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**ON AN INTERPOLATION PROBLEM OF A
SYMMETRIC RANDOM FIELD WITH
DISCRETE PARAMETERS**

By

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Let $\{X(t), t \in T\}$ be a random field such that

$$(1) \quad T = \{\theta\} \cup A, \text{ where } \theta = (0, 0),$$

$A = \{(n, m) | n = 1, 2, \dots, N; m = 1, 2, \dots, M\}$, and N and M are finite positive integers,

$$(2) \quad E\{X(t)\} = 0, \quad \text{for all } t \in T,$$

$$(3) \quad E\{X(t)^2\} < \infty, \quad \text{for all } t \in T.$$

Let us denote the covariance function of the random field by

$$(4) \quad \begin{aligned} K(t, s) &\equiv K((n, m), (k, l)) \\ &= E\{X(t) \cdot X(s)\}, \quad t, s \in T, \end{aligned}$$

where $t = (n, m)$ and $s = (k, l)$.

Let G be a cyclic group with a generator g of order N acting on T in such a manner that:

$$(5) \quad g\theta = \theta,$$

$$(6) \quad \text{for any } t = (n, m) \in A, \quad g \cdot (n, m) = (n(g), m) \text{ for each } m, \text{ where } 1 \leq n(g) \leq N, \\ n + 1 \equiv n(g), \text{ mod } (N).$$

Let us assume that

$$(7) \quad \text{for any } n = 0, 1, 2, \dots, N-1,$$

$K(g^n t, g^n s) = K(t, s)$ for any $t, s \in T$, where $g^0 = e \in G$, and the covariances $K(t, s)$, $t, s \in T$, are known.

Let $\{X(t), t \in A\}$ be observations of the random field and $Z(\theta)$ be the best linear estimator of $X(\theta)$ based on the observations $\{X(t), t \in A\}$.

Then, since A is a finite set, we know that $Z(\theta)$ may be written as follows:

$$(8) \quad Z(\theta) = \sum_{n=1}^N \sum_{m=1}^N a(n, m) \cdot X((n, m)),$$

where $a(n, m), (n, m) \in A$, are constants.

In order to determine $Z(\theta)$, or the coefficients $\{a(n, m), (n, m) \in A\}$, we usually have to solve the $N \times M$ simultaneous equations with respect to the coefficients $\{a(n, m), (n, m) \in A\}$, derived from

$$(9) \quad E\left\{\left(X(\theta) - \sum_{t \in A} a(t) X(t)\right) X(s)\right\} = 0, \quad \text{for all } s \in A.$$

However, by the assumption (7), we can reduce the equation (9) to a smaller one. We now consider this problem.

Let us put

$$(10) \quad \xi(n)' = (X(n, 1), X(n, 2), \dots, X(n, M)),$$

$$(11) \quad A(n)' = (a(n, 1), a(n, 2), \dots, a(n, M)),$$

$$(12) \quad F(n) = A(n)' \xi(n).$$

We then define an operator $U_h, h \in G$, on the space of all linear combinations of the random field $\{X(t), t \in T\}$ in such a manner that:

$$(13) \quad \text{For each } t \in T,$$

$$U g^\nu X(t) = X(g^\nu t), \quad \nu = 0, 1, 2, \dots, N-1.$$

$$(14) \quad \text{For any } u = \sum_{\mu=1}^k C_\mu \cdot X(t_\mu), \quad t_\mu \in T, \quad \mu = 1, 2, \dots, k,$$

$$U g^\nu u = \sum_{\mu=1}^k C_\mu \cdot X(g^\nu t_\mu), \quad \nu = 0, 1, \dots, N-1,$$

where k is a finite integer.

It is clear that $\{U g^\nu, \nu = 0, 1, \dots, N-1\}$ are well-defined linear operators satisfying the conditions:

$$(15) \quad E\{U g^\nu u_1 \cdot U g^\mu u_2\} = E\{u_1 \cdot u_2\} \text{ for any linear combinations } u_1, u_2 \text{ of } \{X(t), t \in T\}.$$

$$(16) \quad U g^\nu \cdot U g^\mu = U g^{\nu+\mu},$$

$$(17) \quad (U g^\nu)^{-1} = U g^{-\nu},$$

where $U g^N = I, U g^k = U g^m, 1 \leq m \leq N, k \equiv m, \text{ mod } (N), U g^{-k} = U^k g^{-1}$.

We now prove:

LEMMA 1. For all $n = 0, 1, 2, \dots, N-1$,

$$(18) \quad U g^n \cdot Z(\theta) = Z(\theta).$$

(PROOF) Since $Z(\theta)$ is the projection of $X(\theta)$ on the Hilbert space $L_2(X(t), t \in A)$ spanned by the observations $\{X(t), t \in A\}$ with the scalar product defined by $(u, v) = E(u \cdot v), u, v \in L_2(X(t), t \in T)$, we can write

$$(19) \quad X(\theta) = Z(\theta) + Y,$$

where Y is a component of $X(\theta)$ orthogonal to $L_2(X(t), t \in A)$.

Since for all $n = 0, 1, 2, \dots, N-1$,

$$(20) \quad \begin{aligned} U g^n X(\theta) &= X(g^n \theta) = X(\theta) \\ &= U g^n Z(\theta) + U g^n Y, \end{aligned}$$

we have $Z(\theta) - U g^n \cdot Z(\theta) = U g^n Y - Y$.

$$\begin{aligned} \text{Hence,} \quad E\{(Z(\theta) - U g^n Z(\theta))^2\} &= E\{(Z(\theta) - U g^n Z(\theta)) \cdot (U g^n Y - Y)\} \\ &= E\{Z(\theta) \cdot U g^n Y\} = E\{U g^{-n} Z(\theta) \cdot Y\}. \end{aligned}$$

Since $Z(\theta)$ may be written as (8), we know that $U g^{-n} \cdot Z(\theta)$ is still an element of $L_2(X(t), t \in A)$. Therefore we have $E\{U g^{-n} Z(\theta) \cdot Y\} = 0$. This implies that $U g^n Z(\theta) = Z(\theta)$, $n = 0, 1, 2, \dots, N-1$.

THEOREM 1.

$$(21) \quad Z(\theta) = Q \cdot R(1),$$

where $R(1)$ is a linear combination of the observations $\{X(t), t = (1, m), m = 1, 2, \dots, M\}$, and

$$(22) \quad Q = \frac{1}{N} (I + U g + U g^2 + \dots + U g^{N-1}).$$

(PROOF) By (8) and (12), $Z(\theta)$ may be written as follows:

$$(23) \quad Z(\theta) = \sum_{n=1}^N F(n).$$

By lemma 1, for all $\nu = 0, 1, 2, \dots, N-1$, we have

$$(24) \quad Z(\theta) = U g^\nu Z(\theta) = \sum_{n=1}^N U g^\nu F(n).$$

Summing up (24) with respect to all $\nu = 0, 1, \dots, N-1$, we obtain

$$(25) \quad NZ(\theta) = \sum_{m=0}^{N-1} \sum_{n=1}^N U g^m \cdot F(n).$$

Let us put

$$(26) \quad R(1) = F(1) + U g F(N) + U g^2 F(N-1) + \dots + U g^{N-1} F(2).$$

Then, we see that $R(1)$ is a linear combination of the observations $\{X(t), t = (1, m), m = 1, 2, \dots, M\}$. By rearranging (25), we have,

$$\begin{aligned} NZ(\theta) &= R(1) + U g R(1) + U g^2 R(1) + \dots + U g^{N-1} R(1), \\ &= (I + U g + U g^2 + \dots + U g^{N-1}) \cdot R(1). \end{aligned}$$

Thus we have proved (21).

Let us now write

$$(27) \quad \begin{aligned} R(1) &= a_1 X((1, 1)) + \cdots + a_M X((1, M)), \\ &= A' \cdot \xi(1), \end{aligned}$$

where

$$A' = (a_1, a_2, \dots, a_M).$$

Then, we have

$$(28) \quad Z(\theta) = Q \cdot R(1) = A' \cdot Q \xi(1) = (Q \xi(1))' \cdot A, \quad \text{where we put } Q(u_1, u_2, \dots, u_M)' = (Qu_1, Qu_2, \dots, Qu_M)'$$

Since $Q = (I + Ug + Ug^2 + \cdots + Ug^{N-1})/N$, we see that

$$(Q \xi(1))' = (\overline{\eta(1)}, \overline{\eta(2)}, \dots, \overline{\eta(M)}) \equiv \overline{X}', \quad \text{say, where}$$

$$\overline{\eta(k)} = \frac{1}{N} \{X((1, k)) + X((2, k)) + \cdots + X((N, k))\}, \quad k=1, 2, \dots, M.$$

Hence, we can write,

$$(29) \quad Z(\theta) = \overline{X}' \cdot A.$$

THEOREM 2. *To determine the best linear estimate $Z(\theta)$ of $X(\theta)$, it is sufficient for us to solve M simultaneous equations*

$$(30) \quad K(\theta, (1, m)) = E\{X((1, m)) \cdot \overline{X}'\} \cdot A, \quad m=1, 2, \dots, M.$$

(PROOF) Since $Z(\theta)$ is the projection of $X(\theta)$ on $L_2(X(t), t \in \Lambda)$, $X(\theta) - Z(\theta)$ is orthogonal to $\xi(1)$, that is, for all $m=1, 2, \dots, M$, $E\{(X(\theta) - Z(\theta)) \cdot X((1, m))\} = 0$, $m=1, 2, \dots, M$.

From this, we obtain M -simultaneous equation (30) with M variables A . By (7), (5) and (15), we have for each $n=1, 2, \dots, N$,

$$E\{(X(\theta) - Z(\theta)) \cdot X((n, m))\} = E\{(X(\theta) - Z(\theta)) \cdot X(1, m)\},$$

$m=1, 2, \dots, M$. Therefore by solving only (30), we can determine the best weights A of $Z(\theta)$.

Reference

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