ON AN INTERPOLATION PROBLEM OF A SYMMETRIC RANDOM FIELD WITH DISCRETE PARAMETERS

By

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Let $\{X(t),\ t \in T\}$ be a random field such that

(1) $$T=\{\theta\}\cup\Lambda, \text{ where } \theta=(0,\ 0),$$

$\Lambda=\{(n,\ m)\,|\,n=1,\ 2,\ \cdots,\ N;\ m=1,\ 2,\ \cdots,\ M\}$, and N and M are finite positive integers,

(2) $$E\{X(t)\}=0, \qquad \text{for all} \quad t \in T,$$

(3) $$E\{X(t)^2\}<\infty, \quad \text{for all} \quad t \in T.$$

Let us denote the covariance function of the random field by

(4) $$K(t,\ s)\equiv K((n,\ m),\ (k,\ l)) = E\{X(t)\cdot X(s)\}, \qquad t,\ s \in T,$$

where $t=(n,\ m)$ and $s=(k,\ l)$.

Let G be a cyclic group with a generator g of order N acting on T in such a manner that:

(5) $$g\theta=\theta,$$

(6) for any $t=(n,\ m) \in \Lambda$, $g\cdot(n,\ m)=(n(g),\ m)$ for each m, where $1\leqq n(g)\leqq N$, $n+1\equiv n(g)$, *mod* (N).

Let us assume that

(7) for any $n=0,\ 1,\ 2,\ \cdots,\ N-1$,

$K(g^n t,\ g^n s)=K(t,\ s)$ for any $t,\ s \in T$, where $g^0=e \in G$, and the covariances $K(t,\ s)$, $t,\ s \in T$, are known.

Let $\{X(t),\ t \in \Lambda\}$ be observations of the random field and $Z(\theta)$ be the best linear estimator of $X(\theta)$ based on the observations $\{X(t),\ t \in \Lambda\}$.

Then, since Λ is a finite set, we know that $Z(\theta)$ may be written as follows:

(8) $$Z(\theta)=\sum_{n=1}^{N}\ \sum_{m=1}^{N} a(n,\ m)\cdot X((n,\ m)),$$

where $a(n, m)$, $(n, m) \in \Lambda$, are constants.

In order to determine $Z(\theta)$, or the coefficients $\{a(n, m), (n, m) \in \Lambda\}$, we usually have to solve the $N\times M$ simultaneous equations with respect to the coefficients $\{a(n, m), (n, m) \in \Lambda\}$, derived from

$$E\{(X(\theta)-\sum_{t\in\Lambda} a(t)\, X(t))\, X(s)\}=0, \qquad \text{for all} \quad s \in \Lambda. \tag{9}$$

However, by the assumption (7), we can reduce the equation (9) to a smaller one. We now consider this problem.

Let us put

$$\xi(n)' = (X(n, 1), X(n, 2), \cdots, X(n, M)), \tag{10}$$

$$A(n)' = (a(n, 1), a(n, 2), \cdots, a(n, M)), \tag{11}$$

$$F(n) = A(n)'\xi(n). \tag{12}$$

We then define an operator U_h, $h \in G$, on the space of all linear combinations of the random field $\{X(t), t \in T\}$ in such a manner that:

(13) For each $t \in T$,

$$Ug^{\nu}X(t)=X(g^{\nu}t), \qquad \nu=0, 1, 2, \cdots, N-1.$$

(14) For any $u=\sum_{\mu=1}^{k} C_{\mu}\cdot X(t_{\mu})$, $\quad t_{\mu} \in T$, $\quad \mu=1, 2, \cdots, k$,

$$Ug^{\nu}u=\sum_{\mu=1}^{k} C_{\mu}\cdot X(g^{\nu}t_{\mu}), \qquad \nu=0, 1, \cdots, N-1,$$

where k is a finite integer.

It is clear that $\{Ug^{\nu}, \nu=0, 1, \cdots, N-1\}$ are well-defined linear operators satisfying the conditions:

(15) $E\{Ug^{\nu}u_1\cdot Ug^{\mu}u_2\}=E\{u_1\cdot u_2\}$ for any linear combinations u_1, u_2 of $\{X(t), t \in T\}$.

$$Ug^{\nu}\cdot Ug^{\mu}=Ug^{\nu+\mu}, \tag{16}$$

$$(Ug^{\nu})^{-1}=Ug^{-\nu}, \tag{17}$$

where $Ug^{N}=I$, $Ug^{k}=Ug^{m}$, $1\leq m\leq N$, $k\equiv m$, $mod\ (N)$, $Ug^{-k}=U^{k}g^{-1}$.
We now prove:

LEMMA 1. For all $n=0, 1, 2, \cdots, N-1$,

$$Ug^{n}\cdot Z(\theta)=Z(\theta). \tag{18}$$

(PROOF) Since $Z(\theta)$ is the projection of $X(\theta)$ on the Hilbert space $L_2(X(t), t \in \Lambda)$ spanned by the observations $\{X(t), t \in \Lambda\}$ with the scalar product defined by $(u, v)=E(u\cdot v)$, $u, v \in L_2(X(t), t \in T)$, we can write

(19) $$X(\theta)=Z(\theta)+Y,$$

where Y is a component of $X(\theta)$ orthogonal to $L_2(X(t),\ t\in\Lambda)$.

Since for all $n=0, 1, 2, \cdots, N-1$,

(20) $$\begin{aligned}Ug^nX(\theta)&=X(g^n\theta)=X(\theta)\\&=Ug^nZ(\theta)+Ug^nY,\end{aligned}$$

we have $Z(\theta)-Ug^n\cdot Z(\theta)=Ug^nY-Y$.

Hence, $$\begin{aligned}E\{(Z(\theta)-Ug^nZ(\theta))^2\}&=E\{(Z(\theta)-Ug^nZ(\theta))\cdot(Ug^nY-Y)\}\\&=E\{Z(\theta)\cdot Ug^nY\}=E\{Ug^{-n}Z(\theta)\cdot Y\}.\end{aligned}$$

Since $Z(\theta)$ may be written as (8), we know that $Ug^{-n}\cdot Z(\theta)$ is still an element of $L_2(X(t),\ t\in\Lambda)$. Therefore we have $E\{Ug^{-n}Z(\theta)\cdot Y\}=0$. This implies that $Ug^nZ(\theta)=Z(\theta)$, $n=0, 1, 2, \cdots, N-1$.

THEOREM 1.

(21) $$Z(\theta)=Q\cdot R(1),$$

where $R(1)$ *is a linear combination of the observations* $\{X(t),\ t=(1, m),\ m=1, 2, \cdots, M\}$, and

(22) $$Q=\frac{1}{N}(I+Ug+Ug^2+\cdots+Ug^{N-1}).$$

(PROOF) By (8) and (12), $Z(\theta)$ may be written as follows:

(23) $$Z(\theta)=\sum_{n=1}^{N}F(n).$$

By lemma 1, for all $\nu=0, 1, 2, \cdots, N-1$, we have

(24) $$Z(\theta)=Ug^\nu Z(\theta)=\sum_{n=1}^{N}Ug^\nu F(n).$$

Summing up (24) with respect to all $\nu=0, 1, \cdots, N-1$, we obtain

(25) $$NZ(\theta)=\sum_{m=0}^{N-1}\sum_{n=1}^{N}Ug^m\cdot F(n).$$

Let us put

(26) $$R(1)=F(1)+UgF(N)+Ug^2F(N-1)+\cdots+Ug^{N-1}F(2).$$

Then, we see that $R(1)$ is a linear combination of the observations $\{X(t),\ t=(1, m),\ m=1, 2, \cdots, M\}$. By rearranging (25), we have,

$$\begin{aligned}NZ(\theta)&=R(1)+UgR(1)+Ug^2R(1)+\cdots+Ug^{N-1}R(1),\\&=(I+Ug+Ug^2+\cdots+Ug^{N-1})\cdot R(1).\end{aligned}$$

Thus we have proved (21).

Let us now write

$$(27) \qquad R(1)=a_1X((1, 1))+\cdots+a_MX((1, M)),$$
$$=A'\cdot\xi(1),$$

where $$A'=(a_1, a_2, \cdots, a_M).$$

Then, we have

(28) $Z(\theta)=Q\cdot R(1)=A'\cdot Q\xi(1)=(Q\xi(1))'\cdot A,$ where we put $Q(u_1, u_2, \cdots, u_M)'=(Qu_1, Qu_2, \cdots, Qu_M)'$.

Since $Q=(I+Ug+Ug^2+\cdots+Ug^{N-1})/N$, we see that

$$(Q\xi(1))'=(\overline{\eta(1)}, \overline{\eta(2)}, \cdots, \overline{\eta(M)})\equiv\bar{X}', \qquad \text{say, where}$$

$$\overline{\eta(k)}=\frac{1}{N}\{X((1, k))+X((2, k))+\cdots+X((N, k))\}, \quad k=1, 2, \cdots, M.$$

Hence, we can write,

$$(29) \qquad Z(\theta)=\bar{X}'\cdot A.$$

THEOREM 2. *To determine the best linear estimate $Z(\theta)$ of $X(\theta)$, it is sufficient for us to solve M simultaneous equations*

$$(30) \qquad K(\theta, (1, m))=E\{X((1, m))\cdot\bar{X}'\}\cdot A, \quad m=1, 2, \cdots, M.$$

(PROOF) Since $Z(\theta)$ is the projection of $X(\theta)$ on $L_2(X(t), t\in A)$, $X(\theta)-Z(\theta)$ is orthogonal to $\xi(1)$, that is, for all $m=1, 2, \cdots, M$, $E\{(X(\theta)-Z(\theta))\cdot X((1, m))\}=0$, $m=1, 2, \cdots, M$.

From this, we obtain M-simultaneous equation (30) with M variables A. By (7), (5) and (15), we have for each $n=1, 2, \cdots, N$,

$$E\{(X(\theta)-Z(\theta))\cdot X((n, m))\}=E\{(X(\theta)-Z(\theta))\cdot X(1, m)\},$$

$m=1, 2, \cdots, M$. Therefore by solving only (30), we can determine the best weights A of $Z(\theta)$.

Reference

[1] NANNAN, E. J.: *Time series analysis.* Methuen & Co., London. (1960)