rés} : I^{n+1}(X, (n+2)Y) \to I^n(Y, (n+1)Z)_0$$

and prove that

$$\begin{aligned} F^k H^n(Y,\mathbb{C})_0 &\simeq I_k^n(Y,(n+1)Z)_0 \\ &\simeq Rés(I_{k+1}^{n+1}(X,(n+2)Y)) \oplus r^n(I_k^n(X,(n+1)Y')_0)), \end{aligned}$$

where r^n denotes the map induced by the natural map $H^n(X,\mathbb{C})_0 \to H^n(Y,\mathbb{C})_0$ (Thorem 4.1). These results might be considered as a generalization of those by P. A. Griffith in the case of a hypersurface in a complex projective space (cf. [9]).

1 Some remarks on primitive cohomology and homology of algebraic manifolds

Let X be a non-singular irreducible algebraic variety of dimension $n+1$ embedded in a higher dimensional complex projective space $\mathbb{P}^N(\mathbb{C})$ and Y a generic hyperplane section of X. In what follows we call such Y a *prime section* of X. We denote by Ω the restriction to X of the fundamental form of the Fubini-Study metric on $\mathbb{P}^N(\mathbb{C})$. Ω is a closed 2-form whose cohomology class $[\Omega] \in H^2(X,\mathbb{C})$ is the Poincaré dual of the homology class $[Y] \in H_{2n}(X,\mathbb{C})$ associated to the the prime section Y. We define $L(\omega) := \Omega \wedge \omega$ for a ($\mathbb{C}$-valued) C^∞ diferential q-form ω on X. If ω is a closed form (resp. detived form), then $L(\omega)$ is also a closed form (resp. derived form) for Ω is a closed form. Hence L define a homomorphism $H^q(X,\mathbb{C}) \to H^{q+2}(X,\mathbb{C})$ $(0 \le q \le 2n)$. Throughput this paper we always idetify the ordinary cohomology with the De Rham cohomology. We call this cohomology operator *Hodge operator* and denote it by the same letter L.

Definition 1.1. A C^∞ differential q-form $(0 \le q \le n+1)$ ω is said to be *primitive* if $L^{n-q+2}(\omega) = 0$ $(L^{n-q+2} = \underbrace{L \circ \cdots \circ L}_{n-q+2 \text{ times}})$. A (De Rham) cohomology class containing a closed, primitive C^∞ differential form is said to be a *primitive* cohomology class.

We call the subgroup of $H^q(X,\mathbb{C})$ which consists of all primitive cohomology classes the *q-th primitive cohomology group* of X, which we denote by $H^q(X,\mathbb{C})_0$.

Remark 1.1. Originarlly, a C^∞ differetial q-form $(\le q \le n+1)$ ω on X is defined to be primitive if $\Lambda\omega = 0$, Λ is the adjoint operator of L with respect to the Hodge metric on X which is the restriction of the Fubini-Study metric on $\mathbb{P}^N(\mathbb{C})$. The above definition of primitive forms is equivalent to the original one (cf. [11]).

The following facts are fundamental for the Hodge operator L.

Theorem 1.1. (Hard Lefshets Theorem)

$$L^k : H^{n+1-k}(X,\mathbb{C}) \simeq H^{n+1+k}(X,\mathbb{C}) \quad (1 \le k \le n+1)$$

Theorem 1.2. (Lefshets decomposition)

(i) $L : H^{q-2}(X,\mathbb{C}) \to H^q(X,\mathbb{C})$ *is injective and*
$H^q(X,\mathbb{C}) \simeq LH^{q-2}(X,\mathbb{C}) \oplus H^q(X,\mathbb{C})_0 \qquad (2 \le q \le n+1).$

(ii) $H^{n+1+k}(X,\mathbb{C}) \simeq L^k H^{n+1+k}(X,\mathbb{C})_0 \oplus L^{k+1} H^{n-1-k}(X,\mathbb{C})$

By restriction C^∞ differential q-forms on X to Y, we obtain a cohomology map $r^q : H^q(X,\mathbb{C}) \to H^q(Y,\mathbb{C})$, for which the folowing holds.

Theorem 1.3. (Weak Lefshetz Theorem)

(i) $r^q : H^q(X,\mathbb{C}) \simeq H^q(Y,\mathbb{C}) \qquad (0 \le q \le n-1).$

(ii) $r^n : H^n(X,\mathbb{C}) \to H^n(Y,\mathbb{C})$ *is injective.*

For the proofs of the theorems above we refer to [11].

Corollary 1.4.

$$0 \to H^{n+1}(X,\mathbb{C})_0 \to H^{n+1}(X,\mathbb{C}) \xrightarrow{r^{n+1}} H^{n+1}(Y,\mathbb{C}) \to 0. \quad \textit{(exact)}$$

Proof. By (1.2), (i) and (1.1), we have

$$\begin{array}{ccccc} 0 & \to & H^{n-1}(X,\mathbb{C}) & \xrightarrow{L} & H^{n+1}(X,\mathbb{C}) \\ & & \simeq\Big\downarrow & & \Big\downarrow r^{n+1} \\ & & H^{n-1}(Y,\mathbb{C}) & \underset{\simeq}{\xrightarrow{L}} & H^{n+1}(Y,\mathbb{C}) \end{array}$$

and,

$$H^{n+1}(X,\mathbb{C}) = H^{n+1}(X,\mathbb{C})_0 \oplus LH^{n-1}(X,\mathbb{C}).$$

Therefore,

$$Ker\ r^{n+1} = H^{n+1}(X,\mathbb{C})_0$$

□

Corollary 1.5.

$$0 \to H^{n+1}(X,\mathbb{C})_0 \to H^{n+1}(X,\mathbb{C}) \xrightarrow{r^{n+1}} H^{n+1}(Y,\mathbb{C}) \to 0 \quad \text{(exact)}$$

In what follows, homology and cohomology are with coefficient in the complex number field if otherwise explicitly mentioned. Taking a topological tublar neighborhood U of Y in X, we consider the homology exact sequence concerning a pair of the topological spaces $(X, X-U)$, which is written as follows:

$$(1.1) \qquad \cdots \to H_q^c(X-U) \xrightarrow{i_q} H_q(X) \xrightarrow{j_q} H_q(X, X-U) \xrightarrow{\partial_q} H_{q-1}^c(X-U) \to \cdots,$$

where H_*^c denotes compact support homology groups. Since $X-U$ is a deformation retract of $X-U$, $H_q^c(X-U) \simeq H_q^c(X-Y)$. By the *excision axiom*, $H_q^c(X, X-U) \simeq H_q^c(U, \partial U)$. By the *Thom isomorphism*, $H_q^c(U,\partial U) \simeq H_{q-2}^c(Y)$ for $q \geq 2$. We obviously have $H_q(U, X-U) = 0$ for $0 \underline{<} q \underline{<} 1$. Therefore the homology exact sequence (1.1) is rewritten as follows:

$$(1.2) \qquad \cdots \to H_q^c(X-Y) \xrightarrow{\iota_q} H_q(X) \xrightarrow{G_q} H_{q-2}(Y) \xrightarrow{\tau_{q-2}} H_{q-1}^c(X-Y) \to \cdots,$$

where

(i) the map $\iota_q : H_q^c(X-U) \to H_q(X)$ is the one induced by the natural inclusion map $\iota : X - Y \to X$,

(ii) the map $G_q : H_q(X) \to H_{q-2}(Y)$ is the one which assignes each q-cycle on M to its intersection cycle with Y, and

(iii) the map $\tau_{q-2} : H_{q-2}(Y) \to H_{q-1}^c(X-U)$ is the one which assighns each $(q-2)$ cycle on Y, say γ, to the cycle $\partial U_{|\gamma}$ on $X-Y$, the restriction of ∂U over γ.

In the subsequence we denote the cycle $\partial U_{|\gamma}$ in (iii) above by $\tau(\gamma)$. Taking the cohomology exact sequence dual to (1.2), we have

$$(1.3) \qquad \cdots \to H^q(X-Y) \xleftarrow{r^q} H^q(X) \xleftarrow{G^{q-2}} H^{q-2}(Y) \xleftarrow{R^{q-1}} H^{q-1}(X-Y) \to \cdots,$$

Here the map $G^{q-2} : H^{q-2}(Y) \to H^q(X)$ is the so-called *Gysin map*. We are now going to describe the Gysin map G^{q-2} by use of differential forms. We take a sufficiently fine, finite open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X such that, in each open subset U_i, Y is defined by a holomorphic equation $\sigma_i = 0$. We put $t_{ij} = \sigma_i/\sigma_j$ for each pair of indexes (i,j) with $U_i \cap U_j \neq \emptyset$. Then the system of transition functions, with respect to the covering $\mathcal{U}$, of the line bundle $[Y]$ associated to Y are given by $\{t_{ij}\}$, and $\sigma = \{\sigma_i\}$ give rise to a cross-section of $[Y]$ whose zero locus is Y. We take a system $\{a_i\}$ of real positive functions a_i of class C^∞ defined in U_i, respectively, satisfying

$$\frac{a_i}{a_j} = |t_{ij}|^2, \quad \text{in } U_i \cap U_j \neq \emptyset.$$

The system $\{a_i\}$ defines a fiber metric on the line bundle $[Y]$. The length function $|\sigma|$ of the cross-section $\sigma = \{\sigma_i\}$ of $[Y]$ with respect to this fiber metric is given by

$$\begin{aligned} |\sigma| &= \sqrt{\sigma_i a_i \overline{\sigma_i}} \\ &= |\sigma_i|\sqrt{a_i} \end{aligned}$$

in each U_i. Note that $|\sigma|^2$ is a globally defined real non-negative function of class C^∞. We define

$$\begin{aligned} \eta &:= \frac{1}{2\pi i}\partial \log |\sigma|^2, \\ \omega &= \overline{\partial}\eta = \frac{1}{2\pi i}\overline{\partial}\partial \log |\sigma|^2. \end{aligned}$$

On each U_i, η and ω are written as

$$\begin{aligned} \eta &:= \frac{1}{2\pi i}(d\log \sigma_i + \partial \log a_i), \\ \omega &= \frac{1}{2\pi i}\overline{\partial}\partial \log a_i. \end{aligned}$$

Note that ω is a globally defined closed C^∞ form of type $(1,1)$ on X, representing the first Chern class $c_1([Y])$ of the line bundle $[Y]$. We denote by $A^*(X)$, $A^*(X-Y)$ and $A^*(Y)$ the De Rham complexes of $\mathbb{C}$-valued, C^∞ differential forms on X, $X-Y$ and Y, respectively.

Definition 1.2. $A^*(\log Y)$ is defined to be the sub-complex of $A^*(X-Y)$ generated by $A^*(X)$ and η.

A form $\varphi \in A^*(\log Y)$ may be (non-uniquely) written as

$$\varphi = \alpha \wedge \eta + \beta \tag{1.4}$$

where $\alpha, \beta \in A^*(V)$. The restriction $\alpha_{|Y} \in A^*(Y)$ is, however, not anbiguous. Hence we may define $R^* : A^*(\log Y) \to A^{*-1}(Y)$ by

$$R^*(\varphi) := 2\pi\sqrt{-1}\alpha_{|Y}, \tag{1.5}$$

which we call *Résidue map*. Let $W^* \subset A^*(\log Y)$ be the kernel of R^*. There is an obvious inclusion

$$A^*(X) \overset{\iota}{\subset} W^*$$

Proposition 1.6. *The inclusion ι induces isomorphisms on d and $\overline{\partial}$ cohomologys.*

For the proof we refer to ([9]), p.49$\sim$p.50.

Proposition 1.7. *The Gysin map $G^{q-2} : H^{q-2}(Y,\mathbb{C}) \to H^q(X,\mathbb{C})$ is described using differential forms as follows: For $\alpha \in A^{q-2}(Y)$, choose $\tilde{\alpha} \in A^{q-2}(X)$ with $\tilde{\alpha}_{|Y} = \alpha$ and set*

$$\gamma(\alpha) = d(\tilde{\alpha} \wedge \eta) = d\tilde{\alpha} \wedge \eta \wedge \eta + (-1)^{q-2}\tilde{\alpha} \wedge \omega.$$

If α is a closed form (resp. deived from), then $\gamma(\alpha)$ is a closed form (resp. derived form) in W^q. Furthermore $\gamma(\alpha)$ is independent of the choice of $\tilde{\alpha}$ modulo derived form in W^q. Hence, by virtue of Proposition 1.7, the correspondence $[\alpha] \to [\gamma(\alpha)]$ defines a map

$$H^{q-2}(Y,\mathbb{C}) \simeq H^{q-2}(A^*(X)) \to H^q(X,\mathbb{C}) \simeq H^q(W^*),$$

which coincides, up to a factor of ± 1, with the Gysin map G.

Proof. By the definition of $W^a st$, $\gamma(\alpha) \in W^a st$. It is obvious that if α is a closed form, then $\gamma(\alpha)$ is also closed in W^q. Assume α is wriiten as $d\beta = \alpha$ for $\beta \in A^{q-3}(Y)$. We choose $\tilde{\beta} \in A^{q-3}(X)$ with $\tilde{\beta}_{|Y} = \beta$ and set

$$\xi = (\tilde{\alpha} - d\tilde{\beta}) \wedge \eta + (-1)^{q-2}\tilde{\beta} \wedge d\eta.$$

Then $\xi \in W^{q-1}$ and

$$\begin{aligned} d\xi &= d\tilde{\alpha} \wedge \eta + (-1)^{q-2}(\tilde{\alpha} - d\tilde{\beta}) \wedge d\eta + (-1)^{q-2} d\tilde{\beta} \wedge d\eta \\ &= d\tilde{\alpha} \wedge \eta + (-1)^{q-2}\tilde{\alpha} \wedge d\eta \\ &= \gamma(\alpha) \end{aligned}$$

Thus $\gamma(\alpha)$ is a derived form in W^*.

The fact that $\gamma(\alpha)$ is independent of the choice of $\tilde{\alpha}$ modulo derived forms in W^* is almost trivial. In fact, if $\tilde{\alpha}'$ is another form in $A^{q-2}(X)$ with $\tilde{\alpha}'_{|Y} = \alpha$, then $(\tilde{\alpha} - \tilde{\alpha}') \wedge \eta \in W^{q-1}(X)$ and $d((\tilde{\alpha} - \tilde{\alpha}') \wedge \eta) = d\tilde{\alpha} \wedge \eta - d\tilde{\alpha}' \wedge \eta$, which shows $\gamma(\alpha)$ is uniquely determined up to derivede forms in W^*. we wre now going to show that the correspondence $[\alpha] \to [\gamma(\alpha)]$ coincides with the Gysin map G. To do this it suufices to show that for any q-cycle c_q on X, the integral $\int_\Gamma \gamma(\alpha)$ converges and

$$\int_{c_q} \gamma(\alpha) = \pm \int_{c_q \cdot Y} \alpha \tag{1.6}$$

holds, where $\Gamma \cdot Y$ denotes the intersection cycle of Γ with Y. We may assume that c_q intersects Y normally in a $(q-2)$ cycle c_{q-2} with respect to some given hermitian metric on X. For a sufficiently small positive ε, we take a ε-*tube* with axis c_{q-2}, and lying in c_q, normally,

$$T_\varepsilon(c_{q-2}) := \{\ p \in c_q \mid d_X(p, c_{q-2}) \leq \varepsilon\ \}$$

where $d_X(\ ,\ )$ denotes the distance function on X defined by the given hermitian metric. We give natural orientationto $T_\varepsilon(c_{q-2})$. Then,

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{c_q - T_\varepsilon(c_{q-2})} \gamma(\alpha) &= \lim_{\varepsilon \to 0} \int_{c_q - T_\varepsilon(c_{q-2})} d(\tilde{\alpha} \wedge \eta) \\ &= \lim_{\varepsilon \to 0} \int_{\partial T_\varepsilon(c_{q-2})} \tilde{\alpha} \wedge \eta \qquad \text{(by Stokes's Theorem)} \end{aligned} \tag{1.7}$$

Using local coordinates $(z_1, \cdots, z_n, z_{n+1})$ on X such that Y is defined by $z_{n+1} = 0$, $\tilde{\alpha} \wedge \eta$ and $\partial T_\varepsilon(c_{q-2})$ are locally written as

$$\begin{aligned} \tilde{\alpha} \wedge \eta &= \frac{1}{2\pi i}\tilde{\alpha} \wedge \frac{dz_{n+1}}{z_{n+1}} + (\textit{regular form}) \\ \pm\partial T_\varepsilon(c_{q-2}) &= c_{q-2} \times \{\ z_{n+1} \in \mathbb{C} \mid |z_{n+1}| = \varepsilon\ \} \\ &\qquad\qquad (\textit{with natural orientation}) \end{aligned}$$

Hence,

$$\int_{\partial T_\varepsilon(c_{q-2})} \tilde{\alpha} \wedge \eta = \pm \int_{c_{q-2}} \tilde{\alpha} + \int_{\partial T_\varepsilon(c_{q-2})} (\textit{regular form}),$$

and since $\lim_{\varepsilon \to 0} \int_{\partial T_\varepsilon(c_{q-2})} (regular form) = 0$,

$$\int_{\partial T_\varepsilon(c_{q-2})} \tilde{\alpha} \wedge \eta = \pm \int_{c_{q-2}} \tilde{\alpha} \tag{1.8}$$

From (1.7) and (1.8) it follows that the integral $\int_{c_q} \gamma(\alpha)$ converges and the equality in (1.6) holds as requied. □

Proposition 1.8. *We have the following commutative diagram:*

$$
\begin{tikzcd}
H^q(X,\mathbb{C}) \arrow[d, "r^q"'] \arrow[dr, "L"] & \\
H^q(Y,\mathbb{C}) \arrow[r, "G^q"] \arrow[dr, "L'"'] & H^{q+2}(X,\mathbb{C}) \arrow[d, "r^{q+2}"] \\
& H^{q+2}(Y,\mathbb{C})
\end{tikzcd}
\tag{1.9}
$$

where L' denotes the Hodge operator on $H^*(Y,\mathbb{C})$ associated to the fundamental form on Y, the restriction $\Omega_{|Y}$ of the fundamental form Ω to Y.

Proof. We first show that the commutativity of the upper triangle. Let α be a closed C^∞ q-form on X. We denote by $[\alpha] \in H^q(X,\mathbb{C})$ its cohomology class. Then,

$$
\begin{aligned}
(G^q \circ r^q)([\alpha]) &= [d(\alpha \wedge \eta)] \\
&= [d\alpha \wedge \eta + (-1)^q \alpha \wedge d\eta] \\
&= [d\eta \wedge \alpha].
\end{aligned}
$$

Now, we recall that $\omega := d\eta$ is a closed (1.1)-form which represents the first Chern class of the line bundle $[Y]$. Hence, ω is cohomologus to Ω in $H^2(X,\mathbb{C})$. From this it follows that

$$[d\eta \wedge \alpha] = [\Omega \wedge \alpha] = L([\alpha]).$$

Thus $(G \circ r^*)([\alpha]) = L([\alpha])$ as required. Similarly, the commutativity of the lower triangle can be proved. $\square$

We now return to the long exact sequence of cohomology (1.3). By Theorem 1.1, Theorem 1.2, Theorem 1.3, Proposition 1.8 and Grothendieck's theorem in [12] which tells us (among other things) that $H^q(X - Y,\mathbb{C}) = 0$ for $q \geq n+2$, we can easily see that the long exact sequence of cohomology (1.3) breaks down into the short exact sequences as follows:

$$0 \to H^q(X,\mathbb{C}) \xrightarrow{r^q} H^q(X-Y,\mathbb{C}) \to 0 \qquad \text{for } 0 \leq q \leq 1, \tag{1.10}$$

$$0 \to H^{q-2}(Y,\mathbb{C}) \xrightarrow{G^{q-2}} H^q(X,\mathbb{C}) \xrightarrow{r^q} H^q(X-Y,\mathbb{C}) \to 0 \qquad \text{for } 2 \leq q \leq n, \tag{1.11}$$

$$0 \to H^{n-1}(Y,\mathbb{C}) \xrightarrow{G^{n-1}} H^{n+1}(X,\mathbb{C}) \xrightarrow{r^{n+1}} H^{n+1}(X-Y,\mathbb{C}) \xrightarrow{R^{n+1}} H^n(Y,\mathbb{C}) \xrightarrow{G^n} H^{n+2}(X,\mathbb{C}) \to 0, \tag{1.12}$$

$$0 \to H^q(Y,\mathbb{C}) \xrightarrow{G^q} H^{q+2}(X,\mathbb{C}) \to 0 \qquad \text{for } n+1 \leq q \leq 2n. \tag{1.13}$$

We now define the notions of *primitive cycles* and *finite cycles* on X with respect to the prime section Y.

Definition 1.3. A q-cycle c_q on X is defined to be *primitive* if its intersection cycle $c_q \cdot Y$ with Y is zero in $H_{q-2}(Y,\mathbb{C})$. A q-cycle c_q on X is defined to be *finite* if its support is contained is contained in $X - Y$.

We call homology classes of primitive cycles *primitive homology classes* and those of finite cycles *finite homology calsses*. We denote the subspace of primitive (resp. finite) q-homology classes by $H_q(X,\mathbb{C})_0$ (resp. $H_q(X,\mathbb{C})_f$) and call it the *primitive q-homology group* of(resp. *finite q-homology groups* of X. Then by the definitions,

$$
\begin{aligned}
H_q(X,\mathbb{C})_0 &:= \operatorname{Ker}\{\ H_q(X,\mathbb{C}) \xrightarrow{\cdot[Y]} H_{q-2}(Y,\mathbb{C})\ \} \\
H_q(X,\mathbb{C})_f &:= \operatorname{Im}\{\ H_q(X-Y,\mathbb{C}) \xrightarrow{\iota_*} H_q(X,\mathbb{C})\ \}.
\end{aligned}
$$

Proposition 1.9. *Primitive q-cycles possibly exist on X only for q with* $0\leq q\leq n+1$, *and*

$$H_q(X,\mathbb{C})_0 = H_q(X,\mathbb{C})_f \qquad \textit{for} \;\; 0\leq q\leq n+1.$$

Proof. From the homology sequences dual to the cohomology sequences in (1.10) through (1.12) the assertion easily follows. □

To state about the relation between primitive cohomology and homology groups, we introduce the notation for a subspace S of $H^q(X,\mathbb{C})$ (resp. $H_q(X,\mathbb{C})$) as follows:

$$Ann(S) := \{\ [\alpha]\in H_q(X,\mathbb{C}) \mid | <[\omega],[\alpha]>=0 \quad \text{for any } [\omega]\in S\ \},$$

where $<\ ,\ >$ denotes the pairing between cohomology and homology. We call this the *annihilator* subspace of $H_q(X,\mathbb{C})$ by the subspace S.

Proposition 1.10.

(i) $H_q(X,\mathbb{C}) \simeq H_q(X,\mathbb{C})_0 \quad (0\leq q\leq 1)$

(ii) $H_q(X,\mathbb{C}) \simeq H_q(X,\mathbb{C})_0 \oplus Ann(H_q(X,\mathbb{C}_0) \quad (2\leq q\leq n+1)$

Proof. The assertion (i) follows from the definition of primitive homology. We will now prove the assertion (ii). By (i) of 1.3 and Proposition 1.9, $G^{q-2}H^{q-2}(X,\mathbb{C}) = LH^{q-2}(X,\mathbb{C})$. Hence, by (ii) of Theorem 1.3,

$$H^{q-2}(X,\mathbb{C}) \simeq G^{q-2}H^{q-2}(Y,\mathbb{C}) \oplus H^q(X,\mathbb{C})_0 \tag{1.14}$$

Therefore, by duality

$$H_q(X,\mathbb{C}) \simeq Ann(G^{q-2}H^{q-2}(Y,\mathbb{C})) \oplus Ann(H^{q-2}(X,\mathbb{C})_0). \tag{1.15}$$

By considering the paring between the exact sequences of cohomology (1.10), (1.11) and their dual exact sequences of homology,

$$Ann(G^{q-2}H^{q-2}(Y,\mathbb{C}) \simeq \iota_* H_q(X-Y,\mathbb{C}) = H_q(X,\mathbb{C})_f \tag{1.16}$$

From (1.15), (1.16) and Proposition 1.11 follows the assertion (ii). □

Proposition 1.11. *For* $0\leq q\leq n+1$, $r^q : H^q(X,\mathbb{C}) \to H^q(X-Y,\mathbb{C})$ *is injective on the subspace* $H^q(X,\mathbb{C})_0$ *and*

$$H^q(X,\mathbb{C})_0 \simeq r^q H^q(X,\mathbb{C}) \hookrightarrow H^q(X-Y,\mathbb{C}).$$

Proof. By the exactness of the cohomology sequences (1.11) and (**??**), $Im\ G = ker\ r^*$. Hence the assertion follows from (1.14). □

Definition 1.4. Cycles with compact support in $X-Y$ is defined to be *résidue cycle* if they bounds in X. We call their homology classes *résidue homology classes.*

We denote the subspace of $H^c_q(X-Y,\mathbb{C})$ comprising résidue homology classes by $H^c_q(X-Y,\mathbb{C})_{rés}$. By the definition,

$$H^c_q(X-Y,\mathbb{C})_{rés} = Ker\ \{\ H^c_q(X-Y,\mathbb{C}) \xrightarrow{\iota_*} H_q(X,\mathbb{C})\ \}.$$

Actually, $H^c_q(X-Y,\mathbb{C})_{rés} \neq 0$ only for $q=n+1$ and

$$H^c_{n+1}(X-Y,\mathbb{C})_{rés} = \tau_n H_n(Y,\mathbb{C}) \tag{1.17}$$

because of the exact homology sequence (1.2) which is dual to (1.3).

Proposition 1.12.

$$r^{n+1}H^{n+1}(X,\mathbb{C}) = Ann(H^c_{n+1}(X-Y,\mathbb{C})_{rés}).$$

Proof. By considering the paring between the cohomology exact sequence (**??**) and its dual homology sequence, we have

$$r^{n+1}H^{n+1}(X,\mathbb{C}) = Ann(\tau_n H_n(Y,\mathbb{C})).$$

Hence the assertion follows from (1.17). $\square$

We denote by $H^q(X,\mathbb{C})_0$ the primitive cohomology group with respect to the Hodge operator L' on Y which is associated to $\Omega_{|Y}$, the restriction of the fundamental form Ω on X. We are now going to discuss the primitive cohomology and homology of Y. For use later we wish to make clear the relation between the image of the map $R^{n+1} : H^{n+1}(X-Y,\mathbb{C}) \to H^n(Y,\mathbb{C}))$ in the exact sequence (**??**) and the primitive cohomology group $H^n(Y,\mathbb{C})_0$. The result is as follows:

Lemma 1.13. *The restriction map $r^n : H^n(X,\mathbb{C}) \to H^n(Y,\mathbb{C})$, which is injective by the Weak Lefshetz Thorem, give rise to an isomorphism from $H^n(Y,\mathbb{C})_0$ into $H^n(Y,\mathbb{C})_0$ and*

$$r^n(H^n(X,\mathbb{C})) \cap H^n(Y,\mathbb{C})_0 = r^n(H^n(X,\mathbb{C})_0)$$

Proof. By the definition of primitive cohomology, the isomorphism in (1.13) for $n+2$, and 1.3, (ii), we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & H^n(X,\mathbb{C})_0 & \longrightarrow & H^n(X,\mathbb{C}) & \xrightarrow{L^2} & H^{n+4}(X,\mathbb{C}) \\ & & & & {\scriptstyle r^n}\downarrow & & \simeq\uparrow{\scriptstyle G^{n+2}} \\ 0 & \longrightarrow & H^n(Y,\mathbb{C})_0 & \longrightarrow & H^n(Y,\mathbb{C}) & \xrightarrow{L'} & H^{n+2}(Y,\mathbb{C}) \end{array} \tag{1.18}$$

From this we infer that $r^n(H^n(Y,\mathbb{C}))_0 \hookrightarrow H^n(Y,\mathbb{C})_0$ and $r^n_{|H^n(Y,\mathbb{C})_0}(H^n(Y,\mathbb{C})_0 \to H^n(Y,\mathbb{C})$ is an isomorphism into. To show the latter part, we consider the following Lefshetz decompositions of $H^n(Y,\mathbb{C})$ and $H^n(Y,\mathbb{C})$:

$$H^n(X,\mathbb{C}) = H^n(X,\mathbb{C})_0 \oplus LH^{n-2}(X,\mathbb{C}), \tag{1.19}$$

$$H^n(Y,\mathbb{C}) = H^n(Y,\mathbb{C})_0 \oplus L'H^{n-2}(Y,\mathbb{C}). \tag{1.20}$$

Note that, since $r^{n-2} : H^{n-2}(X,\mathbb{C}) \to H^{n-2}(Y,\mathbb{C})$ is an isomorphism by the Weak Lefshetz Theorem, $r^n : H^n(X,\mathbb{C}) \to H^n(Y,\mathbb{C})$ maps $LH^{n-2}(X,\mathbb{C})$ onto $L'H^{n-2}(X,\mathbb{C})$ isomorphically. The inclusion

$$r^n H^n(X,\mathbb{C})_0 \hookrightarrow r^n H^n(X,\mathbb{C}) \cap H^n(Y,\mathbb{C})_0 \tag{1.21}$$

is obvious, since $r^n H^n(X,\mathbb{C})_0 \hookrightarrow H^n(Y,\mathbb{C})_0$ as has been proved just above. We will prove the reverse inclusion. Given $x \in r^n H^n(X,\mathbb{C}) \cap H^n(Y,\mathbb{C})_0$, there exists a $y \in H^n(X,\mathbb{C})$ with $r^n(y) = x$. We write y as $y = y_1 + y_2$ where $y_1 \in H^n(X,\mathbb{C})_0$ and $y_2 \in LH^{n-2}(X,\mathbb{C})$. Then $x = r^n(y) = r^n(y_1) + r^n(y_2)$, and $r^n(y_1) \in H^n(Y,\mathbb{C})_0$, $r^*_n(y_2) \in L'H^{n-2}(Y,\mathbb{C})_0$. Hence

$$x - r^n(y_1) = r^n(y_2) \in H^n(Y,\mathbb{C})_0 \cap L'H^{n-2}(Y,\mathbb{C})_0 = 0.$$

Thus $r^n(y_2) = 0$, from which $y_2 = 0$ follows since r^n maps $LH^{n-2}(X,\mathbb{C})$ onto $L'H^{n-2}(Y,\mathbb{C})$ isomorphically. Hence $x = r^n(y_1)$. This shows that

$$r^n H^n(X,\mathbb{C}) \cap H^n(Y,\mathbb{C})_0 \hookrightarrow r^n H^n(X,\mathbb{C})_0 \tag{1.22}$$

By (1.21) and (1.22), $r^n H^n(X,\mathbb{C}) \cap H^n(Y,\mathbb{C})_0 = r^n H^n(X,\mathbb{C})_0$ as required. $\square$

Lemma 1.14. *There is an exact sequence*

$$0 \to LH^n(X,\mathbb{C})_0 \to H^{n+2}(X,\mathbb{C}) \xrightarrow{r^{n+2}} H^{n+2}(Y,\mathbb{C}) \to 0. \tag{1.23}$$

Proof. To see the surjectivity of r^*_{n+2}, we consider the following commutative diagram:

$$\begin{tikzcd}
H^{n+2}(X,\mathbb{C}) \arrow[r, "r^{n+2}"] & H^{n+2}(Y,\mathbb{C}) & \\
H^n(X,\mathbb{C}) \arrow[u, "L", "\simeq"'] & H^{n-2}(Y,\mathbb{C}) \arrow[l, "G^{n-2}"] \arrow[u, "\simeq", "L'^2"'] & 0 \arrow[l]
\end{tikzcd}$$

the commutativity of which follows from the description of the Gysin map G^{n-2} using differential forms (1.8). From this diagram the surjectivity of r^{n+2} follows, since L'^2 is an isomorphism (Hard Lefshetz for Y). The injectivity of $L : H^n(X,\mathbb{C})_0 \to H^{n+2}(X,\mathbb{C})$ follows from the fact that $L : H^n(X,\mathbb{C})_0 \to H^{n+2}(X,\mathbb{C})$ is an isomorphism (Hard Lefshetz for X).

To prove the exactness at the term $H^{n+2}(X,\mathbb{C})$, we consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & \\
 & H^n(X,\mathbb{C})_0 \arrow[r, "inclusion"] \arrow[d, "r^n", "(\text{Lemma 1.13})"'] & H^n(X,\mathbb{C})_0 \arrow[r, "\simeq", "L"'] \arrow[d, "r^n"] & H^{n+2}(X,\mathbb{C}) \arrow[d, "r^{n+2}"] & & \\
0 \arrow[r] & H^n(Y,\mathbb{C})_0 \arrow[r] & H^n(Y,\mathbb{C}) \arrow[r, "L'"] & H^{n+2}(Y,\mathbb{C}) \arrow[r] \arrow[d] & 0 & \text{(exact)} \\
 & & & 0 & &
\end{tikzcd} \tag{1.24}$$

By this diagram we can easily see $LH^n(X,\mathbb{C})_0 \subset Ker\ r^n$. We will prove the converse inclusion by casing the diagram (1.24). Given $x \in Ker\ r^n$, there exists a $y \in H^n(X,\mathbb{C})$ with $L(y) = x$. Then $L'(r^n(y)) = r^{n+2}(L(y)) = r^{n+2}(x) = 0$, hence $r^n(y) \in r^nH^n(X,\mathbb{C}) \cap H^n(X,\mathbb{C})_0$. We should now recall that $r^nH^n(X,\mathbb{C}) \cap H^n(Y,\mathbb{C})_0 = r^nH^n(X,\mathbb{C})_0$ (Lemma 1.13) and r^n is injective. This implies $y \in H^n(X,\mathbb{C})_0$, that is, $x = L(y) \in LH^n(X,\mathbb{C})_0$, which means $Ker\ r^n \subset LH^n(X,\mathbb{C})_0$. Consequently, we conclude $Ker\ r^n = LH^n(X,\mathbb{C})_0$ as requied. □

Theorem 1.15.

$$H^n(Y,\mathbb{C})_0 = R^{n+1}(H^{n+1}(X - Y,\mathbb{C}) \oplus r^n(H^n(X,\mathbb{C})_0)$$

Proof. Let us consider the Lefshetz decompositions of $H^n(Y,\mathbb{C})_0$ and $H^{n+2}(X,\mathbb{C})$:

$$H^n(Y,\mathbb{C}) = H^n(Y,\mathbb{C})_0 \oplus L'H^{n-2}(Y,\mathbb{C})$$
$$H^{n+2}(X,\mathbb{C}) = LH^n(X,\mathbb{C})_0 \oplus L^2H^{n-2}(X,\mathbb{C}).$$

Claim: Concerning the Gysin map $G^n : H^n(Y,\mathbb{C}) \to H^{n+2}(X,\mathbb{C})$, we have

(a) $G^n(L'H^{n-2}(Y,\mathbb{C}) \subset L^2H^{n-2}(X,\mathbb{C})$ G^n maps $L'H^{n-2}(Y,\mathbb{C})$ onto $L^2H^{n-2}(X,\mathbb{C})$ isomorphically,

(b) $G^n(H^n(Y,\mathbb{C})_0) = LH^n(X,\mathbb{C})_0$, and

(c) $Ker\ G^n \subset H^n(Y,\mathbb{C})_0$.

Proof of (a): By Proposition 1.9, we have the following commutative diagram:

$$
\begin{tikzcd}
H^{n-2}(X,\mathbb{C}) \arrow[d, "r^{n-2}"', "\simeq"] \arrow[dr, "L"] & & \\
H^{n-2}(Y,\mathbb{C}) \arrow[r, "G^{n-2}"] \arrow[dr, "L'"'] & H^n(X,\mathbb{C}) \arrow[d, "r^n"] \arrow[dr, "L"] & \\
& H^n(Y,\mathbb{C}) \arrow[r, "G^n"] & H^{n+2}(X,\mathbb{C})
\end{tikzcd}
$$

where $r^{n-2} : H^{n-2}(X,\mathbb{C}) \to H^{n-2}(Y,\mathbb{C})$ is an isomorphism by the Weak Lefshetz Theorem. From this diagram $G^n(L'H^{n-2}(Y,\mathbb{C}) \subset L^2H^{n-2}(X,\mathbb{C})$ follows. The fact that G^n maps $L'H^{n-2}(Y,\mathbb{C})$ onto $L^2H^{n-2}(X,\mathbb{C})$ isomorphically is proved as follows: Since $L : H^{n-2}(X,\mathbb{C}) \to H^n(X,\mathbb{C})$ is injective, and since $L : H^n(X,\mathbb{C}) \to H^{n+2}(X,\mathbb{C})$ is an isomorphism (Hard Lefshetz Theorem), $L^2 : H^{n-2}(X,\mathbb{C}) \to L^2H^{n+2}(X,\mathbb{C})$ is an isomorphism. Besides, since $L' : H^{n-2}(Y,\mathbb{C}) \to H^n(Y,\mathbb{C})$ is injective, $L' : H^{n-2}(Y,\mathbb{C}) \to L'H^{n-2}(Y,\mathbb{C})$ is also an isomorphism. Therefore, taking into account that $r^{n-2} : H^{n-2}(X,\mathbb{C}) \to H^{n-2}(Y,\mathbb{C})$ is an isomorphism, we conclude that the Gysin map G^n maps $L'H^{n-2}(Y,\mathbb{C})$ onto $L^2H^{n-2}(X,\mathbb{C})$ isomorphically.

Proof of (b): Combining (1.9) for $q = n$, Proposition 1.9, (1.23) and (1.13), we have the following commutative diagram:

$$
\begin{tikzcd}
& & & & \text{(exact)} & \\
& & & & 0 & \\
0 \arrow[r] & H^n(Y,\mathbb{C})_0 \arrow[r] & H^n(Y,\mathbb{C}) \arrow[r, "L'"] \arrow[d, "G^n"] & H^{n+2}(Y,\mathbb{C}) \arrow[r] \arrow[ur] & 0 & \text{(exact)} \\
& & H^{n+2}(X,\mathbb{C}) \arrow[ur] \arrow[d] & & & \\
& LH^n(X,\mathbb{C})_0 \arrow[ur] & 0 & & & \\
0 \arrow[ur] & & & & &
\end{tikzcd}
$$

From this it follows

$$G^n(H^n(Y,\mathbb{C})_0) \subset Ker\ r^*_{n+2} = LH^n(X,\mathbb{C})_0$$

Actually, they coincides with each other, since G^n is surjective and (a) holds.

Proof of (c)*:* Let $x \in Ker\ G^n$. We write it as $x = x_1 + x_2$, where $x_1 \in H^n(Y,\mathbb{C})_0$ and $x_2 \in L'H^{n-2}(Y,\mathbb{C})_0$. Then $G^n(x) = G^n(x_1) + G^n(x_2) = 0$, and by (a) and (b), $G^n(x_1) \in LH^n(X,\mathbb{C})_0$ and $G^n(x_2) \in L^2H^{n-2}(X,\mathbb{C})_0$. Hence $G^n(x_2) = -G^n(x_1) \in LH^n(X,\mathbb{C})_0 \cap L^2H^{n-2}(X,\mathbb{C})_0 = 0$. Thus $G^n(x_2) = 0$, whence $x_2 = 0$. This is because G^n maps $L'H^{n-2}(Y,\mathbb{C})$ onto $L^2H^{n-2}(Y,\mathbb{C})$ isomorphically. Therefore $x = x_1 \in H^n(Y,\mathbb{C})_0$, which means $Ker\ G^n \subset H^n(Y,\mathbb{C})_0$.

q.e.d. for the Claim.

Now we can easily deduce the Proposition. In fact, by Lemma 1.13 and the claim (a), (b) (c) above, we have the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] \\
 & & Ker\ G^n \arrow[d] \\
0 \arrow[r] & H^n(X,\mathbb{C})_0 \arrow[r, "r^n"] \arrow[dr, "L"', "\simeq"] & H^n(Y,\mathbb{C})_0 \arrow[d, "G^n"] \\
 & & LH^n(Y,\mathbb{C})_0 \arrow[d] \\
 & & 0,
\end{tikzcd}$$

which implies

$$H^n(Y,\mathbb{C})_0 \simeq Ker\ G^n \oplus r^n(H^n(X,\mathbb{C})_0).$$

Here, recall that $Ker\ G^n = Im\ R^{n+1}$ by (1.12), then we are done. $\square$

We wish to identify the subspace of $H^n(Y,\mathbb{C})_0$ which is dual to $Im\ R^{n+1}$. For this puopose we need to introduce the following notion.

Definition 1.5. Cycles in Y is defined to be *vanishing cycles* with respect to X if they bound in X. We call their homology classes *vanishing homology classes*.

We denote the subspace $H_q(Y,\mathbb{C})$ comprising vanishing homology classes by $H_q(Y,\mathbb{C})_v$. Note that $H_q(Y,\mathbb{C})_v$ may not be zero only if $q=n$.

Proposition 1.16. *$H_q(Y,\mathbb{C})_v$ is included in $H_n(Y,\mathbb{C})_0$ and*

$$H_n(Y,\mathbb{C}) = H_n(Y,\mathbb{C})_v \oplus Ann(Im\ R^{n+1})$$

or, equivalently

$$H_n(Y,\mathbb{C})_0 = H_n(Y,\mathbb{C})_v \oplus [Ann(Im\ R^{n+1}) \cap H_n(Y,\mathbb{C})_0]$$

Proof. By virture of Theorem 1.15, it suffices to show that

$$H_n(Y,\mathbb{C})_0 \cap Ann(r^n H^n(X,\mathbb{C})_0) = H_n(Y,\mathbb{C})_v.$$

The inclusion $H_n(Y,\mathbb{C})_v \subset Ann(r^n H^n(X,\mathbb{C})_0)$ is trivial. To see that $H_n(Y,\mathbb{C})_v \subset H_n(Y,\mathbb{C})_0$, consider the following diagram:

$$\begin{tikzcd}
H_n(Y,\mathbb{C}) \arrow[r, "\cdot[Z]"] \arrow[d, "\iota_n"'] & H_{n-2}(Z,\mathbb{C})_v \arrow[d, "\simeq"', "\iota_{n-2}"] \\
H_n(X,\mathbb{C}) \arrow[r, "\cdot[Y]"'] & H_{n-2}(Y,\mathbb{C})_v,
\end{tikzcd} \tag{1.25}$$

where Z is the intersection of a generic member $|Y|$ (linear sysytem of effective divisors which are linearly equivalent to Y) with Y, which is a non-singular, irreducible hypersurface of Y and for which $c_1([Z]) \sim \Omega_{|Y}$ (cohomologous), where ι^n (resp. ι^{n-2}) is the homomorphism induced by the inclusion map $\iota : Y \hookrightarrow X$ (resp. $\iota : Z \hookrightarrow Y$, and where $\cdot[Z]$ (resp. $[Y]$) is the map which assignes each n-cycle in Y (resp. X) to its intersection cycle with Z (resp. Y). By the diagram (1.25), $H_n(Y,\mathbb{C})_v \hookrightarrow Ker\ (\cdot[Z])$. Meanwhile, $Ker\ (\cdot[Z]) = H_n(Y,\mathbb{C})_0$ by definition. Thus we have $H_n(Y,\mathbb{C})_v \hookrightarrow H_n(Y,\mathbb{C})_0$. Hence

$$H_n(Y,\mathbb{C})_v \hookrightarrow H_n(Y,\mathbb{C})_0 \cap Ann(r^n H^n(X,\mathbb{C})_0). \tag{1.26}$$

Next we will prove the converse inclusion. It suffices to show that if $[\gamma] \in H_n(Y,\mathbb{C})_0 \cap Ann(r^n H^n(X,\mathbb{C})_0)$, then $\int_\gamma \omega = 0$ for any $[\omega] \in H^n(X,\mathbb{C})$. To see this, we use the Lefshetz decomposition

$$H^n(X,\mathbb{C}) = H^n(Y,\mathbb{C})_0 \oplus LH^{n-2}(X,\mathbb{C}).$$

Assume $[\gamma] \in H_n(Y,\mathbb{C})_0 \cap Ann(r^n H^n(X,\mathbb{C})_0)$. Then $\int_\gamma \omega = 0$ for any $[\omega] \in H^n(X,\mathbb{C})_0$, and for any $[\Omega \wedge \omega'] \in LH^{n-2}(X,\mathbb{C})$ $(\omega'] \in H^{n-2}(X,\mathbb{C})$,

$$\int_\gamma \Omega \wedge \omega' = \int_{[\gamma \cdot Y]} \omega' = 0$$

since $[\gamma \cdot Y] = 0$ by the assumption. Thus $\int_\gamma \omega = 0$ for any $[\omega] \in H^n(X,\mathbb{C})$ if $[\gamma] \in H_n(Y,\mathbb{C})_0 \cap Ann(r^n H^n(X,\mathbb{C})_0)$. This implies

(1.27) $$H_n(Y,\mathbb{C})_v \hookleftarrow H_n(Y,\mathbb{C})_0 \cap Ann(r^n H^n(X,\mathbb{C})_0)$$

By (1.26) and (1.27), $H_n(Y,\mathbb{C})_v = H_n(Y,\mathbb{C})_0 \cap Ann(r^n H^n(X,\mathbb{C})_0)$ as requied. □

2 Rational De Rham groups of an algebraic manifold and Integrals of the second kind on it

As in §1 we let X be a non-singular irreducible algebraic variety of dimension $n+1$ embedded in a higher dimensional complex projecyive space $\mathbb{P}^N(\mathbb{C})$ and Y a generic hyperplane section of X. By a *meromorphic q-form on X* we shall mean an exterior differential form ω of degree q, which has the form

$$\omega = \sum f_{i_1 i_2 \cdots i_q} dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_q}$$

where $(z_1, \cdots, z_{n+1})$ is a complex analytic local coordinate system on X and $f_{i_1 i_2 \cdots i_q}{}'s$ are meromorphic functions of the variables $(z_1, \cdots, z_{n+1})$. We denote by $\Omega^q_X(kY)$ the sheaf of germs of meromorphic q-forms having poles of order k (at most) along Y as their only sngularities. The direct limit of the sheaves $\Omega^q_X(kY)$ a $k \to \infty$ we denote by $\Omega^q_X(*Y)$. It is just the sheaf of germs of meromorphic q-forms with poles of arbitrary order along Y. We put $\Omega^\cdot_X(*Y) := \sum \Omega^q_X(*Y)$, which forms a complex of sheaves with respect to the exterior derivative d. We define

$$\Phi^q(kY) := Ker\ \{\ \Omega^q_X(kY) \xrightarrow{d} \Omega^{q+1}_X((k+1)Y)\ \}$$

and call it the sheaf of germs of *closed meromorphic q-forms* having poles of order k (at most) along Y as their only singularities. We define the sheaf $\Omega^q_X(\log Y)$ to be the subsheaf of $\Omega^q_X(*Y)$ consisting of the germs of such local meromorphic q-forms that both of $f\omega$ and $df \wedge \omega$ are holomorphic if f is a local holomorphic defining equation of Y. If $g = 0$ is another defining equation of Y, then $g = uf$ where u is a non-vanishing local holomorphic function and the relation $g\omega = uf\omega$, $dg \wedge \omega = udf \wedge \omega + fdu \wedge \omega$ shows that $\Omega^q_X(\log Y)$ is well-defined. We call the sheaf of germs of meromorphic q-forms having *logarithmic poles (at most) along* Y as thier only singularities. The reason for this naming is that a meromorphic q-form ω $(q \geq 1)$ has logarithmic poles (at most) along Y as its only singularities if and only if ω is locally written as

$$\omega = \varphi \wedge \frac{df}{f} + \psi,$$

where ϕ, ψ are holomorphic forms and $f = 0$ is a local holomorphic equation of Y. The following lemma is fundametal for calculations in the subsequel.

Lemma 2.1.

(i) *The following sheaf sequences are exact:*

(a) $$0 \to \Phi^{q-1}((k-1)Y) \to \Omega^{q-1}_X((k-1)Y) \xrightarrow{d} \Phi^q(kY) \to 0 \quad (q \geq 2, k \geq 2)$$

(b) $$0 \to \Phi^{q-1}(Y) \to \Omega^{q-1}_X(\log Y) \xrightarrow{d} \Phi^q(Y) \to 0 \quad (q \geq 2)$$

(ii) *There exist naturally the following exact sequences of sheaves:*

(c) $$0 \to \mathbb{C}_X \to \mathcal{O}((k-1)Y) \xrightarrow{d} \Phi^1(kY) \xrightarrow{\alpha} \mathbb{C}_Y \to 0 \quad (k \geq 1)$$

(d) $$0 \to \Omega^q_X(Y) \to \Omega^q_X(\log Y) \xrightarrow{R} \Omega^{q-1}_X(Y) \to 0 \quad (q \geq 1)$$

(e) $$0 \to \Phi^q_X \to \Phi^q_X(Y) \xrightarrow{R} \Phi^{q-1}_Y \to 0 \quad (q \geq 1)$$

Proof. We take a local coordinate system $(z_1, \cdots, z_n, w)$ on X such that Y is defined by $w = 0$. First, we prove for all pairs of integers (q, k) with $q \geq 1,\ k \geq 1$ that if φ is a local holomorphic section of the sfeaf $\Phi^q(kY)$, then φ is written as

$$\varphi = \frac{A \wedge dw}{w^k} + \frac{B}{w^{k-1}} \tag{2.1}$$

where A, B are holomorphic and involve only $dz_1, \cdots, dz_n$. In fact, as to such φ, since $w^k\varphi$ is holomorphic, we may write

$$\varphi = \frac{A \wedge dw}{w^k} + \frac{B'}{w^k}$$

where A, B' are holomorphic and do not involve dw. Since φ is closed,

$$d\varphi = \frac{dA \wedge dw}{w^k} + \frac{dB'}{w^k} + (-k)\frac{dw \wedge B'}{w^{k+1}} = 0$$

so that $B := B'/w$ is holomorphic. Hence we have locally the expression in (2.1) as required. Now we prove the exactness of (i)-(a) and (i)-(b). For a local holomorphic section φ of $\Phi^q(kY)$ ($q \geq 1,\ k \geq 2$ and $q \geq 2,\ k = 1$), we take such an expression as in (2.1). If $k \geq 2$, letting $\psi_1 = -(1/(k-1))(A/w^{k-1})$, $\varphi - d\psi_1$ is a local section of $\Phi^q(k-1)$. Repeating this argument, we may find a local section ψ of $\Omega_X^{q-1}((k-1)Y)$ such that $\varphi - d\psi$ is a section of $\Phi_X^q(Y)$. Thus

$$\varphi - d\psi = E \wedge \frac{dw}{w} + F,$$

where E, F are holomorphic and involve only $dz_1 \cdots, dz_n$. We express E as follows:

$$E = E_0(z) + wE_1(z, w)$$

where $E_0(z)$ does not involve w. Then,

$$\varphi - d\psi = E_0(z) \wedge \frac{dw}{w} + F_0$$

where $F_0 = E_1 + F$. Since $d(Edw/w + F) = 0$, $d_zE_0(z)dw/w + dF_0 = 0$. Hence $d_zE_0(z)dw + wdF_0 = 0$. From this it follows that $d_zE_0(z) = 0$, $dF_0 = 0$. Therefore, there exist $D(z)$ and G such that $d_zD = E_0$ and $dG = F_0$, and so

$$d(D\frac{dw}{w} + G) = E_0 \wedge \frac{dw}{w} + F_0.$$

Hence,

$$\varphi = d(\psi + D \wedge \frac{dw}{w} + G), \tag{2.2}$$

namely, φ is a derived form. This shows the exactness of the sequence (i)-(a). If $k = 1$, then ψ does not appear in the expression of φ in (2.2). This shows the exactness of (i)-(b).

Next we prove the exactness of the sequence (ii)-(c). If φ is a local section $\Phi^1(Y)$, then it is written as

$$\varphi = A \wedge \frac{dw}{w} + B,$$

where A is a holomorphic function and B is a holomorphic 1-form, involving only $dz_1 \cdots, dz_n$ (cf. (2.1)). Writting A as

$$A(z, w) = A_0(z) + wA_1(z, w),$$

where $A_0(z)$ is a function of $z_1, \cdots, z_n$, we have

$$\varphi = A_0(z) \wedge \frac{dw}{w} + B_0,$$

where $B_0 = A_1(z,w)dw + B$. Since

$$d\varphi = \frac{d_z A_0(z) \wedge dw}{w} + dB_0 = 0,$$

we have

$$d_z A_0(z) dw + w \; dB_0 = 0,$$

Hence $d_z A_0(z) = dB_0 = 0$. From these it follows that $A_0(z)$ is constant and $B_0 = dC$ for some holomorphic function $C(z,w)$. Thus φ is written as

$$\varphi = A_0 \wedge \frac{dw}{w} + dC,$$

This means $\Phi^1_X(Y)/d\Omega^0_X$ is locally a constant sheaf. At each point $y \in Y$, we take $[(1/2\pi i)dw/w]_y$, the class of $(\Phi^1_X(Y)/d\Omega^0_X)_y$ determined by $(1/2\pi i)dw/w$, as a generator of $(\Phi^1_X(Y)/d\Omega^0_X)_y$. We can easily see that the class $[(1/2\pi i)dw/w]_y$ is uniquely determined, not depending on the choice of a local defining equation of Y. We denote by $\alpha_y : (\Phi^1_X(Y)/d\Omega^0_X)_y \to \mathbb{C}_{Y,y}$ defined by

$$\left[\frac{1}{2\pi i}\frac{dw}{w}\right]_y \to 1_{Y,y}$$

at each point $y \in Y$, which gives rise to a well-defined sheaf homomorphism $\alpha : \Phi^1_X(Y)/d\Omega^0_X \to \mathbb{C}_Y$ as easily seen. The surjectivity of the map α and that the kernel of the homomorphism $d : \Omega^0_X \to \Phi^1_X(Y)$ coincides with $\mathbb{C}_X$ is obvious. The sheaf homomorphism $R^q : \Omega^q_X(\log Y) \to \Omega^{q-1}_Y$, which we call *Résidues map* is defined as follows (resp. $R : \Phi^q_X(Y) \to \Phi^{q-1}_Y$): A local cross-section φ of the sheaf $\Omega^q_X(\log Y)$ (resp. $\Phi^q_X(Y)$) is written as

$$\omega = \varphi \wedge \frac{dw}{w} + \psi,$$

where φ is a holomorphic $(q-1)$-form and ψ is a holomorphic q-form, involving only $dz_1, \cdots, dz_n$. For such ω, we define $R(\omega) := \varphi_{|Y}$. We can easily seen that this map is well-deined and the sequences (c) and (d) are exact. Thus we are done. $\square$

Notation. We denote by $\Omega^{\cdot}_X((k_0+\cdot)Y)$ (k_0: a non-negative integer), $\Omega^{\cdot}_X(\log Y)$ and $L^{\cdot}(Y)$ the complexes of sheaves of $\mathbb{C}$-modules described as follows:

$$\begin{aligned}\Omega^{\cdot}_X((k_0+\cdot)Y) : \Omega^0_X(k_0Y) \to \Omega^1_X((k_0+1)Y) \to \cdots \to &\Omega^p_X((k_0+p)Y) \to \\ &\cdots \to \Omega^n_X((k_0+n)Y),\end{aligned}$$

$$\Omega^{\cdot}_X(\log Y) : \mathcal{O}_X \to \Omega^1_X(\log Y) \to \cdots \to \Omega^p_X(\log Y) \to \cdots \to \Omega^n_X(\log Y),$$

$$L^{\cdot}(Y) : \Omega^0_X \to \Phi^1_X(Y).$$

Proposition 2.2. *The natural homomorphisms of the complexes of sheaves of $\mathbb{C}$-vector spaces*

$$L^{\cdot}(Y) \to \Omega^{\cdot}_X(\log Y) \to \Omega^{\cdot}_X((k_0+\cdot)Y) \to \Omega^{\cdot}_X(*Y)$$

give rise to quasi-isomorphisms among them, and so all of the hypercohomology of these are isomorphic to $H^p(X-Y,\mathbb{C})$.

Proof. The former part of the proposition follows directly from Lemma 2.1. The latter part is proved as follows: What we shall prove is that $\mathbb{H}^p(X, \Omega_X(\log Y)) \simeq H^p(X-Y,\mathbb{C}) \quad (p \geq 0)$. To do this we form a fine resolution of $\Omega^{\cdot}_X(\log Y)$, using *semi-meromorphic forms* which have poles only on Y. Here, after J. Leray ([18]), we call a C^∞-differential form φ on $X - Y$ *semi-meromorphic form* on X, having poles of order k (at most) along Y if $w^k\varphi$ is locally a C^∞ regular differential form at every point of Y, where $w = 0$ is a local defining equation of Y. Similarly, as in the case of meromorphic forms, semi-meromorphic forms having *logarithmic poles* on Y is defined. We denote by $\mathfrak{A}^{p,q}_X(\log Y)$ the sheaf of germs of semi-meromorphic forms of type (p,q), having *logarithmic poles* on Y. Using these sheves, we obtain a fine resolution of $\Omega_X(\log Y)$ as follows:

$$
\begin{array}{c}
\tag{2.3}
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\mathfrak{A}_X^{0,1} & \xrightarrow{\partial^{0,1}} & \mathfrak{A}_X^{1,1}(\log Y) & \xrightarrow{\partial^{1,1}} & \mathfrak{A}_X^{2,1}(\log Y) & \xrightarrow{\partial^{2,1}} & \cdots & \xrightarrow{\partial^{n,1}} & \mathfrak{A}_X^{n+1,1}(\log Y) \\
\uparrow{\scriptstyle \overline{\partial}^{0,0}} & & \uparrow{\scriptstyle \overline{\partial}^{1,0}} & & \uparrow{\scriptstyle \overline{\partial}^{2,0}} & & & & \uparrow{\scriptstyle \overline{\partial}^{n+1,0}} \\
\mathfrak{A}_X^{0,0} & \xrightarrow{\partial^{0,0}} & \mathfrak{A}_X^{1,0}(\log Y) & \xrightarrow{\partial^{1,0}} & \mathfrak{A}_X^{2,0}(\log Y) & \xrightarrow{\partial^{2,0}} & \cdots & \xrightarrow{\partial^{n,0}} & \mathfrak{A}_X^{n+1,0}(\log Y) \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\mathcal{O}_X & \xrightarrow{d} & \Omega_X^1(\log Y) & \xrightarrow{d} & \Omega_X^2(\log Y) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega_X^{n+1}(\log Y) \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
0 & & 0 & & 0 & & & & 0
\end{array}
\end{array}
$$

where $\mathfrak{A}_X^{p,q}$ denotes the sheave of germs of C^∞ differential forms of type (p,q) on X. We put

$$
\begin{aligned}
A_X^{p,q}(\log Y) &:= \Gamma(X, \mathfrak{A}_X^{p,q}(\log Y)) \qquad (p \geq 0,\ q \geq 0), \\
A_X^k(\log Y) &:= \oplus_{p+q=k} A_X^{p,q}(\log Y), \quad d^{p,q} := \partial^{p,q} + (-1)^p \overline{\partial}^{p,q} \quad \text{and} \\
A_X^{\cdot}(\log Y) &:= \oplus_k \oplus_{p+q=k} A_X^{p,q}(\log Y), \qquad d^k := \oplus_{p+q=k} d^{p,q}.
\end{aligned}
$$

Then $(A_X^{\cdot}(\log Y), d)$ forms a complex of $\mathbb{C}$-vector spaces and

$$\mathbb{H}^p(X, \Omega_X(\log Y)) \simeq H^p(A_X^{\cdot}(\log Y)) \qquad (p \geq 0).$$

By Lemma 2.1,(d), we have the exact sequence of complexes of sheaves of $\mathbb{C}$-vector spaces:

$$0 \to \Omega_X^{\cdot} \to \Omega_X^{\cdot}(\log Y) \xrightarrow{R} \Omega_Y^{\cdot}[-1] \to 0. \tag{2.4}$$

From this the following long exact sequence of hypercohomology is derived:

$$\to \mathbb{H}^p(\Omega_X^{\cdot}) \to \mathbb{H}^p(\Omega_X^{\cdot}(\log Y)) \to \mathbb{H}^{p-1}(\Omega_Y^{\cdot}) \to \mathbb{H}^{p+1}(\Omega_X^{\cdot}) \to \cdots \tag{2.5}$$

Letting $A_X^{\cdot}$ and $A_Y^{\cdot}$ be the complexes of $\mathbb{C}$-vector spaces of global C^∞ differential forms on X and Y, respectively, we have $\mathbb{H}^p(\Omega_X^{\cdot}) \simeq H^p(A_X^{\cdot})$ and $\mathbb{H}^p(\Omega_Y^{\cdot}) \simeq H^p(A_Y^{\cdot})$. Hence the sequence (2.5) is rewritten as:

$$\to H^p(A_X^{\cdot}) \xrightarrow{r^p} H^p(A_X^{\cdot}(\log Y)) \xrightarrow{R^p} H^{p-1}(A_Y^{\cdot}) \xrightarrow{G^{p-1}} H^{p+1}(A_X^{\cdot}) \to \cdots \tag{2.6}$$

We claim that this is the dual of the homology sequence

$$\leftarrow H_p(X, \mathbb{C}) \xleftarrow{r_p} H_p^c(X - Y, \mathbb{C}) \xleftarrow{R_{p-1}} H_{p-1}(Y, \mathbb{C}) \xleftarrow{G_{p+1}} H_{p+1}(X, \mathbb{C}) \leftarrow \cdot \tag{2.7}$$

(cf. (1.3). In fact, since $A_X^{\cdot}(\log Y)$ is a subcomplex of $A_{X-Y}^{\cdot}$ which is the complex of $\mathbb{C}$-vector spaces of global C^∞ differential forms on $X-Y$, we can define parings by integrations between the terms corresponding to each other in (2.6) and (2.7). Furthermore, these pairings commute with the homomorphisms in (2.6) and (2.7), since we can easily see $A_X^{\cdot}(\log Y)$ is the same one as defined in Definition 1.2 and the map $R^p : H^p(A_X^{\cdot}(\log Y)) \to H^{p-1}(A_Y^{\cdot})$ is the *Résidue map* defined just after Definition 1.2, and since $G^{p-1} : H^{p-1}(A_Y \cdot) \to H^{p+1}(A_X^{\cdot})$ is the *Gysin map* whose description by use of differential forms has been given in Proposition 1.7. Therefore, by *Five Lemma*, we conclude that the paring between $H^p(A_X^{\cdot}(\log Y))$ and $H^p(X - Y, \mathbb{C})$ is non-degenerated. Hence $H^p(A_X^{\cdot}(\log Y)) \simeq H^p(X - Y, \mathbb{C})$. ∎

Definition 2.1. We define

$$I^p(X, *Y) := \Gamma(X, \Phi_X^p(*Y))/d\Gamma(X, \Omega_X^{q-1}(*Y))$$

and

$$I^p(X, kY) := \Gamma(X, \Phi_X^p(kY))/d\Gamma(X, \Omega_X^{q-1}((k-1)Y)).$$

We call them the *p-th $*Y$-rational De Rham group of X* and *p-th $*Y$-rational De Rham group of X with pole order k* , respectively.

Then, by Proposition 2.2, we have the following:

Proposition 2.3. *Let k_0 be a positive integer such that*

$$H^p(X, \Omega_X^q((k_0+q)Y)) = 0 \qquad \textit{for} \quad p \geq 1,\ q \geq 0,$$

then,

$$I^p(X, (k_0+p)Y) \simeq I^p(X, *Y) \simeq H^p(X - Y, \mathbb{C}) \qquad \text{for} \quad p \geq 0.$$

Remark 2.1. The result in the propoition above is a special case of the theorem of Grothendieck (cf. [12]).

Now we are going to expalin the notion of closed meromorphic forms of the *second kind*, having poles only along Y. There are the following three different definitions for this:

Definition 2.2. A cosed meromorphic q-form φ is of the *second kind* if

(A) (Picard-Lefshetz definition) at any point x of X, there exists a meromorphic $q-1$ form on X such that $\varphi - d\omega$ is holomorphic in a neighborhood of x,

(B) (Geometric Résidue definition) it has no periods on *résidue cycles* (cf. Definition 1.4) of $X - Y$, if Y is sufficiently large subvariety (depending on φ),

(C) Hodge and Atiyah's *algebaric definition*, using spectral sequences associated to the complex of sheaves of $\mathbb{C}$-vector spaces $\Omega_X^{\cdot}(*Y)$ (or $\Omega_X^{\cdot}((k_0+\cdot)Y)$.

We shall explain the last Hodge and Atiyah's definition ([16]) more precisely by use of the fine resolution $\mathfrak{A}_X^{\cdot\cdot}(*Y)$ of $\Omega_X^{\cdot}(*Y)$, where $\mathfrak{A}_X^{\cdot\cdot}(*Y)$ denotes the double complex of $\mathbb{C}$-vector spaces comprising $\mathfrak{A}_X^{p,q}(*Y)$, the sheaf of germs of semi-meromorphic forms of type (p,q) on X, having poles only along Y. In the same manner as for $\mathfrak{A}_X^{p,q}(\log Y)$, we define $\mathfrak{A}_X^{p,q}(*Y)$ and $\mathfrak{A}_X^{k}(*Y)$. We form the complex of $\mathbb{C}$-vector spaces $(A_X^{\cdot}(*Y), d)$ for $\mathfrak{A}_X^{k}(*Y)$. Then we have

$$I^p(X, *Y) \simeq \mathbb{H}^p(X, \Omega_X^{\cdot}(*Y)) \simeq H^p(A_X^{\cdot}(*Y)) \qquad (p \geq 0).$$

under these isomorphisms, we identify $I^p(X, *Y)$ with $H^p(A_X^{\cdot}(*Y))$ in the following. We set

$$''F^k A_X^{\cdot}(*Y) := \oplus_{q \geq k} A_X^{\cdot q}(*Y),$$

then $\{''F^k\}_{k \geq 0}$ give a finite decreasing filtration to $A_X^{\cdot}(*Y)$ and $A_X^{\cdot}(*Y)$ becomes a filtered complex of $\mathbb{C}$-vector spaces. We define

$$I_k^p(X, *Y) := Im\ \{H^p(''F^k(A_X^{\cdot}(*Y))) \to H^p(A_X^{\cdot}(*Y)) \simeq I^p(X, *Y)\ \}$$

then we have a filtration on $I^p(X, *Y)$:

$$I^p(X, *Y) := I_0^p(X, *Y) \supset I_1^p(X, *Y) \supset \cdots I_p^p(X, *Y) \supset I_{p+1}^p(X, *Y) = \{0\}.$$

Hodge and Atiyah have defined that a closed meromorphic p-form φ, having poles only along Y, is of the *second kind* if its cohomology class $[\varphi] \in I^p(X, *Y)$ belongs to the subspace $I_p^p(X, *Y)$, i.e., it has the maximum filtration, and they have proved that the definitions (B) and (C) are equivalent in general. They have also proved that the definition (A) is equivalent to other definitions if Y is a prime section of X.

Notation. We put

$$I^p(X,*Y)_0 = I^p_p(X,*Y)$$

Then we have:

Theorem 2.4.

(i) $I^p(X,*Y)_0 \simeq r^p H^p(X,\mathbb{C}) \simeq H^p(X,\mathbb{C})_0 \quad (1 \leq p \leq n+1)$,

where $r^p : H^p(X,\mathbb{C}) \to H^p(X-Y,\mathbb{C})$ *is the map induced by restricting closed forms on* X *to* $X-Y$,

(ii) $I^p(X,*Y)_0 = I^p(X,*Y) \quad 1 \leq p \leq n$,

(iii) $I^n(X,*Y)/I^n(X,*Y)_0 \simeq Ker\ \{\ H^{n-1}(Y,\mathbb{C})_0 \xrightarrow{G^{n-1}} H^{n+2}(X,\mathbb{C})\ \}$,

where G^{n-1} *denotes the Gysin map.*

Proof. Replacing $H^*(A^{\cdot}_X(\log Y))$ by $I^*(X,*Y)$ in the exact sequence (2.6), we obtain the exact sequence

$$\to H^p(A^{\cdot}_X) \xrightarrow{r^p} I^p(X,*Y) \xrightarrow{R^p} H^{p-1}(A^{\cdot}_Y) \xrightarrow{G^{p-1}} H^{p+1}(A^{\cdot}_X) \to \cdots, \tag{2.8}$$

which is dual to the homology sequence in (2.7). By the *Résidue definition* of the second kind, we have

$$I^p(X,*Y)_0 \simeq Ann(R_{p-1}(H_{p-1}(Y,\mathbb{C}))),$$

where the right hand side above denotes the annihilator subspace of $I^p(X,*Y)$ by $R_{p-1}(H_{p-1}(Y,\mathbb{C}))$ through the paring defined by integration between $I^p(X,*Y)$ and $H^c_p(X-Y,\mathbb{C})$. By the duality between (2.8) and (2.7),

$$Ann(R_{p-1}(H_{p-1}(Y,\mathbb{C}))) = r^p H^p(A^{\cdot}_X) \simeq r^p H^p(X,\mathbb{C}).$$

By Proposition 1.11, $r^p H^p(X,\mathbb{C}) \simeq H^p(X,\mathbb{C})_0$. Thus we have proved (i). By (i), (ii) follows from that $r^q : H^q(X,\mathbb{C}) \to H^q(X-Y,\mathbb{C})$ is surjective for $0 \leq q \leq n$ (cf. (1.10) and (1.11)). By the duality between (2.8) and (2.7), (iii) is trivial if we note that $R^p(I^p(X,*Y)) \subset H^{p-1}(X-Y,\mathbb{C})_0$ (Theorem 1.15). $\square$

Remark 2.2. As in the case of $A^{\cdot}_X(*Y)$, we define a finite decreasing filtration $\{''F^k\}_{k\geq 0}$ on the complex $A^{\cdot}_X(\log Y)$ by

$$''F^k A^{\cdot}_X(\log Y) := \oplus_{q\geq k} A^{\cdot q}_X(\log Y)$$

Then, as is well known in the homological algebra, there arises a spectral sequence from the filtered complex $(A^{\cdot}_X(\log Y), F'')$ as follows:

$$E_2^{p,q} := H^p(X, \mathcal{H}^q(\Omega^{\cdot}_X(\log Y))) \Longrightarrow E_\infty^{p,q} = Gr^p_{F''}\mathbb{H}^{p+q}(X,\Omega^{\cdot}_X(\log Y)) = Gr^p_{F''} I^{p+q}(X,*Y),$$

where $\mathcal{H}^q(\Omega^{\cdot}_X(\log Y)) \quad (q \geq 0)$ are the cohomology sheaves of the complex of sheaves $\Omega^{\cdot}_X(\log Y)$. From Lemma 2.1 it follows

$$E_2^{p,q} = \begin{cases} H^p(X,\mathbb{C}) & q=0 \\ H^p(X,\mathbb{C}) & q=1 \\ 0 & \text{otherwise} \end{cases}$$

Hence we have

$$E_r^{q,p-q} = E_{r+1}^{q,p-q} = \cdots = E_\infty^{q,p-q} = Gr_{F''} I^p(X,*Y) = 0 \qquad \text{for } q \neq p, p-1, \text{ and } r \geq 2 \tag{2.9}$$

This amounts to

$$I^p(X,*Y) = I^p_0(X,*Y) = I^p_1(X,*Y) = \cdots = I^p_{p-1}(X,*Y),$$

namely, the filtration of $I^p(X,*Y)$ induced by $\{''F^k\}_{f\geq 0}$ of $A^{\cdot}_X(\log Y)$ is given by a single subspace $I^p_p(X,*Y)$. From this we can derive the following exact sequence (cf. [7] Chapitre I, Théorème 4.6.2, p.85):

$$\begin{array}{ccccccc} \cdots \longrightarrow & E_2^{p-2,1} & \xrightarrow{d_2^{p-1}} & E_2^{p,0} & \xrightarrow{\iota^p} \cdots \\ & \simeq\downarrow & & \simeq\downarrow & \\ \cdots \longrightarrow & H^{p-2}(Y,\mathbb{C}) & \longrightarrow & H^p(X,\mathbb{C}) & \longrightarrow \cdots \end{array} \tag{2.10}$$

$$
\begin{array}{ccccccc}
\xrightarrow{\iota^p} & E_\infty^p & \xrightarrow{j^p} & E_2^{p-1,1} & \xrightarrow{d_2^p} & \cdots & \\
 & \simeq\Big\downarrow & & \simeq\Big\downarrow & & & \\
\longrightarrow & I^p(X,*Y) & \longrightarrow & H^{p-1}(Y,\mathbb{C}) & \longrightarrow & \cdots, &
\end{array}
$$

where the maps appeared in this exact sequence are described as follows:

(i) d_2^{p-1} and $d_2^p \cdots$ are the differentials of the second term $\{E_2^{p,q}\}$ of the spectral sequence,

(ii) Since

$$
\begin{aligned}
E_{r+1}^{q,0} &= Ker\ \{E_r^{q,0} \xrightarrow{d_r} E_r^{q+r,1-r}\}/Im\ \{E_r^{q-r,r-1} \xrightarrow{d_r} E_r^{q,0}\} \\
&= \begin{cases} E_r^{q,0}/Im\ \{E_r^{q-r,r-1} \xrightarrow{d_r} E_r^{q,0}\} & r=2 \\ 0 \quad r \geq 3, & \end{cases}
\end{aligned}
$$

there is a surjection from $E_2^{q,0}$ onto $E_\infty^{q,0} = Gr_{''F}^q I^q(X,*Y) \simeq I_q^q(X,*Y)$. The map ι^p is the composite of this surjection and the natural injection $I_q^q(X,*Y) \hookrightarrow E_\infty^q = I^q(X,*Y)$.

(iii) Since

$$
\begin{aligned}
E_{r+1}^{q-1,1} &= Ker\ \{E_r^{q-1,1} \xrightarrow{d_r} E_r^{q+r-1,2-r}\}/Im\ \{E_r^{q-r-1,r} \xrightarrow{d_r} E_r^{q-1,1}\} \\
&= \begin{cases} Ker\ \{E_r^{q-1,1} \xrightarrow{d_r} E_r^{q+r-1,2-r}\} & r=2 \\ E_r^{q-1,1} & r \geq 3, \end{cases}
\end{aligned}
$$

there is an injection from $E_\infty^{q-1,1} = Gr_{''F}^{q-1} I^q(X,*Y)$ into $E_2^{q-1,1}$. The map j^p is the composite of the natural surjection $E_\infty^q = I^q(X,*Y)$ onto $E_\infty^{q-1,1} = Gr_{''F}^{q-1} I^q(X,*Y)$ and the injection above from $E_\infty^{q-1,1} = Gr_{''F}^{q-1} I^q(X,*Y)$ into $E_2^{q-1,1}$.

Chasing these maps more precisely by direct calculation, using differential forms, we can conclude that the exact sequence (2.8) is dual to the homology sequence (2.7). Thus we have proved that $I^q(X,*Y) \simeq H^q(X-Y,\mathbb{C})$ again. Besides, since the image of ι^p is $I_q^q(X,*Y)$ as explained above, this shows that *Résidue definition* and *Hodge-Atiyah's algebraic definition* of the closed meromorphic forms of the second kind coincide.

3 Mixed Hodge structures on $*Y$-rational De Rham groups of X

We call the attention of the readers to that $\Omega_X^\cdot(\log Y)$ is the most simple example of a cohomological mixed Hodge complex (CMHC) in the sense of Deligne and it induces mixed Hodge structures (MHS) on $\mathbb{H}^\cdot(X,\Omega_X^\cdot(\log Y)) \simeq H^\cdot(X-Y,\mathbb{C}) \simeq I^\cdot(X,*Y)$. Concerning these MHS's a non-trivial weight filtration comes out only on $I^{n+1}(X,*Y)$ ($n+1 = \dim X$), and it is given by a single subspace. We shall now show that this subspace is nothing but $I^{n+1}(X,*Y)_0$. First, let us recall the definition of CMHC from [4]. A *CMHC K* on a topological space X is given by

(i) A complex $K \in$ Ob $D^+(X,\mathbb{Z})$ such that $H^q(X,K) := H^q(\mathbb{R}\Gamma(X,K))$ (hypercohomology of K) is a finite $\mathbb{Z}$-module and $H^q(X,K)\otimes\mathbb{Q} \simeq H^q(X,K\otimes\mathbb{Q})$, where $D^+(X,\mathbb{Z})$ denotes the derived category of lower bounded complexes of sheaves of $\mathbb{Z}$-modules over X.

(ii) A filtered complex $(K_\mathbb{Q},W) \in$ Ob $D^+F(X,\mathbb{Q})$ and an isomorphism $K_\mathbb{Q} \simeq K\otimes\mathbb{Q}$ in $D^+F(X,\mathbb{Q})$ (W increasing).

(iii) A bifiltered complex $(K_\mathbb{C},W,F) \in$ Ob $D^+F_2(X,\mathbb{C})$ (W increasing and F decreasing) and $\alpha : (K_\mathbb{C},W) \simeq (K_\mathbb{Q},W)\otimes\mathbb{C}$ in $D^+F(X,\mathbb{C})$, i.e., $Gr^W(K_\mathbb{C})$ and $Gr^W(K_\mathbb{Q})$ are quasi-isomorphic as graded comlexes, satisfying the following axioms:

(A) $\mathbb{R}\Gamma(X, Gr_k^W K_{\mathbb{Q}})$, $(\mathbb{R}\Gamma(X, Gr_k^W K_{\mathbb{C}}), F)$ and $\mathbb{R}\Gamma(X, Gr_k^W \alpha) : \mathbb{R}\Gamma(X, Gr_k^W K_{\mathbb{C}}) \simeq \mathbb{R}\Gamma(X, Gr_k^W K_{\mathbb{Q}}) \otimes \mathbb{C}$ is a *Hodge complex (HC) of weight* k,

where *HC of weight* k is defined as follows: A Hodge complex (HC) K of weight k is given by

(i) A complex $K \in \mathrm{Ob}\ D^+(X, \mathbb{Z})$ such that the cohomology $H^q(K)$ is a $\mathbb{Z}$-module of finite type for each q.

(ii) A filtered complex $(K_{\mathbb{C}}, F) \in \mathrm{Ob}\ D^+F\mathbb{C}$ and an isomorphism $\alpha : K_{\mathbb{C}} \simeq K \otimes \mathbb{C}$ in $D^+\mathbb{C}$, satisfying the following axioms:

(AI) The differential d of $K_{\mathbb{C}}$ is strictly compatible to the filtration F, i.e., $F^i \cap Im\ d = Im\ (d/F^i)$ or equivalently the spectral sequence defined by $(K_{\mathbb{C}}, F)$ degenerates at E_1 ($E_1 = E_\infty$).

(AII) The filtration F induced on $H^q(K_{\mathbb{C}}) \simeq H^q(K) \otimes \mathbb{C}$ defines a *HS of weight* $q + k$.

In our case, we take $K \in \mathrm{Ob}\ D^+(X, \mathbb{Z})$, $(K_{\mathbb{Q}}, W) \in \mathrm{Ob}\ D^+F(X, \mathbb{Q})$ and $(K_{\mathbb{C}}, W, F) \in \mathrm{Ob}\ D^+F_2(X, \mathbb{C})$ in the definition above as follows:

$$K := \mathbb{R}j_*\mathbb{Z},$$

where $j : X - Y \hookrightarrow X$ is the open immersion,

$$K_{\mathbb{Q}} := \mathbb{R}j_*\mathbb{Q}_{X-Y},$$
$$W_p(K_{\mathbb{Q}}) := \tau_{\leq p}(K_{\mathbb{Q}}),$$

where $\tau_{\leq p}(K_{\mathbb{Q}})$ denotes the subcomplex of $K_{\mathbb{Q}}$ defined by

$$\tau_{\leq p}(K_{\mathbb{Q}})^n = \begin{cases} K^n & q = 0 \\ \mathrm{Ker}\ d & q = 1 \\ 0 & n > p \end{cases}$$

(which we call the *canonical filtration*)

$$K_{\mathbb{C}} := \Omega^{\cdot}_X(\log Y),$$
$$W_0(K_{\mathbb{C}}) = \Omega^{\cdot}_X,$$
$$W_1(K_{\mathbb{C}}) = \Omega^{\cdot}_X(\log Y),$$
$$F^q(K_{\mathbb{C}}) := \sigma_{\geq q}(\Omega^{\cdot}_X(\log Y)),$$

where $\sigma_{\geq q}(\Omega^{\cdot}_X(\log Y))$ denotes the subcomplex of $\Omega^{\cdot}_X(\log Y)$ defined by

$$(\sigma_{\geq q}(\Omega^{\cdot}_X(\log Y)))^\ell = \begin{cases} 0 & \ell < q \\ \Omega^\ell_X(\log Y) & q \leq \ell, \end{cases}$$

which we call the *stupid filtration*. Instead of the filtartion W, we shall use the filtraion $W[q]$ defined by

$$W[q]_p := W_{p-q},$$

namely, a shift by q to the right on the degree of W. Then $(W[q], F)$ induces a mixed Hodge structure on $H^q(\mathbb{R}\Gamma(X, \Omega^{\cdot}_X(\log Y)) := \mathbb{H}^q(X, \Omega^{\cdot}_X(\log Y)) \simeq I^q(X, *Y)$. We shall calculate $Gr_k^{W[q]} I^q(X, *Y)$ ($k = q, q+1$) by use of spectral sequences. We put

$$K^{\cdot} := A^{\cdot}_X(\log Y), \quad \text{and}$$
$$W_0(K^{\cdot}) = A^{\cdot}_X, \quad W_1(K^{\cdot}) = A^{\cdot}_X(\log Y).$$

$\{W_0(K^{\cdot}) \subset W_1(K^{\cdot}) = A^{\cdot}_X(\log Y)\}$ is the filtration induced by the filtration $\{W_0 \subset W_1 = \Omega^{\cdot}_X(\log Y)\}$ on $\Omega^{\cdot}_X(\log Y)$. We define

$$W'_p(K^{\cdot}) := W[q]_{-p}(K^{\cdot}) = W_{-p-q}(K^{\cdot}) \quad (p \leq -q).$$

Then $\{W'_p(K^{\cdot})\}$ is a decreasing filtration of $K^{\cdot}$. Hence we can consider the spectral sequence concerning the filtration complex $(K^{\cdot}, W'(K^{\cdot}))$, whose 0-th term and 1-st one are computed as follows:

$$\begin{aligned}
{}_{W'}E_0^{r,s} &= Gr^r_{W'}(K^{r+s}) \\
&= \begin{cases} W_0(K^{s-q}) & r=-q \\ W_1(K^{s-q-1})/W_0(K^{s-q-1}) & r=-q-1 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} A_X^{s-q} & r=-q \\ A_X^{s-q-1}(\log Y)/A_X^{s-q-1} \simeq A_Y^{s-q-2} & r=-q-1 \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

where the isomorphism $A_X^{s-q-1}(\log Y)/A_X^{s-q-1} \simeq A_Y^{s-q-2}$ comes from the exact sequence of sheaves

$$0 \to \mathcal{W}_X^{\cdot} \to \mathcal{A}_X^{\cdot}(\log Y) \xrightarrow{R} \mathcal{A}_Y[-1]^{\cdot} \to 0,$$

(cf. Proposition 1.7) which is the C^∞ version of the exact sequence (2.4);

$$\begin{aligned}
{}_{W'}E_1^{r,s} &= \frac{\mathrm{Ker}\{{}_{W'}E_1^{r,s} \xrightarrow{d_1} {}_{W'}E_1^{r+1,s}\}}{\mathrm{Im}\{{}_{W'}E_1^{r-1,s} \xrightarrow{d_1} {}_{W'}E_1^{r,s}\}} \\
&= \begin{cases} H^{s-q}(A_X^{\cdot}) \simeq H^{s-q}(X, \mathbb{C}_X) \quad r=-q,\ s \geq q \\ H^{s-q-1}(A_X^{\cdot}(\log Y)/A_X^{\cdot}) \simeq H^{s-q-1}(A_X^{\cdot}(\log Y)/W_X^{\cdot}) \\ \simeq H^{s-q-2}(A_Y^{\cdot}) \simeq H^{s-q-2}(Y, \mathbb{C}_Y) \quad r=-q-1,\ s \geq q+1, \\ 0 \quad \text{otherwise.} \end{cases}
\end{aligned}$$

Hence we have

$$\begin{aligned}
{}_{W'}E_t^{r,p-r} = {}_{W'}E_{t+1}^{r,p-r} = \cdots = {}_{W'}E_\infty^{r,p-r} = Gr_r^{W'} I^p(X, *Y) = 0 \\
\text{for} \quad r \neq -q, -q-1 \quad \text{and} \quad t \geq 2
\end{aligned} \tag{3.1}$$

This is equivalent to $I^p(X,*Y) = W'_{-q-1}(I^p(X,*Y)) \supset W'_{-q}(I^p(X,*Y)) = E_\infty^{-q,p+q}$. From these we obtain the following exact sequnece:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & {}_{W'}E_1^{-q,p} & \xrightarrow{\iota^p} & {}_{W'}E_\infty^{p,-q} & \xrightarrow{\jmath^p} & {}_{W'}E_1^{-q-1,p+1} \\
& \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
\cdots \longrightarrow & H^{p-q}(X,\mathbb{C}) & \xrightarrow{r^{p-q}} & I^{p-q}(X,*Y) & \xrightarrow{R^{p-q}} & H^{p-q-1}(Y,\mathbb{C})
\end{array} \tag{3.2}$$

$$\begin{array}{ccccccc}
\xrightarrow{d_1^{p+1}} & {}_{W'}E_1^{-q,p+1} & \xrightarrow{\iota^{p+1}} & {}_{W'}E_\infty^{p,-q+1} & \longrightarrow \cdots \\
& \simeq\downarrow & & \simeq\downarrow & \\
\xrightarrow{G^{p-q-1}} & H^{p-q+1}(X,\mathbb{C}) & \xrightarrow{r^{p-q+1}} & I^{p-q+1}(X,*Y) & \longrightarrow \cdots,
\end{array}$$

where the maps in this diagram are described as follows:

(i) d_1^{p+1} is the differential at the first term of $\{E_1^{p-q+1,p+1}\}$ of the spectral sequnece,

(ii) Since

$$
\begin{aligned}
{}_{W'}E_{r+1}^{-q,p} &= \frac{\mathrm{Ker}\{{}_{W'}E_r^{-q,p} \xrightarrow{d_r} {}_{W'}\ E_r^{-q+r,p-r+1}\}}{\mathrm{Im}\{{}_{W'}E_r^{-q-r,p+r-1} \xrightarrow{d_r} {}_{W'}\ E_r^{-q,p}\}} \\
&= \begin{cases} {}_{W'}E_r^{-q,p}/\mathrm{Im}\{{}_{W'}E_r^{-q-1,p} \xrightarrow{d_r} {}_{W'}\ E_r^{-q,p}\}, & r = 1 \\ {}_{W'}E_r^{-q,p}, \quad r \geq 2, \end{cases}
\end{aligned}
$$

there is a surjection from ${}_{W'}E_1^{-q,p}$ onto ${}_{W'}E_\infty^{-q,p} = Gr_{W'}^{-q} I^{p-q}(X, *Y) = W'_{-q} I^{p-q}(X, *Y)$. The map ι^p is the composite of this surjection and the natural injection $W'_{-q} I^{p-q}(X, *Y) \hookrightarrow I^{p-q}(X, *Y) ={}_{W'} E_\infty^{-q,p}$.

(iii) Since

$$
\begin{aligned}
{}_{W'}E_{r+1}^{-q-1,p+1} &= \frac{\mathrm{Ker}\{{}_{W'}E_r^{-q-1,p+1} \xrightarrow{d_r} {}_{W'}\ E_r^{-q-1+r,p-r+2}\}}{\mathrm{Im}\{{}_{W'}E_r^{-q-1-r,p+r} \xrightarrow{d_r} {}_{W'}\ E_r^{-q-1,p+1}\}} \\
&= \begin{cases} \mathrm{Ker}\{{}_{W'}E_1^{-q-1,p+1} \xrightarrow{d_1} {}_{W'}\ E_1^{-q,p+1}\}, & r = 1 \\ {}_{W'}E_r^{-q-1,p+1}, \quad r \geq 2, \end{cases}
\end{aligned}
$$

there is an injection from ${}_{W'}E_\infty^{-q-1,p+1} = Gr_{W'}^{-q-1} I^{p-q}(X, *Y)$ into ${}_{W'}E_1^{-q-1,p+1}$. The map j^p is the composite of the natural surjection ${}_{W'}E_\infty^{p-q} = I^{p-q}(X, *Y)$ onto ${}_{W'}E_\infty^{-q-1,p+1} Gr_{W'} I^{p-q}(X, *Y)$ and the injection above from ${}_{W'}E_\infty^{-q-1,p+1}$ into ${}_{W'}E_1^{-q-1,p+1}$.

Chasing these maps more precisely by direct calculation, using differential forms, we can conclude that the exact sequence (**??**) is dual to the homology sequence (2.7). By the definition of the map ι^p and j^p, we have

$$
\begin{aligned}
\iota^p({}_{W'}E_1^{-q,p}) &= W'_{-q} I^{p-q}(X, *Y) \quad \text{and} \\
j^p({}_{W'}E_\infty^{p+q}) &= Gr_{W'}^{-q-1} I^{p-q}(X, *Y),
\end{aligned}
$$

which are rewritten as

$$
\begin{aligned}
r^{p-q}(H^{p-q}(X, \mathbb{C}) &= W'_{-q} I^{p-q}(X, *Y) \quad \text{and} \\
R^{p-q}(I^{p-q}(X, *Y)) &= Gr_{W'}^{-q-1} I^{p-q}(X, *Y),
\end{aligned}
$$

If we put $p = 2q$, then we have

$$
H^q(X, \mathbb{C})_0 = r^q(H^q(X, \mathbb{C})) = W'_{-q} I^q(X, *Y) = W[q]_q I^q(X, *Y)
$$

and

$$
\begin{aligned}
\mathrm{Ker}\{G^{q-1} : H^{q-1}(Y, \mathbb{C})_0 \to F^k H^{q+2}(X, \mathbb{C})\} &= R^q(I^q(X, *Y)) \\
&\simeq Gr_{W'}^{-q-1} I^q(X, *Y) \\
&\simeq = Gr_{q+1}^{W[q]} I^q(X, *Y).
\end{aligned}
$$

Therefore, combining these results with those of Theorem 2.4, we have

Theorem 3.1.

(i) $Gr_q^{W[q]} H^q(X - Y, \mathbb{C}) = W[q]_q H^q(X - Y, \mathbb{C}) = I^q(X, *Y)_0$,

(ii) $Gr_{q+1}^{W[q]} H^q(X - Y, \mathbb{C}) = I^q(X, *Y)/I^q(X, *Y)_0$,

(iii) $F^k Gr_q^{W[q]} H^q(X - Y, \mathbb{C}) \simeq F^k H^q(X, \mathbb{C})_0$,

(iv) $F^k Gr^{W[q]}_{q+1} H^q(X-Y,\mathbb{C}) \simeq \mathrm{Ker}\{F[-1]^k H^{q-1}(Y,\mathbb{C})_0 \xrightarrow{G^{q-1}} F^k H^{q+2}(Y,\mathbb{C})\}$

From now on, we consider the following complex of sheaves of $\mathbb{C}$-vector spaces:

$$\Omega^{\cdot}_X((1+\cdot)Y) : \mathcal{O}_X(Y) \to \Omega^1_X(2Y) \to \cdots \to \Omega^p_X((p+1)Y) \to \cdots \to \Omega^{n+1}_X((n+2)Y)$$

We define a decreasing filtration $\{F'^k\}_{0\leq k\leq n}$ by

$$\begin{aligned} F'^k(\Omega^{\cdot}_X((1+\cdot)Y)) := \{\cdots \to 0 \to \Omega^k_X(Y) \to \Omega^{k+1}_X(2Y) \to \\ \cdots \to \Omega^p_X((p-k+1)Y) \to \cdots \to \Omega^{n+1}_X((n-k+2)Y)\}, \end{aligned} \tag{3.3}$$

and an increasing filtartion $\{W'_0 \subset W'_1\}$ by

$$\begin{aligned} W'_0(\Omega^{\cdot}_X((1+\cdot)Y)) &: \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^p_X \to \cdots \to \Omega^{n+1}_X, \\ W'_1(\Omega^{\cdot}_X(1+\cdot)) &: \Omega^{\cdot}_X(1+\cdot). \end{aligned}$$

Then we have

Proposition 3.2. *The bi-filtered complexes of sheaves of $\mathbb{C}$-vector spaces*

$$(\Omega^{\cdot}_X(\log Y), W, F) \quad \textit{and} \quad (\Omega^{\cdot}_X((1+\cdot)Y), W', F')$$

are quasi-isomorphic, i.e., bi-graded complex of sheaves of $\mathbb{C}$-vector spaces

$$Gr_F Gr^W(\Omega^{\cdot}_X(\log Y)) \quad \textit{and} \quad Gr_{F'} Gr^{W'}(\Omega^{\cdot}_X((1+\cdot)Y))$$

are quasi-isomorphic where the filtration F of $\Omega^{\cdot}_X(\log Y)$ is defined by

$$F^k(\Omega^{\cdot}_X(\log Y)) := \{\cdots \to 0 \to \Omega^k_X(\log Y) \to \Omega^{k+1}_X(\log Y) \to \cdots \to \Omega^{n+1}_X(\log Y)\} \quad (0\leq k\leq n+1),$$

Proof. First, we have

$$\begin{aligned} Gr^k_{F'} Gr^{W'}_0(\Omega^{\cdot}_X((1+\cdot)Y)) &= \Omega^k_X[-k], \\ Gr^k_{F'} Gr^{W'}_1(\Omega^{\cdot}_X((1+\cdot)Y)) &= (\Omega^{k+\cdot}_X((1+\cdot)Y)/\Omega^{k+\cdot}_X(\cdot Y))[-k], \\ Gr^k_F Gr^W_0(\Omega^{\cdot}_X(\log Y)) &= \Omega^k_X[-k], \quad \text{and} \\ Gr^k_F Gr^W_1(\Omega^{\cdot}_X(\log Y)) &= (\Omega^{k+\cdot}_X(\log Y)/\Omega^{k+\cdot}_X)[-k] \end{aligned}$$

Thus $Gr^k_{F'} Gr^{W'}_0(\Omega^{\cdot}_X((1+\cdot)Y))$ and $Gr^k_F Gr^W_0(\Omega^{\cdot}_X(\log Y))$ are quasi-isomorphic and

$$H^p(Gr^k_F Gr^W_1(\Omega^{\cdot}_X(\log Y)) = \begin{cases} \Omega^p_X(\log Y)/\Omega^p_X & p = k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

We shall calculate $H^p(Gr^k_{F'} Gr^{W'}_1(\Omega^{\cdot}_X((1+\cdot)Y))$. Obviously,

$$H^p(Gr^k_F Gr^{W'}_1(\Omega^{\cdot}_X((1+\cdot)Y)) = 0 \quad \text{for} \quad 0\leq p\leq k-1, \quad 1\leq k.$$

Assume $p \geq k+1$. Let $[\omega] \in \Omega^p_X((p-k+1)Y)/\Omega^p_X((p-k)Y)$ be an element with $d[\omega] = 0$ in $\Omega^p_X((p-k+2)Y)/\Omega^p_X((p-k+1)Y)$ where ω is an element of $\Omega^{p+1}_X((p-k+1)Y)$. Since $d\omega$ is a closed form, by Lemma 2.1, (i)-(a), there exists $\varphi \in \Omega^p_X((p-k)Y)$ such that $d\varphi = d\omega$. Since $\omega - \varphi \in \Phi^p_X((p-k+1)Y)$, by the same reason, there exists $\psi \in \Omega^{p-1}_X((p-k)Y)$ such that $d\psi = \omega - \varphi$. This means $d[\psi] = [\omega]$. Thus $H^p(Gr^k_F Gr^{W'}_1(\Omega^{\cdot}_X((1+\cdot)Y)) = 0$ for $p \geq k+1$. Let $[\omega] \in \Omega^k_X(Y)/\Omega^k_X$ be an element with $d[\omega] = 0$ in

$\Omega_X^{k+1}(2Y)/\Omega_X^{k+1}(Y)$. This amounts to $d\omega \in \Omega_X^{k+1}(Y)$. If $k \geq 1$, we can easily see that this is the case if and only if $\omega \in \Omega_X^k(\log Y)$. This fact tells us that

$$H^k(Gr^k_{F'}Gr_1^{W'}(\Omega^{\cdot}_X((1+\cdot)Y)) \simeq \Omega^k_X(\log Y)/\Omega^k_X \quad \text{for} \quad k \geq 1.$$

If $k=0$, we can easily see that $\omega \in \mathcal{O}_X$, since $\omega \in \mathcal{O}_X(Y)$, $d\omega \in \Omega^1_X(Y)$. Hence $H^0(Gr^0_{F'}Gr_1^{W'}(\Omega^{\cdot}_X(1+\cdot))) = 0$. This completes the proof. $\square$

We define

$$I^p_k(X,(p+1)Y) := \frac{\Gamma(X, \Phi^p_X((p-k+1)Y))}{d\Gamma(X, \Omega^{p-1}_X((p-k)Y))} \quad (0 \leq k \leq p)$$

and denote by $I^p_k(X,(p+1)Y)_0$ the subspace of $I^p_k(X,(p+1)Y)$ generated by closed moromorphic of p-forms of the second kind. The CMHC $(\Omega^{\cdot}_X(\log Y), W, F)$ induces a mixed Hodge structure on $H^p(X-Y,\mathbb{C})$ ($\simeq \mathbb{H}^p(X, \Omega_X(\log Y))$). We denote by $\{F^kH^p(X-Y,\mathbb{C})\}_{0\leq k\leq p}$ the Hodge filtration of $H^p(X-Y,\mathbb{C})$ concerning this mixed Hodge structure, and by $\{F^kH^p(X,\mathbb{C})_0\}_{0\leq k\leq p}$ the ordinary Hodge filtration of $H^p(X,\mathbb{C})_0$, the p-th primitive cohomology group of X. With this notation we have

Theorem 3.3. *If Y is sufficiently ample so that*

$$H^p(X, \Omega^q_X(kY)) = 0 \quad \text{for} \quad p \geq 1, q \geq 0, k \geq 1, \tag{3.4}$$

then we have

$$F^kH^p(X-Y,\mathbb{C}) \simeq I^p_k(X,(p+1)Y) \quad 0 \leq k \leq p \quad \text{and} \tag{3.5}$$

$$F^kH^p(X,\mathbb{C})_0 \simeq I^p_k(X,(p+1)Y)_0 \quad 0 \leq k \leq p \tag{3.6}$$

under the isomorphisms $H^p(X-Y,\mathbb{C}) \simeq I^p(X,(p+1)Y)$ and $H^p(X,\mathbb{C})_0 \simeq I^p(X,(p+1)Y)_0$ in Proposition 2.3 and Theorem 2.4, respectively.

Proof. Using the sheaves $\mathfrak{A}^{p,q}_X(\ell Y)$, the sheaves of germs of semi-meromorphic forms of type (p,q) on X, having poles of order ℓ (at most) alomg Y, we can form a fine resolution of it by use of more small sheaves. Let $\mathfrak{B}^{p,q}_X(\ell Y)$ be the subsheaves of $\mathfrak{A}^{p,q}_X(\ell Y)$ characterized by the following prescription: Letting φ be a local cross-section of $\mathfrak{A}^{p,q}_X(\ell Y)$ if and only if $f^{\ell-1}df \wedge \varphi$ is a C^∞ regular differential form where $f=0$ is a local holomorphic defining equation for Y. Using $\mathfrak{B}^{p,q}_X(\ell Y)$, we obtain a fine resolution of $\Omega^{\cdot}_X((1+\cdot)Y)$ as follows:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\mathfrak{B}^{0,1}_X(Y) & \xrightarrow{\partial^{0,1}} & \mathfrak{B}^{1,1}_X(2Y) & \xrightarrow{\partial^{1,1}} & \mathfrak{B}^{2,1}_X(3Y) & \xrightarrow{\partial^{2,1}} & \cdots & \xrightarrow{\partial^{n,1}} & \mathfrak{B}^{n+1,1}_X((n+2)Y) \\
\uparrow{\scriptstyle\bar\partial^{0,0}} & & \uparrow{\scriptstyle\bar\partial^{1,0}} & & \uparrow{\scriptstyle\bar\partial^{2,0}} & & & & \uparrow{\scriptstyle\bar\partial^{n,0}} \\
\mathfrak{B}^{0,0}_X(Y) & \xrightarrow{\partial^{0,0}} & \mathfrak{B}^{1,0}_X(2Y) & \xrightarrow{\partial^{1,0}} & \mathfrak{B}^{2,0}_X(3Y) & \xrightarrow{\partial^{2,0}} & \cdots & \xrightarrow{\partial^{n,0}} & \mathfrak{B}^{n+1,0}_X((n+2)Y) \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\Omega^0_X(Y) & \xrightarrow{d} & \Omega^1_X(2Y) & \xrightarrow{d} & \Omega^2_X(3Y) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{n+1}_X((n+2)Y) \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
0 & & 0 & & 0 & & & & 0
\end{array} \tag{3.7}$$

We put

$$\begin{aligned}
B^{p,q}_X((p+1)Y) &:= \Gamma(X, \mathfrak{B}^{p,q}_X((p+1)Y) \quad (p \geq 0, q \geq 0), \\
B^k_X((k+1)Y) &:= \oplus_{p+q=k} B^{p,q}_X((p+1)Y) \quad d^{p,q} = \partial^{p,q} + (-1)^p\bar\partial^{p,q} \quad \text{and} \\
B^{\cdot}_X((1+\cdot)Y) &:= \oplus_k \oplus_{p+q=k} B^{p,q}_X((p+1)Y)
\end{aligned}$$

Then $(B^{\cdot}_X((1+\cdot)Y), d)$ forms a complex of $\mathbb{C}$-vector spaces and we have

$$\mathbb{H}^p(X, \Omega_X((1+\cdot)Y)) \simeq H^p(B^{\cdot}_X((1+\cdot)Y)) \quad (p \geq 0).$$

The filtration $\{F'^k\}$ of $\Omega_X((1+\cdot)Y)$ defined in (3.3) induces a filtration on $B^{\cdot}_X((1+\cdot)Y)$, which we denote by $\{F'^k B^{\cdot}_X((1+\cdot)Y)\}$, i.e.,

$$F'^k B^{\cdot}_X((1+\cdot)Y) := \oplus_p \oplus_{p \geq q \geq k} B^{q,p-q}_X((q+1)Y)$$

Since $(\Omega_X((1+\cdot)Y), W^{\cdot}, F')$ is a CMHC by Proposition 3.2, the spectral sequence, associated to the filtration $\{F'^k B^{\cdot}_X((1+\cdot)Y)\}$ and whose final terms are

$${}_{F'}E^{p,q}_{\infty} = Gr^p_{F'} = Gr^p_{F'} H^{p+q}(B^{\cdot}_X((1+\cdot)Y)),$$

is degenerated at the 1-st term (cf. [2], Théorème 3.2.5, [4], Théorème 3.2.1). Therefore, we have

$$\begin{aligned} {}_{F'}E^{k,p-k}_1 &= H^p(F^k(B^{\cdot})/F^{k+1}(B^{\cdot})) \quad (B^{\cdot} = B^{\cdot}_X((1+\cdot)Y)) \\ &\simeq_{F'} E^{k,p-k}_{\infty} = Gr^k_{F'} H^p(B^{\cdot}) \end{aligned} \tag{3.8}$$

Here we should recall that the filtration on $H^p(B^{\cdot})$ induced by $\{F'\}$ on $B^{\cdot}$ is defined by

$$\begin{aligned} F'^k H^p(B^{\cdot}) \quad &:= \quad \mathrm{Im}\{H^p(F^k(B^{\cdot})) \to H^p(B^{\cdot})\} \quad \text{and} \\ Gr^k_{F'} H^p(B^{\cdot}) \quad &= \quad F'^k H^p(B^{\cdot})/F'^{k+1} H^p(B^{\cdot}) \end{aligned}$$

From this and (3.8) it follows that the natural map

$$H^p(F'^k(B^{\cdot})) \to H^p(F'^k(B^{\cdot})/F'^{k+1}(B^{\cdot}))$$

is surjective. Hence the long exact sequence of cohomology associated to the exact sequence of complex

$$0 \to F'^{k+1}(B^{\cdot}) \to F'^k(B^{\cdot}) \to F'^k(B^{\cdot})/F'^{k+1}(B^{\cdot}) \to 0$$

breaks up into the following short exact sequences

$$0 \to H^p(F'^{k+1}(B^{\cdot})) \to H^p(F'^k(B^{\cdot})) \to H^p(F'^k(B^{\cdot})/F'^{k+1}(B^{\cdot})) \to 0$$
$$(0 \leq p \leq n+1, 0 \leq k \leq p)$$

Here $H^p(F'^k(B^{\cdot})/F'^{k+1}(B^{\cdot})) \simeq Gr^k_{F'} H^p(B^{\cdot})$. Hence

$$H^p(F'^k(B^{\cdot}) \simeq F'^k H^p(B^{\cdot}) \simeq F'^k H^p(X-Y, \mathbb{C}) \quad (0 \leq k \leq p, 0 \leq p \leq n+1). \tag{3.9}$$

On the other hand, by the assumption 3.4, we have

$$\begin{aligned} H^p(F'^k(B^{\cdot})) &\simeq \mathbb{H}^p(F'^k(\Omega^{\cdot}_X((1+\cdot)Y))) \\ &= \mathbb{H}^p(\Omega^{k+\cdot}_X((1+\cdot)Y)[-k]) \\ &= \mathbb{H}^{p-k}(\Omega^{k+\cdot}_X((1+\cdot)Y)) \\ &\simeq \frac{\Gamma(X, \Phi^p_X((p-k+1)Y))}{d\Gamma(X, \Omega^{p-1}_X((p-k)Y))} \\ &= I^p_k(X, (p+1)Y). \end{aligned} \tag{3.10}$$

By Proposition 3.2, the ordinary Hodge filtartion $F^k H^p(X-Y, \mathbb{C})$ of the cohomology $H^p(X-Y, \mathbb{C})$ coincides with $F'^k H^p(X-Y, \mathbb{C})$. Therefore, by (3.9) and (3.10), we conclude that (3.5) certainly holds. Noticing that $I^p(X, (p+1)Y)_0 \simeq I^p(X, *Y)_0$, we obtain (3.6) from (3.5) and Theorem 3.1. □

4 Generalized Poincaré résidue map

The setting under which we shall work in this section is as follows: Let X be a non-singular irreducible algebraic variety of dimension $n+1$ embedded in a sufficiently higher complex projective space $\mathbb{P}^N$, Y a generic hyperplane section of X which satisfies the condition (3.4) in Theorem 3.3, and Y' a non-singular, irreducible hypersurface section of sufficiently higher degree such that if we set $Z = Y \cdot Y'$, then

$$H^p(Y, \Omega_Y^q(kZ)) = 0 \quad \text{for} \quad p \geq 1, q \geq 0 \quad \text{and} \quad k \geq 1. \tag{4.1}$$

When we refer to primitive cohomology, we always means the one concerning the Hodge metric whose fundamental forms is dual to the homology class $[Y]$ (resp. $[Z]$). Under this setting and with the same notation as in the previous sections, the purpose of this section is to define the so-called *generalized Poincaé residue map*

$$\textit{Rés} : I^{n+1}(X, (n+2)Y) \to I^n(Y, (n+1)Z)_0$$

and prove the following theorem:

Theorem 4.1. *Under the setting above, we have*

$$\begin{aligned} F^k H^n(Y, \mathbb{C})_0 &\simeq I_k^n(Y, (n+1)Z)_0 \\ &\simeq \textit{Rés}(I_{k+1}^{n+1}(X, (n+2)Y)) \oplus r^n(I_k^n(X, (n+1)Y')_0)), \end{aligned}$$

where r^n denote the map induced by the natural map $H^n(X, \mathbb{C})_0 \to H^n(Y, \mathbb{C})_0$.

We shall prove the theorem after several lemmas and Propositions. We denote by $\Omega^q(kY + *Y')$ the sheaf of germs of meromorphic q-forms having poles of order k (at most) along Y and poles of arbitrary order along Y' as their only singularities. We denote by $\Omega^q(\log Y + kY')$ the sheaf of germs of meromorphic q-forms having logarithmic poles along Y and poles of order k at most as their only singulatities. We consider the following homomorphisms of comlexes of sheaves of $\mathbb{C}$-vector spaces:

$$\begin{array}{lcccccccc}
\Omega^{\cdot}_X((1+\cdot)Y): & \mathcal{O}_X(Y) & \longrightarrow & \Omega^1_X(2Y) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & \\
\Omega^{\cdot}_X((1+\cdot)Y + *Y')): & \mathcal{O}_X(Y + *Y') & \longrightarrow & \Omega^1_X(2Y + *Y') & \longrightarrow \cdots \\
& \uparrow & & \uparrow & \\
\Omega^{\cdot}_X(\log Y + (1+\cdot)Y'): & \mathcal{O}_X(Y') & \longrightarrow & \Omega^1_X(\log Y + 2Y') & \longrightarrow \cdots
\end{array} \tag{4.2}$$

$$\begin{array}{ccccccc}
\longrightarrow & \Omega^p_X((p+1)Y) & \longrightarrow \cdots \longrightarrow & \Omega^{n+1}_X((n+2)Y) \\
& \downarrow & & \downarrow \\
\longrightarrow & \Omega^p_X((p+1)Y + *Y') & \longrightarrow \cdots \longrightarrow & \Omega^{n+1}_X((n+2)Y + *Y') \\
& \uparrow & & \uparrow \\
\longrightarrow & \Omega^p_X(\log Y + (p+1)Y') & \longrightarrow \cdots \longrightarrow & \Omega^{n+1}_X(\log Y + (n+2)Y')
\end{array}$$

Proposition 4.2. *The homomorphism of complexes of sheaves*

$$\Omega^{\cdot}_X(\log Y + (1+\cdot)Y') \to \Omega^{\cdot}_X((1+\cdot)Y + *Y')$$

in the diagram (??) is a quasi-isomorphism.

Proof. By virtue of Proposition 2.2 it suffices to show that the stalks of the cohomology sheves $\mathcal{H}^p(\Omega^{\cdot}_X(\log Y + (1+\cdot)Y'))$ and $\mathcal{H}^p(\Omega^{\cdot}_X((1+\cdot)Y + *Y')$ are isomorphic at a point $x_0 \in Y \cap Y'$. Let $(z_1, \cdots, z_{n+1})$ be a holomorphic local coordinate system at x_0 such that $z_1 = 0$ and $z_2 = 0$ are local defining equations Y and Y', respectively. We are going to show that

$$
\begin{aligned}
&\mathcal{H}^p(\Omega^{\cdot}_X((1+\cdot)Y + *Y')) \\
(4.3)\qquad &\simeq \mathcal{H}^p(\Omega^{\cdot}_X(\log Y + (1+\cdot)Y')) = \begin{cases} \mathbb{C}_X & p = 0 \\ \mathbb{C}\{\frac{dz_1}{z_1}, \frac{dz_2}{z_2}\} & p = 1 \\ \mathbb{C}\{\frac{dz_1 dz_2}{z_1 z_2}\} & p = 2 \\ 0 & \text{otherwize} \end{cases}
\end{aligned}
$$

Now let $\varphi = dz_1 \wedge \alpha + \beta$ be a local cross-section of $\Phi^p_X((p+1)Y + *Y') \quad (p \geq 1)$ in a neighborhood of x_0, where α, β are local meromorphic forms, having poles of order $p+1$ (at most) along Y and poles of arbitrary order along Y' as their only singularities, and not involving dz_1. Then we may write

$$
\begin{aligned}
\alpha &= \alpha_0 + \frac{\alpha_1}{z_1} + \frac{\alpha_2}{z_1^2} + \cdots + \frac{\alpha_{p+1}}{z_1^{p+1}} \\
\beta &= \beta_0 + \frac{\beta_1}{z_1} + \frac{\beta_2}{z_1^2} + \cdots + \frac{\beta_{p+1}}{z_1^{p+1}},
\end{aligned}
$$

where $\alpha_i, \beta_i \ (i \geq 1)$ do not involve z_1 and dz_1, and $\alpha_i, \beta_i \quad (i \geq 0)$ have poles of arbitrary order (at most) along Y' as their only singularities. Since $d\varphi = 0$, we have

$$
\begin{aligned}
d\varphi = -dz_1 \wedge d\alpha_0 + d\beta_0 &- \frac{dz_1 \wedge d\alpha_1 + d\beta_1}{z_1} - \frac{dz_1 \wedge (d\alpha_2 + \beta_1) - d\beta_2}{z_1^2} \\
&- \cdots - \frac{dz_1 \wedge (d\alpha_{p+1} + p\beta_p) - d\beta_{p+1}}{z_1^{p+1}} - (p+1)\frac{dz_1 \wedge \beta_{p+1}}{z_1^{p+2}} \\
= 0.&
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(4.4)\qquad & d\alpha_1 = d\alpha_2 + \beta_1 = d\alpha_3 + 2\beta_2 = \cdots = d\alpha_{p+1} + p\beta_p = 0, \\
& (p+1)\beta_{p+1} = 0, \\
& d\beta_1 = d\beta_2 = \cdots = d\beta_{p+1} = 0, \\
& d\varphi_0 = 0, \quad \text{where} \quad \varphi_0 = dz_1 \wedge \alpha_0 + \beta_0.
\end{aligned}
$$

Put

$$
\theta = -\frac{\alpha_2}{z_1} - \frac{\alpha_3}{2z_1^2} - \cdots - \frac{\alpha_{p+1}}{pz_1^p},
$$

then

$$
(4.5)\qquad \varphi = d\theta + \frac{dz_1}{z_1} \wedge \alpha_1 + \varphi_0, \quad \text{and} \quad d\varphi_0 = 0.
$$

Hence if $p \geq 3$, since $d\alpha_1 = d\varphi_0 = 0$, there exist local cross-sections γ of $\Omega^{p-2}(*Y')$ and φ_1 of $\Omega^{p-1}(*Y')$ with $d\gamma = \alpha_1$ and $d\varphi_1 = \varphi_0$ in a neighborhood of x_0. Put

$$
\theta_1 = \frac{dz_1}{z_1} \wedge \gamma + \varphi_1,
$$

then $\theta + \theta_1$ is a local cross-section of $\Omega^{p-1}(pY + *Y')$ and $\varphi = d(\theta + \theta_1)$. This shows that $\mathcal{H}^p(\Omega^{\cdot}_X((1+\cdot)Y + *Y')) = 0 \quad$ for $\quad p \geq 3$. If $p = 2$, α_1 in the expression (4.5) of φ is a local cross-section of $\Phi^1(*Y')$.

Hence, as shown in the proof of Lemma 2.1 (ii)-(c), there exists a constant $\lambda^{\in}\mathbb{C}$ and a local cross-section γ of $\Omega^0_X(*Y')$ with

$$\alpha_1 = \lambda\frac{dz_2}{z_2} + d\gamma.$$

Furthermore, since φ_0 is a localcross-section of $\Phi^2(*Y')$, by Lemma 2.1 (i)-(a), there exists a local cross-section φ_1 of $\Omega^1(*Y')$ with $d\varphi_1 = \varphi_0$. Put

$$\theta_1 = \frac{dz_1}{z_1}\wedge\gamma + \varphi_1$$

then $\theta + \theta_1$ is a local cross-section of $\Omega^1(2Y + *Y')$ at x_0 and

$$\begin{aligned}\varphi &= d\theta + \lambda\frac{dz_1\wedge dz_2}{z_1 z_2} + \frac{dz_1}{z_1}\wedge d\gamma + \varphi_0 \\ &= \lambda\frac{dz_1\wedge dz_2}{z_1 z_2} + d(\theta + \theta_1)\end{aligned}$$

This shows that

$$\mathcal{H}^2(\Omega^{\cdot}_X((1+\cdot)Y + *Y'))_{x_0} \simeq \mathbb{C}\{\frac{dz_1\wedge dz_2}{z_1 z_2}\}.$$

If $p_1 = 1$, α_1 is a meromorphic function, hence $d\alpha_1 = 0$ implies that $\alpha_1 = \lambda$, a constant. Since φ_0 is a local cross-section of $\Phi^1(*Y')$, by Lemma 2.1 (ii)-(c), there exists $\varphi_1 \in \Omega^0(*Y')_{x_0}$ such that

$$\varphi_0 = \mu\frac{dz_2}{z_2} + d\varphi_1.$$

Hence the expression of φ in (4.5) becomes

$$\varphi = \lambda\frac{dz_1}{z_1} + \mu\frac{dz_2}{z_2} + d(\varphi_1 + \theta).$$

Since $\varphi_1 + \theta \in \Omega^1(Y + *Y')$, this shows

$$\mathcal{H}^1(\Omega^{\cdot}_X((1+\cdot)Y + *Y')) \simeq \mathbb{C}\{\frac{dz_1}{z_1}, \frac{dz_2}{z_2}\}.$$

$\mathcal{H}^0(\Omega^{\cdot}_X((1+\cdot)Y + *Y')) \simeq \mathbb{C}_X$ is obvious. To prove the same for $\mathcal{H}^p(\Omega^{\cdot}_X(\log Y + (1+\cdot)Y'))$ is rather easy. If φ is a local cross-section of $\Phi^p(\log Y + (p+1)Y')$ in a neighborhood of x_0, then φ is written as

$$\varphi = \frac{dz_1}{z_1}\wedge\alpha + \beta,$$

where $\alpha \in \Omega^{p-1}((p+1)Y')$, $\beta \in \Omega^p((p+1)Y')$ do not involve dz_1. Furthermore, we may assume that α does not involve z_1. Then $d\varphi =$ implies $d\alpha = d\beta = 0$, and by the same arguments as in the case of $\Omega^{\cdot}_X((p+1)Y + *Y')$, we can show that (4.3) for $\mathcal{H}^p(\Omega^{\cdot}_X(\log Y + (p+1)Y))$. □

Lemma 4.3. *Assume we are under the setting at the begining of this section. Particularly, we assume that the following conditions are satisfied:*

$$\begin{aligned}H^p(X, \Omega^p_X(kY)) &= 0, \\ H^p(Y, \Omega^p_Y(kZ)) &= 0 \quad \textit{for} \quad p \geq 1, q \geq 0, k \geq 1.\end{aligned}$$

Then we have

$$H^p(X, \Omega^q_X(\log Y + (q+1)Y')) = 0 \quad \textit{for} \quad p \geq 1, q \geq 0.$$

Proof. We consider the following exact sequence

$$0 \to \Omega_X^q \to \Omega_X^q(\log Y) \xrightarrow{R} \Omega_Y^{q-1} \to 0 \quad (q \geq 1),$$

where R is the résidue map (cf. Lemma 2.1 (ii)-(c)). Tensoring $\mathcal{O}_X((q+1)Y')$ to this exact sequence, we have

$$0 \to \Omega_X^q((q+1)Y') \to \Omega_X^q(\log Y + (q+1)Y') \to \Omega_Y^{q-1}((q+1)Z) \to 0.$$

From the long exact sequence of cohomology associated to this sequence, the assertion of the lemma follows. □

We define

$$\begin{aligned} I^p(X, \log Y + (p+1)Y') &:= \frac{\Gamma(X, \Phi_X^p(\log Y + (p+1)Y'))}{d\Gamma(X, \Omega_X^{p-1}(\log Y + pY'))}, \\ I^p(X, (p+1)Y + *Y') &:= \frac{\Gamma(X, \Phi_X^p((p+1)Y + *Y'))}{d\Gamma(X, \Omega_X^{p-1}(pY + *Y'))}. \end{aligned}$$

Combining Proposition 4.2 with Lemma 4.3 implies the following:

Proposition 4.4. *Assume that we are under the setting at the bigining of this section. Then*

$$I^p(X, \log Y + (p+1)Y') \simeq I^p(X, (p+1)Y + *Y') \quad \textit{for} \quad p \geq 0.$$

We are now ready to define the Résidue map

$$\textit{Rés} : I^p(X, (p+1)Y) \to I^{p-1}(Y, pZ)_0$$

Let $\omega \in \Gamma(X, \Phi_X^p((p+1)Y))$ be given. We think of ω as an element of $\Gamma(X, \Phi_X^p((p+1)Y + *Y')$. Then, by Propostion 4.4, there exists a $\varphi \in \Gamma(X, \Omega_X^{p-1}(pY + *Y')))$ such that $\omega - d\varphi \in \Gamma(\Phi_X^p(\log Y + (p+1)Y'))$. We take an open covering $\{U_i\}_{i \in I}$ of X such that there is a local coordinate system $(z_1^i, \cdots, z_{n+1}^i)$ on each U_i, satisfying the following conditions:

(4.6)
(a) If $U_i \cap Y \neq \emptyset, z_1^i = 0$ is a defining equation of Y in U_i.
(b) If $U_i \cap (Y \cap Y') \neq \emptyset, z_1^i = 0$ and $z_2^i = 0$ are defining equations of Y and Y' in U_i, respectively.

In each U_i with $U_i \cap Y \neq \emptyset$, we can write $\omega - d\varphi$ as

$$\omega - d\varphi = \frac{dz_1^i}{z_1^i} \wedge \alpha_i + \beta_i, \tag{4.7}$$

where $\alpha_i \in \Gamma(U_i, \Phi_X^{p-1}((p+1)Y'))$, $\beta_i \in \Gamma(U_i, \Phi_X^p((p+1)Y'))$, α_i and β_i does not involve dz_1^i. We can easily see $\alpha_{i|Y} = \alpha_{j|Y}$ if $U_i \cap U_j \cap Y \neq \emptyset$, hence $\{\alpha_{i|Y}\}$ defines an element of $\Gamma(Y, \Phi_X^{p-1}((p+1)Z))$. We claim that $\{2\pi\sqrt{-1}\alpha_{i|Y}\}$ determine a unique element of $I^{p-1}(Y, (p+1)Z))$, not depending on the chice of φ. In fact, if φ' is another element of $\Gamma(X, \Omega_X^{p-1}((p+1)(Y+Y'))$ with $\omega - d\varphi' \in \Gamma(X, \Phi_X^p(\log Y + (p+1)Y'))$ and

$$\omega - d\varphi' = \frac{dz_1^i}{z_1^i} \wedge \alpha_i' + \beta_i'$$

is the expression of $\omega - d\varphi'$ as in (4.7), then

$$d(\varphi' - \varphi) = \frac{dz_1^i}{z_1^i} \wedge (\alpha_i - \alpha_i') + (\beta_i - \beta_i') \in \Gamma(X, \Phi_X^p(\log Y + (p+1)Y')$$

is zero in $I^p(X, \log Y + (p+1)Y')$. Hence, by Proposition 4.4, there exists an element $\psi \in \Omega_X^{p-1}(\log Y + pY'))$ such that $d\psi = d(\varphi' - \varphi)$. Let

$$\psi = \frac{dz_1^i}{z_1^i} \wedge \gamma_i + \delta_i$$

be the expression of ψ as in (4.7). Then, since $d\psi = d(\varphi' - \varphi)$, we have

$$d\gamma_{i|Y} = d_Y(\gamma_{i|Y}) = \alpha_{i|Y} - \alpha'_{i|Y} \tag{4.8}$$

for each i with $U_i \cap Y \neq \emptyset$, where d_Y denotes the exterior derivative on Y. Since $\{\gamma_{i|Y}\}$ is a global cross-section of $\Gamma(Y, \Omega_X^{p-1}(pZ))$, (4.8) shows that $\{\alpha_{i|Y}\} = \{\alpha'_{i|Y}\}$ in $I^{p-1}(Y, (p+1)Z)$. Furthermore, the arguments above also show that if ω is a derived form, then so is $\{\alpha'_{i|Y}\}$. Therefore, we conclude that the correspondence

$$\omega \longmapsto \{\alpha'_{i|Y}\}$$

determine a map $I^p(X, (p+1)Y) \to I^{p-1}(Y, (p+1)Z)$. Since $I^{p-1}(Y, (p+1)Z) \simeq I^{p-1}(Y, pZ)$ by Proposition 2.3, this map is thought of as a map from $I^p(X, (p+1)Y)$ to $I^{p-1}(Y, pZ)$, which we define to be the *generalized Poncaré* résidue map and denote it *Rés*. We denote $\{\alpha'_{i|Y}\}$ by $\textit{rés}[\omega]$ (determined up to derived forms) and call *résidue form* of ω.

Proposition 4.5.

$$\textit{Rés}(I^p(X, (p+1)Y)) \subset I^{p-1}(Y, pZ)_0$$

Proof. For a $\omega \in \Gamma(X, \Phi^p((p+1)Y))$, we shall show that its résidue form $\textit{rés}[\omega] = \{\alpha_{i|Y}\}$ (precisely speaking, a closed form representing the class $\textit{rés}[\omega]$ of $I^{p-1}(Y, (p+1)Z))$ is of the second kind in the sense of Picard-Lefshetz. From this the assertion of the proposition follows, since $I^{p-1}(Y, (p+1)Z)_0 \simeq I^{p-1}(Y, pZ)_0$. As before we take an open covering $\{U_i\}_{i\in I}$ of X such that there is a local coordinate system $(z_1^i, \cdots, z_{n+1}^i)$ on each U_i, subject to the conditions in (4.6), and take a $\varphi \in \Gamma(X, \Omega^{p-1}(pY + *Y'))$ such that $\omega - d\varphi \in \Gamma(X, \Phi_X^p(\log Y + (p+1)Y'))$. On each U_i with $U_i \cap Y \neq \emptyset$, we write

$$\omega - d\varphi = \frac{dz_1^i}{z_1^i} \wedge \alpha_i + \beta_i \tag{4.9}$$

as in (4.7). We will show that for a point $x_0 \in Z \cap U_i$, $\textit{rés}[\omega]_{|U_i} = \alpha_{i|Y}$ is a holomorphic form modulo derived meromorphic forms in a sufficiently small neighborhood of x_0 in Y. For this end we take a generic prime hypersurface section Y'' which is linearly equivalent to Y', which does not go through x_0 and intersect Y and Y' transversely. We think ω as an element of $\Gamma(X, \Phi^p((p+1)Y + *Y''))$. Since $I^p(X, (p+1)Y + *Y'')) \simeq I^p(X, \log Y + (p+1)Y'')$ by Proposition 4.4, there exists a $\varphi' \in \Gamma(X, \Omega^{p-1}(pY + *Y''))$ with $\omega - d\varphi' \in \Gamma(X, \Phi^p(\log Y + (p+1)Y''))$. Let

$$\omega - d\varphi' = \frac{dz_1^i}{z_1^i} \wedge \alpha'_i + \beta'_i \tag{4.10}$$

be the expression of $\omega - d\varphi'$ as in (4.7) on each $U_i \cap Y \neq \emptyset$. If U_{i_0} is the coordinate neighborhood with $x_0 \in U_{i_0} \cap Z$, since Y'' does not go through x_0, $\alpha'_{i_0|Y}$ is holomorphic in a sufficiently open neighborhood of x_0 in $U_{i_0} \cap Y$. From (4.9) and (4.10),

$$d(\varphi' - \varphi) = \frac{dz_1^{i_0}}{z_1^{i_0}} \wedge (\alpha_{i_0} - \alpha'_{i_0}) + (\beta_{i_0} - \beta'_{i_0}). \tag{4.11}$$

Since $d(\varphi' - \varphi) \in \Gamma(X, \Phi^p(\log Y + *(Y' + Y'')))$ is zero in $I^p(X, (p+1)Y + *(Y' + Y''))$, by Proposition 4.4, there exists a $\psi \in \Gamma(X, \Omega_X^{p-1}(\log Y + *(Y' + Y'')))$ with $d\psi = d(\varphi' - \varphi)$. On each U_i, we write

$$\psi = \frac{dz_1^i}{z_1^i} \wedge \gamma_i + \xi_{i_0} \tag{4.12}$$

as in (4.7). Then $d\psi = d(\varphi' - \varphi)$ implies

$$d\gamma_i = \alpha_i - \alpha'_i.$$

Hence $d_Y(\gamma_{i|Y}) = \alpha_{i|Y} - \alpha'_{i|Y}$ for each i where d_Y denotes the exterior derivation on Y. This means $d_Y(\textit{rés}[\psi]) = \textit{rés}[\alpha] - \textit{rés}[\alpha']$ where $\textit{rés}[\psi] \in \Gamma(Y, \Omega_Y^{p-2}(p(Y' + Y'')))$. Since $\textit{rés}[\alpha']$ is holomorphic at x_0, so is $\textit{rés}[\alpha]$ modulo derived meromorphic forms as requied. $\square$

Proof of Theorem 4.1:

We can now easily deduce Theorem 4.1 from what we have proved till now. First, by Theorem 1.15,

$$H^n(Y,\mathbb{C})_0 = R^{n+1}(H^{n+1}(X-Y,\mathbb{C})) \oplus r^n(H^n(X,\mathbb{C})_0).$$

By Theorem 3.1,

$$F^k(H^n(Y,\mathbb{C})_0 = R^{n+1}(F^{k+1}H^{n+1}(X-Y,\mathbb{C})) \oplus r^n(F^kH^n(X,\mathbb{C})_0). \tag{4.13}$$

By Theorem 3.3, (3.5),

$$F^{k+1}H^{n+1}(X-Y,\mathbb{C})) \simeq I^{n+1}_{k+1}(X,(n+2)Y). \tag{4.14}$$

Applying Theorem 3.3, (3.6) to the pair (X,Y') instead of (X,Y), we have

$$F^kH^n(X,\mathbb{C}))_0 \simeq I^n_k(X,(n+1)Y')_0. \tag{4.15}$$

From (4.13), (4.14) and (4.15) it follows that

$$F^kH^n(Y,\mathbb{C}))_0 = R^{n+1}I^{n+1}_{k+1}(X,(n+2)Y') \oplus r^n(I^n_k(X,(n+1)Y')_0).$$

Here the map $R^{n+1}: I^{n+1}(X,(n+2)Y) \simeq H^{n+1}(X-Y,\mathbb{C}) \to H^n(Y,\mathbb{C})$ should be interpreted in terms of C^∞ De Rham group as follows: By use of isomorphisms

$$H^{n+1}(X-Y,\mathbb{C}) \simeq \mathbb{H}^{n+1}(X,\Omega^{\cdot}_X(\log Y)) \simeq I^{n+1}(X,(n+2)Y) \simeq H^{n+1}(A^{\cdot}(\log Y)),$$

(cf. Proposition 2.2 and its proof), we can take a $\varphi \in \mathrm{Ker}\{(A^{n+1}(\log Y)) \to A^{n+2}(\log Y)\}$ with $\omega = \varphi$ modulo $dA^n(\log Y)$ for a $\omega \in \Gamma(X,\Phi^{n+1}((n+2)Y))$. φ is written as

$$\varphi = \alpha \wedge \eta + \beta,$$

where η is C^∞ form of type $(1,0)$ with the property $\overline{\partial}\eta$ represents the first Chern class $c_1([Y])$, and $\alpha \in A^{n-1}(X)$, $\beta \in A^{n+1}(X)$ (cf. (1.4). $d\varphi = 0$ implies $d_Y(\alpha_{|Y}) = 0$. Then $R^{n+1}([\omega])$ ($[\omega] \in I^{n+1}(X,(n+2)Y))$ is defined by

$$R^{n+1}([\omega]) = 2\pi\sqrt{-1}\Big[\alpha_{|Y}\Big],$$

where $[\alpha_{|Y}]$ denote the De Rham cohomology class represented by $\alpha_{|Y}$. Taking into consideration this fact, we will be done if we see

$$R^{n+1}(I^{n+1}_{k+1}(X,(n+2)Y)) = R\acute{e}s(I^{n+1}_{k+1}(X,(n+2)Y) \tag{4.16}$$

in the De Rham cohomology. To see this, we first note that both of the right and left hand sides of (4.16) are included in $H^n(Y,\mathbb{C})_0$. due to Theorem 1.15 and Theorem 2.4. Hence, by Proposition 1.9 and Proposition 1.10, in order to prove (4.16), it suffices to show that

$$\int_{\tau_\varepsilon(\gamma)} \omega = \int_\gamma r\acute{e}s[\omega] \tag{4.17}$$

for a $\omega \in \Gamma(X,\Phi^{n+1}((n+2)Y))$ and an n cycle γ lying in $Y-Z$, where $\tau_\varepsilon(\gamma)$ is $\partial U_{\varepsilon|\gamma}$, the restriction of the boundary of a topological ε *tublorneighborhood* U_ε of Y in X to γ. We are now going to prove (4.17). We take the local expression (4.7) of ω with respect to some open covering $\{U_i\}_{i\in I}$ of X and a local coordinate system $(z^i_1,\cdots,z^i_{n+1})$ on each U_i, subject to the conditions in (4.6). Let $\{\rho_i\}$ be a partition of unity subordinate to the covering $\{U_i\}_{i\in I}$. Then

$$\begin{aligned}
\int_{\tau_\varepsilon(\gamma)} \omega &= \int_{\tau_\varepsilon(\gamma)} \sum_i \rho_i(\frac{dz^i_1}{z^i_1} \wedge \alpha_i + \beta_i) + d\varphi \\
&= \int_{\tau_\varepsilon(\gamma)} \sum_i \rho_i(\frac{dz^i_1}{z^i_1} \wedge \alpha_i + \beta_i) \\
&= \sum_i \int_{\tau_\varepsilon(\gamma)} \rho_i(\frac{dz^i_1}{z^i_1} \wedge \alpha_i + \beta_i).
\end{aligned}$$

Locally, $\tau_\varepsilon(\gamma)$ looks like $\mathbb{R}^{n+1} \times \{\ |z| = \varepsilon \mid z \in \mathbb{C}\ \}$ $(\varepsilon > 0)$. Hence

$$\begin{aligned}\sum_i \int_{\tau_\varepsilon(\gamma)\cap U_i} \rho_i\Big(\frac{dz_1^{(i)}}{z_1^{(i)}} \wedge \alpha_i + \beta_i\Big) &= \lim_{\varepsilon\to 0} \sum_i \int_{\tau_\varepsilon(\gamma)\cap U_i} \rho_i\Big(\frac{dz_1^i}{z_1^i} \wedge \alpha_i + \beta_i\Big) \\ &= 2\pi\sqrt{-1} \sum_i (\rho_i \alpha_i)_{|\gamma \cap U_i} \\ &= 2\pi\sqrt{-1} r\acute{e}s[\omega]\end{aligned}$$

as required. This completes the proof of Theorem 4.1.

Remark 4.1. For $[\omega] \in I_{k+1}^{n+1}(X, (n+2)Y)$ it can be proved more directly that the Hodge type of $R^{n+1}([\omega]) = R\acute{e}s([\omega])$ is $(n, 0) + (n-1, 1) + \cdots + (k, n-k)$. By virtue of the isomorphism

$$I_{k+1}^{n+1}(X, (n+2)Y) \simeq H^{n+1}(F'^{k+1}(B^{\cdot}))$$

$$= \frac{\mathrm{Ker}\{\sum_{\ell=0}^{n-k} B_X^{n-\ell+1,\ell}(n-\ell-k+1) \xrightarrow{d} \sum_{\ell=0}^{n-k+1} B_X^{n-\ell+2,\ell}(n-\ell-k+2)\}}{\mathrm{Im}\{\sum_{\ell=0}^{n-k-1} B_X^{n-\ell,\ell}(n-\ell-k) \xrightarrow{d} \sum_{\ell=0}^{n-k} B_X^{n-\ell+1,\ell}(n-\ell-k+1)\}}$$

(cf. the proof of Theorem 3.3, (3.5)), $\omega \in \Gamma(X, \Phi_X^{n+1}((n+2)Y)$ is cohomologous to a closd form φ of $\sum_{\ell=0}^{n-k} B_X^{n-\ell+1,\ell}(n-\ell-k+1)$ in the De Rham cohomology. If we wtite φ as

$$\varphi = \varphi^{(n+1,0)} + \varphi^{(n,1)} + \cdots + \varphi^{(k+1,n-k)},$$

where $\varphi^{(n-\ell+1,\ell)} \in B_X^{n+\ell-1,\ell}(n-\ell-k+2)$ $\quad(0 \underline{<} \ell \underline{<} n-k)$, then each $\varphi^{(n-\ell+1,\ell)}$ is written in each U_i as

$$\varphi^{(n-\ell+1,\ell)} = \frac{\alpha_i^{(n-\ell,\ell)} dz_1^i}{(z_1^i)^{n-\ell-k+1}} + \frac{\beta_i^{(n-\ell+1,\ell)}}{(z_1^i)^{n-\ell-k}}$$

where $\alpha_i^{(n-\ell,\ell)}$, $\beta_i^{(n-\ell+1,\ell)}$ are regular C^∞ differential forms of types $(n-\ell, \ell)$, $(n-\ell+1, \ell)$, respectively, not involving z_1^i, where $z_1^i = 0$ is the local defining equation of Y. This is because $(z_1^i)^{n-\ell-k}\varphi^{(n-\ell+1,\ell)}$ and $(z_1^i)^{n-\ell-k} dz_1^i \wedge \varphi^{(n-\ell+1,\ell)}$ are C^∞ regular forms by the definition of $B_X^{n-\ell+1,\ell}(n-\ell-k+1)$. Put

$$\psi_i^{(n-\ell,\ell)} := \frac{\alpha_i^{(n-\ell,\ell)}}{(n-\ell-k)(z_1^i)^{n-\ell-k}} \qquad (0 \underline{<} \ell \underline{<} n-k-1),$$

then

$$\begin{aligned}\eta_i^{(n-\ell+1,\ell)+(n-\ell,\ell+1)} &:= d\psi_i^{(n-\ell,\ell)} + \varphi^{(n-\ell+1,\ell)} \\ &= \frac{d\alpha_i^{(n-\ell,\ell)}}{(n-\ell-k)(z_1^i)^{n-\ell-k}} + \frac{\beta_i^{(n-\ell+1,\ell)}}{(z_1^i)^{n-\ell-k}}\end{aligned}$$

is a semi-meromorphic form of type $(n-\ell+1, \ell) + (n-\ell, \ell+1)$ and has poles of order $n-\ell-k$ along Y. Let $\{\rho_1\}$ be a partition of unity subodinate to the open covering $\{U_i\}_{i\in I}$ as before. We put

$$\begin{aligned}\psi^{(n-\ell,\ell)} &= \sum_i \rho_i \psi_i^{(n-\ell,\ell)}, \\ \eta^{(n-\ell+1,\ell)+(n-\ell,\ell+1)} &= \sum_i \rho_i \eta_i^{(n-\ell+1,\ell)+(n-\ell,\ell+1)}\end{aligned}$$

Now,

$$\begin{aligned}\varphi^{(n-\ell+1,\ell)} - d\psi^{(n-\ell,\ell)} &= \varphi^{(n-\ell+1,\ell)} - \sum_i d\rho_i \psi_i^{(n-\ell,\ell)} + \sum_i \rho_i d\psi_i^{(n-\ell,\ell)} \\ &= \sum_i \rho_i \eta_i^{(n-\ell+1,\ell)+(n-\ell,\ell+1)} - \sum_i d\rho_i \psi_i^{(n-\ell,\ell)} \\ &= \eta^{(n-\ell+1,\ell)+(n-\ell,\ell+1)} - \sum_i d\rho_i \psi_i^{(n-\ell,\ell)}\end{aligned}$$

which is a semi-morphic form of type $(n-\ell+1,\ell)+(n-\ell,\ell+1)$ having poles of order $n-\ell-k$ along Y. Continuing this process, $\varphi^{(n-\ell+1,\ell)}$ $(0\leq\ell\leq n-k)$ is reduced to a semi-meromorphic form of thpe $(n-\ell+1,\ell)+\cdots+(k+1,n-k)$, having poles of order 1 along Y modulo derived forms. Hence φ is reduced to a closed semi-meromorphic form ξ of $A^{n+1,0}(\log Y)+\cdots+A^{k+1,n-k}(\log Y)$ modulo derived forms. Hence the Hodge type of $R^{n+1}([\omega])=R^{n+1}([\xi])$ is $(n,0)+(n-1,1)+\cdots+(k,n-k)$.

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