ON REGRESSION ANALYSIS OF A CERTAIN RANDOM FIELD ON THE UNIT SPHERE

By

Takeaki NAGAI

§ 1. Summary

Let a random field $\{X(t),\ t \in T\}$ be the sum of an unknown mean value function $m(t)=E\{X(t)\},\ t \in T$ and a second-order homogeneous random field on the unit sphere T.

We shall consider in this report a test procedure for the null hypothesis H_0: $m(t)\equiv m(gt)$, for all $g \in G_0,\ t \in T$ against an alternative hypothesis H_1: $m(t) \not\equiv m(gt)$, for $g \in G_0,\ t \in T$, where G_0 is rotations around z-axis.

By making use of A. M. Obukhov's result [4] it can be shown that the statistic S_N defined by (4) follows the χ^2-destribution with $2N$ degrees of freedom when the hypothesis H_0 is true.

§ 2. Regression analysis

Let $T^*=\{(x, y, z)\,|\,x^2+y^2+z^2=1\}\subset R^3$. Let (θ, φ), $0\leq\theta\leq\pi$, $0\leq\varphi\leq 2\pi$, be the polar coordinates of points of T^*, say,

$$x = \sin\theta\, \cos\varphi,$$

$$y=\sin\theta\, \sin\varphi,$$

$$z =\cos\theta.$$

Let us write $\Theta=\{\theta\,|\,0\leq\theta\leq\pi\}$, $\Phi=\{\varphi\,|\,0\leq\varphi<2\pi\}$ and $T\equiv\Theta\times\Phi=\{(\theta, \varphi)\,|\,\theta \in \Theta, \varphi \in \Phi\}$.

Let $G^*=SO(3)$ and G be a group consisting of transformations $g: (\theta, \varphi)\rightarrow(\theta', \varphi')$, from T onto T induced by $g^* \in G^*$ such that

$$g^*\cdot\begin{pmatrix} \sin\theta\ \cos\varphi \\ \sin\theta\ \sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \sin\theta'\ \cos\varphi' \\ \sin\theta'\ \sin\varphi' \\ \cos\theta' \end{pmatrix}.$$

For any metric space $\mathfrak{X}$ we denote by $\mathfrak{B}_{\mathfrak{X}}$ the σ-field of Borel subsets of $\mathfrak{X}$. Let $\mu(\cdot)$ be a measure defined on $\mathfrak{B}_T$ such that

(1) For any $C \in \mathfrak{B}_T$,
$\mu(C)=\mu(g\cdot C)$ for any $g \in G$.

(2) $\mu(T)=1$.

Then, it is well-known that for any $A \in \mathfrak{L}_{\Theta}$, $B \in \mathfrak{L}_{\Phi}$,

$$\mu(A\times B)=\frac{1}{4\pi}\int_A\int_B sin\theta\ d\theta\, d\varphi.$$

Let $\{X(t),\ t \in T\}$ be a real-valued Gaussian random field measurable with respect to $\mathfrak{L}_T$. Let $m(t)=E\{X(t)\}$, $t \in T$, be the mean value function of the random field and let us assume that $m(t)$, $t \in T$ is unknown.

Let $K(t, s)=Cov(X(t), X(s))$, $t, s \in T$, be the covariance function of the random field satisfying the following conditions:

(i) The covariance function $K(t, s)$, $t, s \in T$, is known.

(ii) $K(t, s)$ is a continuous positive definite function on $T\times T$.

(iii) For every $g \in G$,

$$K(gt, gs)=K(t, s), \qquad t, s \in T.$$

Then, by the A. M. OBUKHOV's result [4], we may write $X(t)$, $t \in T$, as follows:

$$X(t)=m(t)+\sum_{n=0}^{\infty}\{Z_{n0}\varphi_{n0}(t)+\sum_{m=1}^{n}\sum_{i=1}^{2}Z_{nm}^{(i)}\cdot\varphi_{nm}^{(i)}(t)\}, \quad t \in T, \tag{3}$$

where

$$\varphi_{n0}(\theta, \varphi)=\sqrt{2n+1}\cdot P_n(cos\theta), \qquad n=0, 1, 2, \cdots,$$

$$\varphi_{nm}^{(1)}(\theta, \varphi)=\sqrt{\frac{2(2n+1)(n-m)!}{(n+m)!}}\cdot P_n^m(cos\theta)\cdot cos\, m\varphi,$$

$$\varphi_{nm}^{(2)}(\theta, \varphi)=\sqrt{\frac{2(2n+1)(n-m)!}{(n+m)!}}\cdot P_n^m(cos\theta)\cdot sin\, m\varphi,$$

$$n=1, 2, 3, \cdots, m=1, 2, \cdots, n.$$

$\{Z_{n0},\ n=0, 1, \cdots;\ Z_{nm}^{(i)},\ i=1, 2,\ n=1, 2, \cdots,\ m=1, 2, \cdots, n\}$ are real Gaussian random variables such that

$$E\{Z_{n0}\}=E\{Z_{nm}^{(i)}\}=0, \qquad n=0, 1, \cdots, m=1, 2, \cdots, n,\ i=1, 2,$$

$$E\{Z_{n0}\cdot Z_{n'\cdot m'}^{(i)}\}=0,$$

$$E\{Z_{n0}\cdot Z_{n'0}\}=\delta_{n\cdot n'}\cdot\lambda_n,$$

$$E\{Z_{nm}^{(i)}\cdot Z_{n'm'}^{(j)}\}=\delta_{n,n'}\delta_{m\cdot m'}\delta_{i\cdot j}\cdot\lambda_n$$

where

$$\delta_{n\cdot m}=1 \quad \text{if} \quad n=m$$
$$=0 \quad \text{otherwise.}$$

$\{\lambda_n,\ n=0, 1, 2, \cdots\}$ are determined by the following equations:

$$\lambda_n \varphi_{n0}(t) = \int_T K(t, s)\varphi_{n0}(s)d\mu(s).$$

$\{P_n(z),\ n=0, 1, \cdots, |z| \leq 1\}$ are Legendre's polynomials and $\{P_n^m(z),\ n=1, 2, \cdots, m=1, 2, \cdots, n;\ |z| \leq 1\}$ are associated Legendre functions.

Let $L_2(X)$ be the Hilbert space consisting of all random variables U which may be represented either as a finite linear combination

$$U = \sum_{i=1}^{n} c_i X(t_i)$$

for some integer n, points $t_1, t_2, \cdots, t_n$ in T and real numbers $c_1, c_2, \cdots, c_n$ or as a limit in quadratic mean of such finite liner combinations under the inner product (U, V) defined by

$$\begin{aligned}(U, V) &= E_m\{U \cdot V\}\\ &= Cov(U, V) + E_m\{U\} \cdot E_m\{V\}.\end{aligned}$$

The subscript m on an expectation operator E is written to indicate that the expectation is computed under the assumption that $m(\cdot)$ is the true mean value function.

We shall now make use of a theorem of Moore-Aronsjazn [1]:

THEOREM I. [See [6], page 7, theorem 2B].

A symmetric non-negaeive kernel K generates a unique Hilbert space, which we denote by $H(K)$, of which K is the reproducing kernel.

Let $H(K)$ be the reproducing kernel Hilbert space with a reproducing kernel $K(t, s)$, $t, s \in T$, that is, $H(K)$ is a Hilbert space such that

(K.1) $\quad K(t, \cdot) \in H(K) \quad$ for each $\quad t \in T$,

(K.2) $\quad$ For any $\quad f \in H(K)$,

$$(f, K(t, \cdot))_K = f(t),$$

where by $(f_1, f_2)_K$ we denote the scalar product of every pair of elements f_1, f_2 in $H(K)$.

Throughout this note we shall assume that $m(\cdot) \in H(K)$.

Now, we have the following:

LEMMA 1. $\quad H(K) \subset L_2(T, \mathfrak{B}_T, \mu)$.

(PROOF) For any $h \in H(K)$, the following inequality holds: For any $t_1, t_2 \in T$,

$$\begin{aligned}|h(t_1) - h(t_2)| &= |(h, K(t_1, \cdot) - K(t_2, \cdot))_K|\\ &\leqq \|h\|_K \cdot \{2(K(t_1, t_1) - K(t_1, t_2))\}^{1/2}.\end{aligned}$$

This implies that $h(t)$, $t \in T$ is a continuous function on T. T is a compact metric space and $\mu(T)=1$. Hence, it follows that $H(K) \subset L_2(T, \mathfrak{B}_T, \mu)$.

For every pair of functions f_1, f_2 in $L_2(T, \mathfrak{B}_T, \mu)$, we denote the scalar product of (f_1, f_2) by

$$(f_1, f_2)_T = \int_T f_1(t)\overline{f_2(t)}d\mu(t).$$

THEOREM II. [[7], page 29, theorem 4A].

Let M be a known class of functions in $H(K)$ and let us assume that $m(\cdot) \in M$.

Then, we have the following results:

(I) *There is a linear one-one mapping ψ from $H(K)$ onto $L_2(X)$ with the following properties:*

(i) $$\psi(K(t, \cdot)) = X(t) \qquad \textit{for each} \quad t \in T,$$

(ii) *For any* $h \in H(K)$,

$$E_m\{\psi(h)\} = (h, m)_K \qquad \textit{for all} \qquad m \in M,$$

(iii) *For any* $h_1, h_2 \in H(K)$,

$$Cov(\psi(h_1), \psi(h_2)) = (h_1, h_2)_K,$$

(II) *A random variable $\psi(h)$, $h \in H(K)$ is said to be an unbiased linear estimate of the value $m(t)$ at a particular point $t \in T$ of the mean value function $m(\cdot)$ if*

$$E_m\{\psi(h)\} = (h, m)_K = m(t) \qquad \textit{for all} \quad m \in M.$$

The uniformly minimum variance unbiased linear estimate $\widehat{m(t)}$ of $m(t)$ is given by

$$\widehat{m(t)} = \psi(E^*(K(t, \cdot) \mid \bar{M})),$$

where $E^*(K(t, \cdot) \mid \bar{M})$ is the projection of $K(t, \cdot)$ onto the smallest subspace of $H(K)$ containing M.

Now, we have the following:

LEMMA 2.

(i) $$\varphi_{n0}, \varphi_{nm}^{(i)} \in H(K), \qquad n = 0, 1, \cdots, m = 1, 2, \cdots, n, i = 1, 2,$$

(ii) $$\psi(\varphi_{n0}) = \lambda_n^{-1} \cdot X_{n0}, \qquad n = 0, 1, 2, \cdots,$$

$$\psi(\varphi_{nm}^{(i)}) = \lambda_n^{-1} \cdot X_{nm}^{(i)}, \qquad n = 1, 2, \cdots, m = 1, 2, \cdots, n, i = 1, 2,$$

where $$X_{n0} = \int_T X(t)\varphi_{n0}(t)d\mu(t) = (m, \varphi_{n0})_T + Z_{n0}$$

$$X_{nm}^{(i)} = \int_T X(t)\varphi_{nm}^{(i)}(t)d\mu(t) = (m, \varphi_{nm}^{(i)})_T + Z_{nm}^{(i)}$$

(iii) $$(\varphi_{n0}, \varphi_{n'0})_K = \delta_{nn'} \cdot \lambda_n^{-1},$$

$$(\varphi_{nm}^{(i)}, \varphi_{n'm'}^{(j)})_K = \delta_{nn'} \cdot \delta_{mm'} \delta_{ij} \cdot \lambda_n^{-1},$$

$$(\varphi_{n0}, \varphi_{n'm}^{(j)})_K = 0, \qquad \textit{for all} \quad n, n', m, j.$$

(PROOF) Let us consider the following integrals:

$$X_{n0} = \int_T X(t)\varphi_{n0}(t)d\mu(t),$$

$$X_{nm}^{(i)} = \int_T X(t)\varphi_{nm}^{(i)}(t)d\mu(t).$$

From (3), we see that $\{X_{n0},\ n=0,\ 1,\ \cdots,\ X_{nm}^{(i)},\ i=1,\ 2,\ n=1,\ 2,\ \cdots,\ m=1,\ 2,\ \cdots,\ n\}$ are well-defined and elements of $L_2(X)$, and it is clear that

$$X_{n0} = (m,\ \varphi_{n0})_T + Z_{n0},$$

$$X_{nm}^{(i)} = (m,\ \varphi_{nm}^{(i)})_T + Z_{nm}^{(i)}.$$

From theorem II, for each $n=0,\ 1,\ 2,\ \cdots$, there is a unique function $k_{n0} \in H(K)$ such that $\psi(k_{n0}) = X_{n0}$ and for each $i=1,\ 2,\ n=1,\ 2,\ \cdots,\ m=1,\ 2,\ \cdots,\ n$, a unique function $k_{nm}^{(i)} \in H(K)$ such that $\psi(k_{nm}^{(i)}) = X_{nm}^{(i)}$.

By virtue of the properties of the reproducing kernel Hilbert space we see that

$$\begin{aligned} k_{n0}(t) &= (k_{n0},\ K(t,\ \cdot))_K \\ &= Cov.(X_{n0},\ X(t)) \\ &= \lambda_n \varphi_{n0}(t), \end{aligned}$$

and

$$\begin{aligned} k_{nm}^{(i)}(t) &= (k_{nm}^{(i)},\ K(t,\ \cdot))_K \\ &= Cov(X_{nm}^{(i)},\ X(t)) \\ &= \lambda_n \cdot \varphi_{nm}^{(i)}(t). \end{aligned}$$

Thus, it follows that $\varphi_{n0} \in H(K)$, $\varphi_{nm}^{(i)} \in H(K)$ and

$$\psi(\varphi_{n0}) = \lambda_n^{-1} \cdot X_{n0} = \lambda_n^{-1} \cdot \{(m,\ \varphi_{n0})_T + Z_{n0}\},$$

$$\psi(\varphi_{nm}^{(i)}) = \lambda_n^{-1} \cdot X_{nm}^{(i)} = \lambda_n^{-1} \cdot \{(m,\ \varphi_{mm}^{(i)})_T + Z_{nm}^{(i)}\}.$$

From theorem II, it is immediate that

$$\begin{aligned} (\varphi_{n0},\ \varphi_{n'0})_K &= Cov(\lambda_n^{-1} X_{n0},\ \lambda_{n'}^{-1} X_{n'0}) \\ &= \delta_{nn'} \cdot \lambda_n^{-1}, \\ (\varphi_{nm}^{(i)},\ \varphi_{n'm'}^{(j)})_K &= Cov(\lambda_n^{-1} X_{nm}^{(i)},\ \lambda_{n'}^{-1} \cdot X_{n'm'}^{(j)}) \\ &= \delta_{nn'} \cdot \delta_{mm'} \cdot \delta_{ij} \cdot \lambda_n^{-1}, \\ (\varphi_{n0},\ \varphi_{n'm'}^{(j)})_K &= Cov(X_{n0} \cdot \lambda_n^{-1},\ \lambda_{n'}^{-1} \cdot X_{n'm'}^{(j)}) \\ &= 0. \end{aligned}$$

Let $G^*{}_0=\{U(\varphi)\,|\,0\leq\varphi<2\pi\}$,
where we write

$$U(\varphi)=\begin{pmatrix} cos\varphi, & sin\varphi, & 0 \\ -sin\varphi, & cos\varphi, & 0 \\ 0\ , & 0\ , & 1 \end{pmatrix},$$

that is, G_0^* is a subgroup of G^* consisting of all rotations around z-axis.

Let G_0 be a subgroup of G isomorphic to G_0^* under the isomorphism between G and G^*.

Let M be such that

$$M=\{f\,|\,f\in H(K),\ f(gt)=f(t) \qquad \text{for all} \quad g\in G_0,\ t\in T\}$$

Then, we have the following:

THEOREM III. *For all $m\in M$, the best linear unbiased estimate of $m(t)$ is givn by*

$$\widehat{m(t)}=\sum_{\nu=0}^{\infty} X_{\nu 0}\varphi_{\nu 0}(t), \qquad t\in T.$$

The estimate $\widehat{m(t)}$ follows the normal distribution with the mean value $m(t)$ and the variance

$$Var\,(\widehat{m(t)})=\sum_{\nu=0}^{\infty}(2\nu+1)\cdot\lambda_\nu\cdot\{P_\nu(cos\theta)\}^2,$$

where $\qquad t=(\theta,\ \varphi)\in T.$

(PROOF) Let $f\in M$. Then we may write

$$f(t)=\sum_{n=0}^{\infty}\{(f,\ \varphi_{n0})_T\cdot\varphi_{n0}(t)+\sum_{m=1}^{n}\sum_{i=1}^{2}(f,\varphi_{nm}^{(i)})_T\cdot\varphi_{nm}^{(i)}(t)\}, \qquad t\in T,$$

since $\{\varphi_{n0},\ n=0,\ 1,\ \cdots;\ \varphi_{nm}^{(i)},\ i=1,\ 2,\ \cdots;\ n=1,\ 2,\ \cdots;\ m=1,\ 2,\ \cdots,\ n\}$ is a complete orthonormal system of $L^2(T,\ \mathfrak{L}_T,\ \mu)$.

If $f\in M$, then $f(\theta,\ \varphi)=f(\theta,\ \varphi')$ for all $\varphi,\ \varphi'\in\Phi$, that is, $f(\theta,\ \varphi)$ is independent of φ. Hence, it follows that for any $f\in M$, $(f,\ \varphi_{n,m}^{(i)})_T=0$, $i=1,\ 2,\ \cdots,\ m=1,\ 2,\ \cdots,\ n$.

Thus, we may write for any $f\in M$,

$$f(t)=\sum_{\nu=0}^{\infty}(f,\ \varphi_{\nu 0})_T\cdot\varphi_{\nu 0}(t),$$

since for all ν, $\varphi_{\nu 0}\in M$, we see that $\bar{M}=L_2(\varphi_{\nu 0},\ \nu=0,\ 1,\ \cdots)$.

The projection of $K(t,\ \cdot)$ onto $\bar{M}$ may be written as follows:

$$E^*(K(t,\ \cdot)\,|\,\bar{M})=\sum_{\nu=0}^{\infty}a_\nu(t)\cdot\varphi_{\nu 0}(\cdot), \qquad t\in T,$$

where $a_\nu(t)$ is a constant for each $t\in T$ and $\nu=0,\ 1,\ \cdots$ such that

$$\sum_{\nu=0}^{\infty} \{a_\nu(t)\}^2/\lambda_\nu < \infty, \qquad \text{for each} \quad t \in T.$$

From theorem II, we have

$$\widehat{m(t)} = \psi\Big(\sum_{\nu=0}^{\infty} a_\nu(t)\cdot\varphi_{\nu 0}(\cdot)\Big)$$

$$= \sum_{\nu=0}^{\infty} a_\nu(t)\cdot\psi(\varphi_{\nu 0})$$

$$= \sum_{\nu=0}^{\infty} a_\nu(t)\cdot\lambda_\nu^{-1}\cdot X_{\nu 0}.$$

Since $\widehat{m(t)}$ is unbiased for all $m \in M$,

$$m(t) = E_m\{\widehat{m(t)}\}$$

$$= \sum_{\nu=0}^{\infty} a_\nu(t)\cdot(m, \varphi_{\nu 0})_K, \qquad \text{for all} \quad m \in M.$$

In particular, let us put $m = \varphi_{\nu 0}$.
Then, we have

$$\varphi_{\nu 0}(t) = \sum_{n=0}^{\infty} a_n(t)\cdot(\varphi_{n0}, \varphi_{\nu 0})_K$$

$$= a_\nu(t)\cdot\lambda_\nu^{-1}, \qquad \text{for} \quad \nu = 0, 1, 2, \cdots.$$

Thus, we obtain

$$\widehat{m(t)} = \sum_{n=0}^{\infty} X_{n0}\varphi_{n0}(t).$$

It is clear that the estimate $\widehat{m(t)}$ follows the normal distribution with the mean value $m(t)$. The variance of $\widehat{m(t)}$ is given from lemma 3 by

$$Var(\widehat{m(\theta, \varphi)}) = \sum_{\nu=0}^{\infty} \lambda_\nu\cdot\{\varphi_{\nu 0}(\theta, \varphi)\}^2$$

$$= \sum_{\nu=0}^{\infty} \lambda_\nu\cdot(2\nu+1)\cdot\{P_\nu(\cos\theta)\}^2.$$

Thus, we have proved theorem III.
Let us consider the following null hypothesis:

H_0: The mean value function $m(t)$ is invariant under every $g \in G_0$, that is, $m \in M$, against an alternative hypothesis:

H_1: $m \notin M$.

Here, we have the following:

THEOREM IV. *Let us put*

$$S_N=\sum_{n=1}^{N}\lambda_n^{-1}\cdot\{\sum_{m=1}^{n}\sum_{i=1}^{2}(X_{nm}^{(i)})^2\},\qquad N=1,\ 2,\ \cdots. \tag{4}$$

Then, under the assumption that H_0 is true, the statistic S_N follows the χ^2-distribution with $2N$ degrees of freedom and for a large N

$$\sqrt{2S_N}-\sqrt{2N-1}$$

follows approximately the normal distribution with zero mean and the unit variance.

(PROOF) If the hypothesis H_0 is true, then, $m \in \bar{M}=L_2(\varphi_{\nu 0},\ \nu=0,\ 1,\ \cdots)$. Hence, $(m, \varphi_{nm}^{(i)})_T=0$ for all $i=1,\ 2,\ n=1,\ 2,\ \cdots,\ m=1,\ 2,\ \cdots,\ n$. Therefore, the random variables $\{X_{nm}^{(i)},\ i=1,\ 2,\ n=1,\ 2,\ \cdots,\ m=1,\ 2,\ \cdots,\ n\}$ follow the normal distribution independently with zero means and $Var(X_{nm}^{(i)})=\lambda_n$. Thus, the random variables $\{X_{nm}^{(i)}/\sqrt{\lambda_n},\ i=1,\ 2,\ n=1,\ 2,\ \cdots,\ m=1,\ 2,\ \cdots,\ n\}$ follow the $N(0.\ 1)$ distribution independently. Now, it it clear that under the hypothesis H_0, the statistic S_N follows the χ^2-distribution with $2N$ degrees of freedom.

References

[1] N. ARONSZAJN: "*Theory of reproducing kernels*", Trans. Amer. Math. Soc. Vol. **68**, (1950), pp. 337-404.
[2] E. J. HANNAN: "*Group representations and Applied probability*", Methuen, London (1965).
[3] T. NAGAI: "*On regression analysis of a homogeneous random field on a compact group*", The Univ. Sheffield. Research Report. No. 42, (1968).
[4] A. M. OBUKHOV: "*Statistically homogeneous random fields on a sphere*", Uspehi Mat. Nauk, Vol. **2**, No. 2, (1947), pp. 196-198.
[5] E. PARZEN: "*Time series analysis papers*", Holden-Day, Calif. (1967).
[6] E. PARZEN: "*Regression analysis of continuous parameter time series, II*", Stanford Univ. Tech. Report. No. 35, (1960).
[7] E. PARZEN: "*A new approach to the synthesis of optimal smoothing and prediction systems*", Stanford Univ. Tech. Report, No. 34, (1960).
[8] A. M. YAGLOM: "*Second-order Homogeneous random fields*", Proc. fourth Berkeley Symp. on Math. Stat. and Prob. Vol. **II**, Univ. Calif. Press, (1961), pp. 493-622.