RLS Fixed-Lag Smoother using Covariance Information in Linear Continuous Stochastic Systems

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Abstract

This paper newly designs the recursive least-squares fixed-lag smoother and filter using covariance information in linear continuous-time stochastic systems. It is assumed that the signal is observed with added white observation noise and the signal is uncorrelated with the observation noise. The estimators require the covariance information of the signal in the semi-degenerate kernel form and the variance of the observation noise. The proposed estimators are appropriate for estimations of stationary or non-stationary stochastic signal generally.

Keywords: Linear Continuous Systems, Fixed-Lag Smoother, Least-Squares Estimation, Covariance Information, Wiener-Hopf Integral Equation, Stochastic Signal, fixed-lag smoother

1 Introduction

There are three types of smoothers known as fixed-interval Kalman smoother, fixed-point Kalman smoother and fixed-lag Kalman smoother. These Kalman smoothers use the information of the state-space model [1], [2]. Usually, the smoother shows better estimation accuracy than the filter.

If the model used in the filter is different from the actual system dynamics, then the filter will diverge. This may also happen if the system has multiple modes of operation; the filter can only describe one of them. In interacting multiple model (IMM) [3], the fixed-lag smoother shows that its accuracy in terms of mean squared error increases proportionally to the lag [4]. In the target tracking on the behavior of fish, the Segmenting Track Identifier, a non-Bayesian curve fitting and segmenting tracker, is shown to be most effective for tracking the unpredictable and complex horizontal motion of fish, while a Kalman fixed-interval and fixed-lag smoothers using a constant-velocity model is shown to be most effective for tracking the more predictable and piecewise linear vertical motion of fish [5].

By the way, in the estimation problems of the signal, there is an approach to use the covariance information of the signal and the observation noise [6], [7], [8]. In [7] the estimation technique on the recursive Wiener estimators etc. are considered. In [9], [10], the recursive Wiener fixed-lag smoother is designed based on the innovations approach. The impulse response function for the fixed-lag smoothing estimate is approximated and the suboptimal fixed-lag smoothing algorithms are proposed. It is a characteristic that the recursive Wiener estimator uses the information of the observation matrix, the system matrix for the state variable and the cross-variance function of the state variable with the signal.

In the fixed-lag smoother [11], the auto-covariance function of the signal is expressed in the degenerate kernel form. The degenerate kernel function has a limitation that it cannot express the auto-covariance functions of stationary or non-stationary stochastic processes in general. Hence, the fixed-lag smoother using the auto-covariance function in the form of the degenerate kernel can not be applied to the estimation of the stochastic signal processes with the auto-covariance function expressed except in the degenerate kernel form. In [12], by assuming that the fixed-lag smoothing estimate is given as a sum of the filtering estimate of the signal and correction term to the filtering estimate, the recursive least-squares (RLS) fixed-lag smoother using the covariance information is designed in linear continuous-time stochastic systems. The auto-covariance function of the signal is expressed in the semi-degenerate kernel form. The semi-degenerate kernel can express the auto-covariance function of the stationary or non-stationary stochastic signal processes generally by a finite sum of nonrandom functions. It is a characteristic that the estimators in [11] and [12] do not use the state-space model for the signal.

In this paper, under the same assumptions for the signal and the observations noise with [12], the RLS fixed-lag smoother is newly designed in linear continuous-time stochastic systems. It is assumed that the signal is observed with additive white observation noise and the signal is uncorrelated with the observation noise. The fixed-lag smoother requires the information of the covariance function of the signal and the variance of the observation noise. The auto-covariance function of the signal is expressed in the semi-degenerate kernel form. Unlike the fixed-lag smoother in [12], the fixed-lag smoothing estimate is not given as a sum of the filtering estimate and the correction term based on the effect of the smoothing. The current fixed-lag smoother is designed based on the linear least-squares criterion (5) and has different calculation structure from the fixed-lag smoother in [12].

A numerical simulation example is shown to compare the estimation accuracy of the proposed RLS fixed-lag smoother with that in [12].

2 Fixed-lag smoothing problem

Let an observation equation be given by

$$y(t) = z(t) + v(t), \tag{1}$$

in linear continuous-time stochastic systems, where $z(t)$ is an $n \times 1$ signal vector and $v(t)$ is a white observation noise. It is assumed that the signal and the observation noise are mutually independent stochastic processes with zero means. Let the auto-covariance function of $v(t)$ be given by

$$E[v(t)v^T(s)] = R(t)\delta(t-s), \quad R(t) > 0. \tag{2}$$

Here, $\delta(\cdot)$ denotes the Dirac δ function.

Let $K(t,s)$ represent the auto-covariance function of the signal and let $K(t,s)$ be expressed in the semi-degenerate kernel [12] form of

$$K(t,s) = \begin{cases} A(t)B^T(s), & 0 \leqslant s \leqslant t, \\ B(t)A^T(s), & 0 \leqslant t \leqslant s. \end{cases} \tag{3}$$

Here, $A(t)$ and $B(s)$ are bounded $n \times m$ matrices.

Let a fixed-lag smoothing estimate $\hat{z}(t-D,t)$ of $z(t-D)$ be given by

$$\hat{z}(t-D,t) = \int_0^t h(t,s)y(s)ds \tag{4}$$

as a linear integral transformation of the observed value $\{\ y(s),\ \ 0 \leqslant s \leqslant t\ \}$, where $h(t,s)$ and D are referred to be an impulse response function and the fixed lag.

The impulse response function, which minimizes the mean-square value of the fixed-lag smoothing error $z(t-D) - \hat{z}(t-D,t)$,

$$J = E[\|z(t-D) - \hat{z}(t-D,t)\|^2], \tag{5}$$

satisfies

$$K(t-D,s)=\int_0^t h(t,\tau)K(\tau,s)d\tau \tag{6}$$

by an orthogonal projection lemma [1], [2]:

$$z(t-D)-\hat{z}(t-D,t)\perp y(s),\ 0\leqslant s\leqslant t. \tag{7}$$

Here, "$\perp$" denotes the notation of the orthogonality. From (1), (2) and (6), the linear least-squares impulse response function satisfies

$$h(t,s)R(s)=K(t-D,s)-\int_0^t h(t,\tau)K(\tau,s)d\tau. \tag{8}$$

3 Fixed-lag smoothing and filtering algorithms

The fixed-lag smoothing problem by starting with (8) has been considered to be difficult in the derivation of the least-squares fixed-lag estimation equations. In this paper, for the values of t and s, let the impulse response function $h(t,s)$ be separated into $h_1(t,s)$ and $h_2(t,s)$ as

$$h(t,s)=\begin{cases} h_1(t,s), & 0\leqslant s\leqslant t-D\leqslant t,\\ h_2(t,s), & 0\leqslant t-D\leqslant s\leqslant t.\end{cases} \tag{9}$$

Hence, from (4) and (9), the fixed-lag smoothing estimate might be written as

$$\hat{z}(t-D,t)=\int_0^{t-D} h_1(t,s)y(s)ds+\int_{t-D}^t h_2(t,s)y(s)ds. \tag{10}$$

For $0\leqslant s\leqslant t-D$, the impulse response function $h_1(t,s)$ satisfies

$$h_1(t,s)R(s)=K(t-D,s)-\int_0^{t-D} h_1(t,\tau)K(\tau,s)d\tau-\int_{t-D}^t h_2(t,\tau)K(\tau,s)d\tau. \tag{11}$$

From (3), by $K(t-D,s)=A(t-D)B^T(s)$, $0\leqslant s\leqslant t-D\leqslant t$, (11) is written as

$$h_1(t,s)R(s)=A(t-D)B^T(s)-\int_0^{t-D} h_1(t,\tau)K(\tau,s)d\tau-\int_{t-D}^t h_2(t,\tau)A(\tau)d\tau B^T(s). \tag{12}$$

By introducing

$$F(t)=\int_{t-D}^t h_2(t,\tau)A(\tau)d\tau, \tag{13}$$

(12) is written as

$$h_1(t,s)R(s)=(A(t-D)-F(t))B^T(s)-\int_0^{t-D} h_1(t,\tau)K(\tau,s)d\tau. \tag{14}$$

Also, for $0\leqslant t-D\leqslant s\leqslant t$, the impulse response function $h_2(t,s)$ satisfies

$$h_2(t,s)R(s)=K(t-D,s)-\int_0^{t-D} h_1(t,\tau)B(\tau)d\tau A^T(s)-\int_{t-D}^t h_2(t,\tau)K(\tau,s)d\tau. \tag{15}$$

From (3), by $K(t-D,s)=B(t-D)A^T(s)$, $0\leqslant t-D\leqslant s\leqslant t$, (15) is written as

$$h_2(t,s)R(s)=B(t-D)A^T(s)-\int_0^{t-D} h_1(t,\tau)B(\tau)d\tau A^T(s)-\int_{t-D}^t h_2(t,\tau)K(\tau,s)d\tau. \tag{16}$$

By introducing

$$G(t)=\int_0^{t-D} h_1(t,\tau)B(\tau)d\tau, \tag{17}$$

(16) is written as

$$h_2(t,s)R(s)=(B(t-D)-G(t))A^T(s)-\int_{t-D}^t h_2(t,\tau)K(\tau,s)d\tau. \tag{18}$$

Introducing an auxiliary function $J_1(t,s)$ which satisfies

$$J_1(t,s)R(s)=B^T(s)-\int_0^{t-D} J_1(t,\tau)K(\tau,s)d\tau, \tag{19}$$

we obtain, from (14) and (19), the impulse response function

$$h_1(t,s)=(A(t-D)-F(t))J_1(t,s). \tag{20}$$

Also, introducing

$$J_2(t,s)R(s)=A^T(s)-\int_{t-D}^t J_2(t,\tau)K(\tau,s)d\tau, \tag{21}$$

from (18) and (21), we obtain

$$h_2(t,s)=(B(t-D)-G(t))J_2(t,s). \tag{22}$$

Substituting (20) into (17), we have

$$G(t)=(A(t-D)-F(t))\int_{0}^{t-D}J_1(t,\tau)B(\tau)d\tau. \tag{23}$$

By introducing a function

$$r_1(t)=\int_{0}^{t-D}J_1(t,\tau)B(\tau)d\tau, \tag{24}$$

(23) is written as

$$G(t)=(A(t-D)-F(t))r_1(t). \tag{25}$$

Substituting (22) into (13), we have

$$F(t)=(B(t-D)-G(t))\int_{t-D}^{t}J_2(t,\tau)A(\tau)d\tau. \tag{26}$$

By introducing

$$r_2(t)=\int_{t-D}^{t}J_2(t,\tau)A(\tau)d\tau, \tag{27}$$

(26) is written as

$$F(t)=(B(t-D)-G(t))r_2(t). \tag{28}$$

Differentiating (19) with respect to t, we have

$$\frac{\partial J_1(t,s)}{\partial t}R(s)=-J_1(t,t-D)K(t-D,s)-\int_{0}^{t-D}\frac{\partial J_1(t,\tau)}{\partial t}K(\tau,s)d\tau. \tag{29}$$

From (3) for $0\leqslant s\leqslant t-D$, comparing (29) with (19), we obtain

$$\frac{\partial J_1(t,s)}{\partial t}=-J_1(t,t-D)A(t-D)J_1(t,s). \tag{30}$$

Differentiating (24) with respect to t, we have

$$\frac{dr_1(t)}{dt}=J_1(t,t-D)B(t-D)+\int_{0}^{t-D}\frac{\partial J_1(t,\tau)}{\partial t}B(\tau)d\tau. \tag{31}$$

Substituting (30) into (31), we have

$$\frac{dr_1(t)}{dt}=J_1(t,t-D)B(t-D)-J_1(t,t-D)A(t-D)\int_{0}^{t-D}J_1(t,\tau)B(\tau)d\tau. \tag{32}$$

From (24), we obtain

$$\frac{dr_1(t)}{dt}=J_1(t,t-D)(B(t-D)-A(t-D)r_1(t)). \tag{33}$$

Here, from (24), the initial condition on the differential equation (33) is $r_1(D)=0$ and $r_1(\tau)=0,\quad 0\leqslant\tau\leqslant D$.

Differentiating (21) with respect to t and using (3), we have

$$\begin{aligned}\frac{\partial J_2(t,s)}{\partial t}R(s)&=-J_2(t,t)K(t,s)+J_2(t,t-D)K(t-D,s)-\int_{t-D}^{t}\frac{\partial J_2(t,\tau)}{\partial t}K(\tau,s)d\tau\\&=-J_2(t,t)A(t)B^T(s)+J_2(t,t-D)K(t-D,s)-\int_{t-D}^{t}\frac{\partial J_2(t,\tau)}{\partial t}K(\tau,s)d\tau.\end{aligned} \tag{34}$$

Since $0\leqslant t-D\leqslant s$, (34) is written as

$$\frac{\partial J_2(t,s)}{\partial t}R(s)=-J_2(t,t)A(t)B^T(s)+J_2(t,t-D)B(t-D)A^T(s)-\int_{t-D}^{t}\frac{\partial J_2(t,\tau)}{\partial t}K(\tau,s)d\tau. \tag{35}$$

By introducing a function

$$J_3(t,s)R(s)=B^T(s)-\int_{t-D}^{t}J_3(t,\tau)K(\tau,s)d\tau, \tag{36}$$

we obtain

$$\frac{\partial J_2(t,s)}{\partial t}=-J_2(t,t)A(t)J_3(t,s)+J_2(t,t-D)B(t-D)J_2(t,s). \tag{37}$$

Differentiating (36) with respect to t and using (3), we have

$$\begin{aligned}\frac{\partial J_3(t,s)}{\partial t}R(s)&=-J_3(t,t)K(t,s)+J_3(t,t-D)K(t-D,s)-\int_{t-D}^{t}\frac{\partial J_3(t,\tau)}{\partial t}K(\tau,s)d\tau\\&=-J_3(t,t)A(t)B^T(s)+J_3(t,t-D)K(t-D,s)-\int_{t-D}^{t}\frac{\partial J_3(t,\tau)}{\partial t}K(\tau,s)d\tau.\end{aligned} \tag{38}$$

Since $0 \leqslant t-D \leqslant s$, (38) is written as

$$\frac{\partial J_3(t,s)}{\partial t}R(s) = -J_3(t,t)A(t)B^T(s) + J_3(t,t-D)B(t-D)A^T(s) - \int_{t-D}^{t} \frac{\partial J_3(t,\tau)}{\partial t}K(\tau,s)d\tau. \tag{39}$$

Comparing (39) with (21) and (36), we obtain

$$\frac{\partial J_3(t,s)}{\partial t} = -J_3(t,t)A(t)J_3(t,s) + J_3(t,t-D)B(t-D)J_2(t,s). \tag{40}$$

Differentiating (27) with respect to t, we have

$$\frac{dr_2(t)}{dt} = J_2(t,t)A(t) - J_2(t,t-D)A(t-D) + \int_{t-D}^{t} \frac{\partial J_2(t,\tau)}{\partial t}A(\tau)d\tau. \tag{41}$$

Substituting (37) into (41), we have

$$\begin{aligned}\frac{dr_2(t)}{dt} &= J_2(t,t)A(t) - J_2(t,t-D)A(t-D) - \\ &J_2(t,t)A(t)\int_{t-D}^{t} J_3(t,\tau)A(\tau)d\tau + J_2(t,t-D)B(t-D)\int_{t-D}^{t} J_2(t,\tau)A(\tau)d\tau.\end{aligned} \tag{42}$$

Introducing a function

$$r_3(t) = \int_{t-D}^{t} J_3(t,\tau)A(\tau)d\tau \tag{43}$$

and using (27), we obtain

$$\frac{dr_2(t)}{dt} = J_2(t,t)(A(t) - A(t)r_3(t)) - J_2(t,t-D)(A(t-D) - B(t-D)r_2(t)). \tag{44}$$

Here, from (27), the initial condition on the differential equation (44) is $r_2(t-D) = 0$.

Differentiating (43) with respect to t, we have

$$\frac{dr_3(t)}{dt} = J_3(t,t)A(t) - J_3(t,t-D)A(t-D) + \int_{t-D}^{t} \frac{\partial J_3(t,\tau)}{\partial t}A(\tau)d\tau. \tag{45}$$

Substituting (40) into (45), we have

$$\begin{aligned}\tfrac{dr_3(t)}{dt} &= J_3(t,t)A(t) - J_3(t,t-D)A(t-D) - \\ &J_3(t,t)A(t)\int_{t-D}^{t} J_3(t,\tau)A(\tau)d\tau + J_3(t,t-D)B(t-D)\int_{t-D}^{t} J_2(t,\tau)A(\tau)d\tau.\end{aligned} \tag{46}$$

From (27) and (43), we obtain

$$\frac{dr_3(t)}{dt} = J_3(t,t)(A(t) - A(t)r_3(t)) - J_3(t,t-D)(A(t-D) - B(t-D)r_2(t)). \tag{47}$$

Here, from (43), the initial condition on the differential equation (47) is $r_3(t-D) = 0$.

Putting $s = t-D$ in (19) and using (3) and (24), we have

$$\begin{aligned}J_1(t,t-D)R(t-D) &= B^T(t-D) - \int_0^{t-D} J_1(t,\tau)K(\tau,t-D)d\tau \\ &= B^T(t-D) - \int_0^{t-D} J_1(t,\tau)B(\tau)d\tau A^T(t-D) \\ &= B^T(t-D) - r_1(t)A^T(t-D).\end{aligned} \tag{48}$$

Putting $s = t$ in (21) and using (3), we have

$$\begin{aligned}J_2(t,t)R(t) &= A^T(t) - \int_{t-D}^{t} J_2(t,\tau)K(\tau,t)d\tau \\ &= A^T(t) - \int_{t-D}^{t} J_2(t,\tau)B(\tau)d\tau A^T(t).\end{aligned} \tag{49}$$

By introducing a function

$$r_4(t) = \int_{t-D}^{t} J_2(t,\tau)B(\tau)d\tau, \tag{50}$$

(49) is written as

$$J_2(t,t)R(t) = A^T(t) - r_4(t)A^T(t). \tag{51}$$

Differentiating (50) with respect to t, we have

$$\frac{dr_4(t)}{dt} = J_2(t,t)B(t) - J_2(t,t-D)B(t-D) + \int_{t-D}^{t} \frac{\partial J_2(t,\tau)}{\partial t}B(\tau)d\tau. \tag{52}$$

Substituting (37) into (52), we have

$$\begin{aligned}\frac{dr_4(t)}{dt} &= J_2(t,t)B(t) - J_2(t,t-D)B(t-D) - \\ &J_2(t,t)A(t)\int_{t-D}^{t} J_3(t,\tau)B(\tau)d\tau + J_2(t,t-D)B(t-D)\int_{t-D}^{t} J_2(t,\tau)B(\tau)d\tau.\end{aligned} \tag{53}$$

From (50), introducing

$$r_5(t) = \int_{t-D}^{t} J_3(t,\tau)B(\tau)d\tau, \tag{54}$$

we obtain

$$\frac{dr_4(t)}{dt} = J_2(t,t)(B(t) - A(t)r_5(t)) - J_2(t,t-D)(B(t-D) - B(t-D)r_4(t)). \tag{55}$$

Here, from (50), the initial condition on the differential equation (55) is $r_4(t-D) = 0$.

Differentiating (54) with respect to t, we have

$$\frac{dr_5(t)}{dt} = J_3(t,t)B(t) - J_3(t,t-D)B(t-D) + \int_{t-D}^{t} \frac{\partial J_3(t,\tau)}{\partial t} B(\tau)d\tau. \tag{56}$$

Substituting (40) into (56), we have

$$\frac{dr_5(t)}{dt} = J_3(t,t)B(t) - J_3(t,t-D)B(t-D) -$$

$$J_3(t,t)A(t)\int_{t-D}^{t} J_3(t,\tau)B(\tau)d\tau + J_3(t,t-D)B(t-D)\int_{t-D}^{t} J_2(t,\tau)B(\tau)d\tau. \tag{57}$$

From (50) and (54), we obtain

$$\frac{dr_5(t)}{dt} = J_3(t,t)(B(t) - A(t)r_5(t)) - J_3(t,t-D)(B(t-D) - B(t-D)r_4(t)). \tag{58}$$

Here, from (54), the initial condition on the differential equation (58) is $r_5(t-D) = 0$.

Putting $s = t-D$ in (21), and using (3) and (27), we have

$$\begin{aligned} J_2(t,t-D)R(t-D) &= A^T(t-D) - \int_{t-D}^{t} J_2(t,\tau)K(\tau,t-D)d\tau \\ &= A^T(t-D) - \int_{t-D}^{t} J_2(t,\tau)A(\tau)d\tau B^T(t-D) \\ &= A^T(t-D) - r_2(t)B^T(t-D). \end{aligned} \tag{59}$$

Putting $s = t$ in (36), and using (3) and (54), we have

$$\begin{aligned} J_3(t,t)R(t) &= B^T(t) - \int_{t-D}^{t} J_3(t,\tau)K(\tau,t)d\tau \\ &= B^T(t) - r_5(t)A^T(t). \end{aligned} \tag{60}$$

Putting $s = t-D$ in (36), and using (3) and (43), we have

$$\begin{aligned} J_3(t,t-D)R(t-D) &= B^T(t-D) - \int_{t-D}^{t} J_3(t,\tau)K(\tau,t-D)d\tau \\ &= B^T(t-D) - \int_{t-D}^{t} J_3(t,\tau)A(\tau)B^T(t-D)d\tau \\ &= B^T(t-D) - r_3(t)B^T(t-D). \end{aligned} \tag{61}$$

Substituting (20) and (22) into (10), we have

$$\hat{z}(t-D,t) = (A(t-D) - F(t))\int_{0}^{t-D} J_1(t,s)y(s)ds + (B(t-D) - G(t))\int_{t-D}^{t} J_2(t,s)y(s)ds. \tag{62}$$

By introducing functions

$$e_1(t) = \int_{0}^{t-D} J_1(t,s)y(s)ds, \tag{63}$$

$$e_2(t) = \int_{t-D}^{t} J_2(t,s)y(s)ds, \tag{64}$$

(62) is written as

$$\hat{z}(t-D,t) = (A(t-D) - F(t))e_1(t) + (B(t-D) - G(t))e_2(t). \tag{65}$$

Differentiating (63) with respect to t, and using (30) and (63), we obtain

$$\begin{aligned} \frac{de_1(t)}{dt} &= J_1(t,t-D)y(t-D) + \int_{0}^{t-D} \frac{\partial J_1(t,s)}{\partial t} y(s)ds \\ &= J_1(t,t-D)y(t-D) - J_1(t,t-D)A(t-D)\int_{0}^{t-D} J_1(t,s)y(s)ds \\ &= J_1(t,t-D)(y(t-D) - A(t-D)e_1(t)). \end{aligned} \tag{66}$$

The initial condition on the differential equation (66) at $t = D$ is $e_1(D) = 0$ and $e_1(\tau) = 0,\quad 0 \leqslant \tau \leqslant D$.

In the calculation of the fixed-lag smoothing estimate, the functions $e_1(t)$ and $e_2(t)$ are used. From (64), the function $e_2(t)$ is calculated as the integral from $t-D$ to t. Here, we set $t_D = t-D$ and consider the integral in the form

$$e_2(t) = \int_{t_D}^{t} J_2(t,s)y(s)ds. \tag{67}$$

Differentiating (67) with respect to t, and using (37), we have

$$\begin{aligned}\frac{de_2(t)}{dt} &= J_2(t,t)y(t) + \int_{t_D}^{t} \frac{\partial J_2(t,s)}{\partial t} y(s)ds \\ &= J_2(t,t)y(t) - J_2(t,t)A(t)\int_{t_D}^{t} J_3(t,s)y(s)ds + J_2(t,t-D)B(t-D)\int_{t_D}^{t} J_2(t,s)y(s)ds.\end{aligned} \tag{68}$$

From (64), introducing a function,

$$e_3(t) = \int_{t-D}^{t} J_3(t,s)y(s)ds, \tag{69}$$

we obtain

$$\frac{de_2(t)}{dt} = J_2(t,t)y(t) - J_2(t,t)A(t)e_3(t) + J_2(t,t-D)B(t-D)e_2(t). \tag{70}$$

The initial condition on the differential equation (70) at $t = t_D$ is $e_2(t_D) = e_2(t-D) = 0$.

Differentiating (69) with respect to t, and using (40), (67) and (69), we obtain

$$\begin{aligned}\frac{de_3(t)}{dt} &= J_3(t,t)y(t) - J_3(t,t-D)y(t-D) + \int_{t-D}^{t} \frac{\partial J_3(t,s)}{\partial t} y(s)ds \\ &= J_3(t,t)y(t) - J_3(t,t-D)y(t-D) - J_3(t,t)A(t)\int_{t-D}^{t} J_3(t,s)y(s)ds + \\ &\quad J_3(t,t-D)B(t-D)\int_{t-D}^{t} J_2(t,s)y(s)ds \\ &= J_3(t,t)y(t) - J_3(t,t-D)y(t-D) - J_3(t,t)A(t)e_3(t) + J_3(t,t-D)B(t-D)e_2(t).\end{aligned} \tag{71}$$

The initial condition on the differential equation (71) at $t = t_D$ is $e_3(t_D) = e_3(t-D) = 0$.

Finally, from (25) and (28), we obtain

$$F(t) = (B(t-D) - A(t-D)r_1(t))r_2(t)(I - r_1(t)r_2(t))^{-1}. \tag{72}$$

Now, let us summarize the above results in [Theorem 1].

[Theorem 1]

Let the observation equation be given by (1). Let the auto-covariance function of the signal $z(t)$ be given by (3) in the semi-degenerate kernel form in continuous-time stochastic systems. Then the fixed-lag smoothing estimate $\hat{z}(t-D,t)$ of $z(t-D)$ is calculated recursively by (73)-(88).

$$\hat{z}(t-D,t) = (A(t-D) - F(t))e_1(t) + (B(t-D) - G(t))e_2(t) \tag{73}$$

$$G(t) = (A(t-D) - F(t))r_1(t) \tag{74}$$

$$F(t) = (B(t-D) - A(t-D)r_1(t))r_2(t)(I - r_1(t)r_2(t))^{-1} \tag{75}$$

$$\frac{de_1(t)}{dt} = J_1(t,t-D)(y(t-D) - A(t-D)e_1(t)), \quad e_1(D) = 0, \quad e_1(\tau) = 0, \quad 0 \leqslant \tau \leqslant D \tag{76}$$

$$J_1(t,t-D) = (B^T(t-D) - r_1(t)A^T(t-D))R^{-1}(t-D) \tag{77}$$

$$\frac{dr_1(t)}{dt} = J_1(t,t-D)(B(t-D) - A(t-D)r_1(t)), \quad r_1(D) = 0, \quad r_1(\tau) = 0, \quad 0 \leqslant \tau \leqslant D \tag{78}$$

$$\frac{de_2(t)}{dt} = J_2(t,t)y(t) - J_2(t,t)A(t)e_3(t) + J_2(t,t-D)B(t-D)e_2(t), \quad e_2(t_D) = e_2(t-D) = 0 \tag{79}$$

$$\frac{de_3(t)}{dt} = J_3(t,t)y(t) - J_3(t,t-D)y(t-D) - J_3(t,t)A(t)e_3(t) + J_3(t,t-D)B(t-D)e_2(t),$$

$$e_3(t-D) = 0 \tag{80}$$

$$J_2(t,t) = (A^T(t) - r_4(t)A^T(t))R^{-1}(t) \tag{81}$$

$$J_2(t,t-D) = (A^T(t-D) - r_2(t)B^T(t-D))R^{-1}(t-D) \tag{82}$$

$$J_3(t,t) = (B^T(t) - r_5(t)A^T(t))R^{-1}(t) \tag{83}$$

$$J_3(t,t-D) = (B^T(t-D) - r_3(t)B^T(t-D))R^{-1}(t-D) \tag{84}$$

$$\frac{dr_2(t)}{dt} = J_2(t,t)(A(t) - A(t)r_3(t)) - J_2(t,t-D)(A(t-D) - B(t-D)r_2(t)), \quad r_2(t-D) = 0 \tag{85}$$

$$\frac{dr_3(t)}{dt} = J_3(t,t)(A(t) - A(t)r_3(t)) - J_3(t,t-D)(A(t-D) - B(t-D)r_2(t)), \quad r_3(t-D) = 0 \tag{86}$$

$$\frac{dr_4(t)}{dt} = J_2(t,t)(B(t) - A(t)r_5(t)) - J_2(t,t-D)(B(t-D) - B(t-D)r_4(t)), \quad r_4(t-D) = 0 \tag{87}$$

$$\frac{dr_5(t)}{dt} = J_3(t,t)(B(t) - A(t)r_5(t)) - J_3(t,t-D)(B(t-D) - B(t-D)r_4(t)), \quad r_5(t-D) = 0 \tag{88}$$

4 Fixed-lag smoothing error variance function

Let $P_{\tilde{z}}(t-D,t)$ represent the fixed-lag smoothing error variance function as

$$\begin{aligned} P_{\tilde{z}}(t-D,t) &= E[(z(t-D) - \hat{z}(t-D,t))(z(t-D) - \hat{z}(t-D,t))^T] \\ &= K(t-D,t-D) - E[\hat{z}(t-D,t)\hat{z}^T(t-D,t)] \\ &= A(t-D)B^T(t-D) - E[\hat{z}(t-D,t)\hat{z}^T(t-D,t)]. \end{aligned} \tag{89}$$

By substituting (73) into the second term on the right hand side of (89), the term is written as

$$\begin{aligned} E[\hat{z}(t-D,t)\hat{z}^T(t-D,t)] = C(t)f_1(t)C^T(t) + Q(t)f_2(t)Q^T(t) + C(t)f_3(t)Q^T(t) + \\ Q(t)f_3^T(t)C^T(t), \end{aligned} \tag{90}$$

here,

$$C(t) = A(t-D) - F(t), \quad Q(t) = B(t-D) - G(t),$$

$$f_1(t) = E[e_1(t)e_1^T(t)], \quad f_2(t) = E[e_2(t)e_2^T(t)], \quad f_3(t) = E[e_1(t)e_2^T(t)]. \tag{91}$$

Differentiating $f_1(t)$ with respect to t, we have

$$\frac{df_1(t)}{dt} = E[\frac{de_1(t)}{dt}e_1^T(t)] + E[e_1(t)\frac{de_1^T(t)}{dt}]. \tag{92}$$

From (76), the first term on the right hand side of (92) is written as

$$E[\frac{de_1(t)}{dt}e_1^T(t)] = E[(J_1(t,t-D)(y(t-D) - A(t-D)e_1(t))e_1^T(t)]. \tag{93}$$

From (3), (24) and (63), $E[y(t-D)e_1^T(t)]$ is written as

$$\begin{aligned} E[y(t-D)e_1^T(t)] &= E[y(t-D)(\int_0^{t-D} J_1(t,s)y(s)ds)^T] \\ &= \int_0^{t-D} E[y(t-D)y^T(s)]J_1^T(t,s)ds \\ &= \int_0^{t-D} (R(t-D)\delta_K(t-D-s) + A(t-D)B^T(s))J_1^T(t,s)ds \\ &= R(t-D)J_1^T(t,t-D) + A(t-D)r_1(t). \end{aligned} \tag{94}$$

From (91-(94), we obtain

$$\begin{aligned} \frac{df_1(t)}{dt} &= 2J_1(t,t-D)R(t-D)J_1^T(t,t-D) + J_1(t,t-D)A(t-D)r_1(t) \\ &\quad - A(t-D)f_1(t) + r_1(t)A^T(t-D)J_1^T(t,t-D) - f_1^T(t)A^T(t-D), \end{aligned}$$

$$f_1(D) = 0. \tag{95}$$

Similarly, by introducing functions

$$f_4(t) = E[e_3(t)e_2^T(t)], \quad f_5(t) = E[e_1(t)e_3^T(t)], \quad f_6(t) = E[e_3(t)e_3^T(t)], \tag{96}$$

the following recursive equations are obtained.

$$\begin{aligned} \frac{df_2(t)}{dt} &= J_2(t,t)(R(t) + A(t)r_4(t)) - J_2(t,t)A(t)f_4(t) + J_2(t,t-D)B(t-D)f_2(t) \\ &\quad + (R(t) + r_4^T(t)A^T(t))J_2^T(t,t) - f_4^T(t)A^T(t)J_2^T(t,t) + f_2^T(t)B^T(t-D)J_2^T(t,t-D), \end{aligned}$$

$$f_2(t-D)=0 \tag{97}$$

$$\begin{aligned}\frac{df_3(t)}{dt} &= J_1(t,t-D)(R(t-D)J_2^T(t,t-D)+B(t-D)r_2^T(t))\\ &- J_1(t,t-D)A(t-D)f_3(t)+(r_1(t)A^T(t)+J_1(t,t)R(t))J_2^T(t,t)\\ &- f_5(t)A^T(t)J_2^T(t,t)+f_3(t)B^T(t-D)J_2^T(t,t-D),\end{aligned}$$

$$f_3(t-D)=0 \tag{98}$$

$$\begin{aligned}\frac{df_4(t)}{dt} &= J_3(t,t)(R(t)+A(t)r_4^T(t))-J_3(t,t-D)(R(t-D)J_2^T(t,t-D)+B(t-D)r_2^T(t))\\ &- J_3(t,t)A(t)f_4(t)+J_3(t,t-D)B(t-D)f_2(t)\\ &+ J_3(t,t)R(t)J_2^T(t,t)+r_5(t)A^T(t)J_2^T(t,t)-f_6(t)A^T(t)J_2^T(t,t)\\ &+ f_4(t)B^T(t-D)J_2^T(t,t-D),\end{aligned}$$

$$f_4(t-D)=0 \tag{99}$$

$$\begin{aligned}\frac{df_5(t)}{dt} &= J_1(t,t-D)(R(t-D)J_3^T(t,t-D)+B(t-D)r_3^T(t))-J_1(t,t-D)A(t-D)f_5(t)\\ &+ (r_1(t)A^T(t)+J_1(t,t)R(t))J_3^T(t,t)-(J_1(t,t-D)R(t-D)\\ &+ r_1(t)A^T(t-D))J_3^T(t,t-D)-f_5(t)A^T(t)J_3^T(t,t)+f_3(t)B^T(t-D)J_3^T(t,t-D),\end{aligned}$$

$$f_5(t-D)=0 \tag{100}$$

$$\begin{aligned}\frac{df_6(t)}{dt} &= 2J_3(t,t)R(t)J_3^T(t,t)+J_3(t,t)A(t)r_5^T\\ &- 2J_3(t,t-D)R(t-D)J_3^T(t,t-D)-J_3(t,t-D)B(t-D)r_3^T(t)\\ &- J_3(t,t)A(t)f_6(t)+J_3(t,t-D)B(t-D)f_4^T(t)\\ &+ r_5(t)A^T(t)J_3^T(t,t)-r_3(t)B^T(t-D)J_3^T(t,t-D+f_6^T(t)A^T(t)J_3^T(t,t)\\ &+ f_4(t)B^T(t-D)J_3^T(t,t-D),\end{aligned}$$

$$f_6(t-D)=0. \tag{101}$$

Hence, the fixed-lag smoothing error variance function is calculated by (77), (78), (81)-(88), (89)-(91), (95), (97)-(101) recursively.

5 A numerical simulation example

Let a scalar observation equation be given by

$$y(t)=z(t)+v(t). \tag{102}$$

Let the observation noise $v(t)$ be a zero-mean white Gaussian process with the variance R, $N(0,R)$. Let the auto-covariance function of the signal $z(t)$ be given by

$$K(t,s)=\frac{3}{16}e^{-|t-s|}+\frac{5}{48}e^{-3|t-s|}. \tag{103}$$

From (103), the functions $A(t)$ and $B(s)$ in (3) are expressed as follows:

$$A(t)=\left[\begin{array}{cc}\frac{3}{16}e^{-t} & \frac{5}{48}e^{-3t}\end{array}\right],\quad B(s)=\left[\begin{array}{cc}e^{s} & e^{3s}\end{array}\right]. \tag{104}$$

If we substitute (104) into the fixed-lag smoothing algorithm of [Theorem 1], we can calculate the fixed-lag smoothing estimate recursively. Figure 1 illustrates the signal $z(t)$ and the fixed-lag smoothing estimate $\hat{z}(t-0.01,t)$ for the white Gaussian observation noise $N(0,0.1^2)$ by the RLS fixed-lag smoother in [Theorem1]. Figure 2 illustrates the mean-square values (MSVs) of the fixed-lag smoothing errors by the proposed RLS fixed-lag smoother in [Theorem 1] and by the fixed-lag smoother in [12] for the observation noises $N(0,0.07^2)$, $N(0,0.1^2)$ and $N(0,0.15^2)$ vs. the fixed lag D, $\Delta \leqslant D \leqslant 10\Delta$. The MSVs of the fixed-lag smoothing errors are evaluated by $\sum_{i=1}^{2000}(z(\Delta i)-z(\Delta i,\Delta i+L))^2/2000$, $\Delta=0.001$. Here, for the numerical integration of the differential equations, the fourth-order Runge-Kutta method is used. In Figure 2, as the fixed lag D becomes large, in the current fixed-lag smoother, there is a tendency that the estimation accuracy of the fixed-lag smoother is improved for the observation noises $N(0,0.1^2)$ and $N(0,0.15^2)$. For the observation noise $N(0,0.07^2)$, the tendency applies in the region of D, $\Delta \leqslant D \leqslant 6\Delta$. From Figure 2, it is seen that the estimation accuracy of the proposed fixed-lag smoother in [Theorem 1] is superior to the fixed-lag smoother in [12]. Also, the smaller the variance of the observation noise becomes, the better the estimation accuracy of the smoother becomes.

For references, the state-space model, which generates the signal process, is given by

$$z(t)=x_1(t)$$

$$\frac{dx_1(t)}{dt}=x_2(t)+u(t),\quad \frac{dx_2(t)}{dt}=-3x_1(t)-4x_2(t)-2u(t),$$

$$E[u(t)u(s)]=\delta(t-s). \tag{105}$$

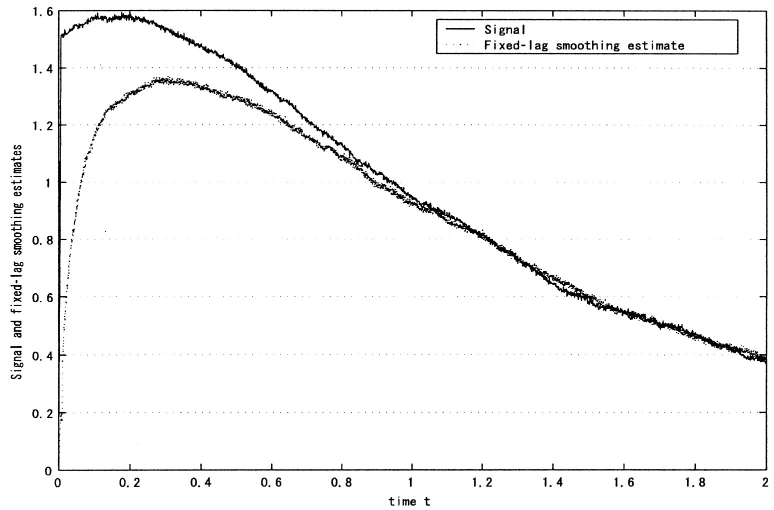

Figure 1: Signal $z(t)$ and the fixed-lag smoothing estimate $\hat{z}(t-0.01,t)$ for the white Gaussian observation noise $N(0,0.1^2)$ by the RLS fixed-lag smoother in [Theorem1

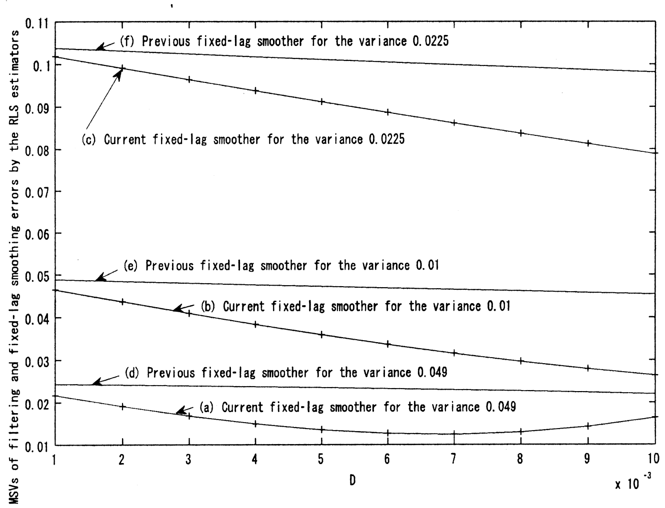

Figure 2: Mean-square values of the fixed-lag smoothing errors by theproposed RLS fixed-lag smoother in [Theorem 1] and by the fixed-lag smoother in [12] for the observation noises $N(0,0.07^2)$, $N(0,0.1^2)$ and $N(0,0.15^2)$ vs. the fixed lag D, $\Delta \leqslant D \leqslant 10\Delta$, $\Delta = 0.001$.

6 Conclusions

In this paper, the RLS fixed-lag smoother, using the information of the covariance function of the signal, in the semi-degenerate kernel form, and the variance of white observation noise, have been devised in linear continous-time stochastic systems. From the simulation result in section 5, the proposed RLS fixed-lag smoother is superior in estimation accuracy to the RLS fixed-lag smoother in [12]. As the fixed lag D becomes large, in the current fixed-lag smoother, there is a tendency that the estimation accuracy of the fixed-lag smoother is improved for the observation noises $N(0,0.1^2)$ and $N(0,0.15^2)$. For the observation noise $N(0,0.07^2)$, the tendency applies in the region of D, $\Delta \leqslant D \leqslant 6\Delta$.

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