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Rep. Fac. Sci., Kagoshima Univ.
No. 30, 7~11 (1997)

Construction and Enumeration of Graphical Sequences Corresponding to Graphs Having Exact Three Vertices with the Same Degree

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(Received August 20, 1997)

Abstract

The aim of this note is to construct all the graphical sequences corresponding to graphs which have exact three vertices with the same degree. This work is a continuation of the first author's paper [2] in this Reports.

Key words: Graph, Graph having exact three vertices with the same degree, Degree sequence, Graphical sequence.

1 Classification

In this note we use freely the terminology and notation concerning graphs in G.Chartrand and L.Lesniak [1]. For any positive integer n and non-negative integer m with $m < n$, we use the following notation:

$[n] := \{1, 2, 3, ..., n\} \quad [m, n] := \{m, m+1, ..., n\}$.

A sequence $s : s_1, s_2, ..., s_n$ of non-negative integers is said to be *graphical* if there exists a simple graph G of order n whose degree sequence is s.

The purpose of this note is to determine all the graphical sequences $s : s_1, s_2, ..., s_n$ with the following property:

(*) $\quad n-1 \geq s_1 > s_2 > ... > s_{k-1} > s_k = s_{k+1} = s_{k+2} > s_{k+3} > ... > s_n \geq 0$

for some $k \in [n-2]$.

For the sake of brevity any sequence $s : s_1, s_2, ..., s_n$ of non-negative integers with the property $(*)$ is said to be $(n,3)$-*admissible* and any sequences with $(*)$ are denoted by $s_n(s_1, s_n; s_k)$. For any fixed s_1 and s_n let $S_n(s_1, s_n)$ be the set of $(n,3)$-admissible sequences given in the form $s_n(s_1, s_n; s_k)$. It is seen easily that the set of all $(n,3)$-admissible sequence is partitioned into the five classes $S_n(n-1, 2)$, $S_n(n-1, 1)$, $S_n(n-2, 1)$, $S_n(n-2, 0)$ and $S_n(n-3, 0)$. We note that $s_n(n-m, 3-m; k), k \in [3-m, n-m]$, expresses a sequence for $m = 1, 2, 3$. Further we denote by $GS(n,3)$ and $GS_n(s_1, s_n)$ the set of all graphical $(n,3)$-admissible sequences and the

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set of all graphical sequences in $S_n(s_1, s_n)$ respectively. Then we have

Lemma 1.1 *$GS(n,3)$ is partitioned into the five classes as follows :*
$GS_n(n-1,2) \cup GS_n(n-1,1) \cup GS_n(n-2,1) \cup GS_n(n-2,0) \cup GS_n(n-3,0)$.

Computing directly, we get the following Lemmas 1.2 – 1.5.

Lemma 1.2 $GS(3,3) = \{(2,2,2),(0,0,0)\}$. *More precisely we have*
$GS_3(2,2) = \{(2,2,2)\}$
$GS_3(2,1) = GS_3(1,1) = GS_3(1,0) =$ *empty*
$GS_3(0,0) = \{(0,0,0)\}$.

Lemma 1.3 $GS(4,3) = \{(3,1,1,1),(2,2,2,0)\}$. *More precisely we have*
$GS_4(3,2) =$ *empty*
$GS_4(3,1) = \{(3,1,1,1)\}$
$GS_4(2,1) =$ *empty*
$GS_4(2,0) = \{(2,2,2,0)\}$
$GS_4(1,0) =$ *empty.*

Lemma 1.4 *$GS(5,3)$ consists of the following five sequences :*
$GS_5(4,2) =$ *empty*
$GS_5(4,1) = \{(4,3,3,3,1)\}$
$GS_5(3,1) = \{(3,2,1,1,1),(3,2,2,2,1),(3,3,3,2,1)\}$
$GS_5(3,0) = \{(3,1,1,1,0)\}$
$GS_5(2,0) =$ *empty.*

Lemma 1.5 *$GS(6,3)$ consists of the following twelve sequences :*
$GS_6(5,2) = \{(5,4,3,2,2,2),(5,4,3,3,3,2),(5,4,4,4,3,2)\}$
$GS_6(5,1) = \{(5,4,2,2,2,1)\}$
$GS_6(4,1) = \{(4,3,2,1,1,1),(4,3,2,2,2,1),(4,3,3,3,2,1),(4,4,4,3,2,1)\}$
$GS_6(4,0) = \{(4,3,3,3,1,0)\}$
$GS_6(3,0) = \{(3,2,1,1,1,0),(3,2,2,2,1,0),(3,3,3,2,1,0)\}$.

2 Construction of $GS(n,3)$

In this section we shall construct inductively all sequences in $GS(n,3)$, $n \geq 4$. The next lemma, noted in [1,Theorem 1.4], plays the essential role in our discussion.

Lemma 2.1 *A sequence $s : s_1, s_2, ..., s_n$ of non-negative integer with $s_1 \geq s_2 \geq ... \geq s_n$, $n \geq 2$, $s_1 \geq 1$, is graphical if and only if the following sequence $h(s)$ with $n-1$ terms is graphical:*
$h(s) : s_2 - 1, s_3 - 1, ..., s_{t+1} - 1, s_{t+2}, s_{t+3}, ..., s_n$
where $t = s_1$.

Now for any sequence $s : s_1, s_2, ..., s_{n-1}, n \geq 2$, of integers with $n-1$ terms we define the sequence $p(s)$ with n terms by

$p(s): n-1, s_1+1, s_2+1, ..., s_{n-1}+1.$

For any set F of sequences of integers, we set $p(F) = \{p(s); s \in F\}$. Some $(n-1,3)$-admissible sequences are mapped injectively to $(n,3)$-admissible ones by the map $p: s \mapsto p(s)$. More precisely we have

Lemma 2.2 *Let n be any positive integer with $n \geq 4$. Then we have*
(1) $h(p(s)) = s$ *for any* $s \in S_{n-1}(n-3,1) \cup S_{n-1}(n-3,0) \cup S_{n-1}(n-4,0)$.
(2) $S_n(n-1,2) = p(S_{n-1}(n-3,1)) \cup \{s_n(n-1,2;n-1)\}$.
(3) $S_n(n-1,1) = p(S_{n-1}(n-3,0) \cup S_{n-1}(n-4,0)) \cup \{s_n(n-1,1;n-1)\}$.

Various criteria for sequences to be graphic are shown in G. Sierksma and H. Hoogeven [3]. We use the next criterion noted in [1,Theorem 1.5].

Lemma 2.3 *A sequence $s: s_1, s_2, ..., s_n (n \geq 2)$ of non-negative integers with $s_1 \geq s_2 \geq s_3 \geq ... \geq s_n$ is graphical if and only if the following two conditions hold:*
(P) $\quad \sum_{k=1}^{n} s_k = even$
and for each integer $k \in [n-1]$,
(E_k) $\quad \sum_{j=1}^{k} s_j \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, s_j\}$.

In what follows, for any sequence s as in Lemma 2.3 the left[resp. right] hand side of (E_k) is denoted by (EL_k)[resp. (ER_k)].

Lemma 2.4 *Let n be any positive integer. Then we have*
(1) $s = s_n(n-1,2;n-1)$ *is not graphical for* $n \geq 4$.
(2) *Any sequences of type* $s_n(n-1,1;n-1)$ *are not graphical for* $n \geq 3$.

Proof For the sequence s in (1), $(EL_3) = 3n-3 > 3n-4 = (ER_3)$. So s is not graphical by Lemma 2.3. We note that $s_3(2,2;2) = (2,2,2)$ is graphical. (2) is seen similarly. □

From Lemmas 2.1-2.4, it follows that $GS_n(n-1,2)$ and $GS_n(n-1,1)$ is constructed from $GS(n-1,3)$ by the map p.

Theorem 2.5 *Let n be any positive integer with $n \geq 4$. Then we have*
(1) $GS_n(n-1,2) = p(GS_{n-1}(n-3,1))$.
(2) $GS_n(n-1,1) = p(GS_{n-1}(n-3,0) \cup GS_{n-1}(n-4,0))$.

For any $(n,3)$-admissible sequence $s: s_1, s_2, ..., s_n$, we define a $(n,3)$-admissible sequence $c(s)$ by :

$c(s): n-1-s_n, n-1-s_{n-1}, ..., n-1-s_2, n-1-s_1$

Considering a graph and its complement graph, we see that s is graphical if and only if so is $c(s)$. For any set F of $(n,3)$-admissible sequences, we set $c(F) = \{c(s); s \in F\}$. Then the next is seen easily

Theorem 2.6 *Let n be any positive integer with $n \geq 3$. Then we have*
(1) $GS_n(n-3,0) = c(GS_n(n-1,2))$.
(2) $GS_n(n-2,0) = c(GS_n(n-1,1))$.

Finally we determine explicitly any sequences in $GS_n(n-2,1)$.

Lemma 2.7 *Let m be any positive integer. Every sequence in $S_n(n-2,1)$ is not graphical for $n=4m-1$ and $n=4m$.*

Proof This follows from the fact that for $n=4m-1$ and $n=4m$, every sequence in $S_n(n-2,1)$ does not satisfy the condition (P) in Lemma 2.3. □

Lemma 2.8 *Let m be any positive integer. Then we have*

(1) *$s_{4m+1}(4m-1,1;t)$ is not graphical for any t, $1 \leq t < m$.*

(2) *$s_{4m+2}(4m,1;t)$ is not graphical for any t, $1 \leq t < m$.*

Proof For the sequence s in (1) we have

$(EL_{2m}) = 6m^2 - m$ and $(ER_{2m}) = 6m^2 - 3m + 2t$.

$(ER_{2m}) - (EL_{2m}) = 2(t-m) < 0$.

Hence s is not graphical. We see (2) similarly. □

Lemma 2.9 *Let m be any positive integer. Then we have*

(1) *$s_{4m+1}(4m-1,1;m)$ is graphical.*

(2) *$s_{4m+2}(4m,1;m)$ is graphical.*

Proof For the sequence s in (1) we have

$$(ER_k) - (EL_k) = \begin{cases} k & \text{if } 1 \leq k \leq m \\ 2m-k & \text{if } m < k \leq 2m \\ 2(2m-k)^2 & \text{if } 2m < k \leq 3m \\ 2\{(k-2m-1)^2 + 2m\} & \text{if } 3m < k \leq 4m. \end{cases}$$

Hence we have (1). Similarly we see (2). □

Lemma 2.10 *Let m be any positive integer. Then we have*

(1) *$s_{4m+1}(4m-1,1;t)$ is graphical for any t, $m \leq t \leq 3m$.*

(2) *$s_{4m+2}(4m,1;t)$ is graphical for any t, $m \leq t \leq 3m+1$.*

Proof We prove (1) by the induction on m. By Lemma 1.4 the assertion is true for the case $m=1$. Let $m>1$, $s(t) = s_{4m+1}(4m-1,1;t)$ and $3m > t > m$. Let us apply twice Lemma 2.1 to $s(t)$. Then we see that $h(h(s(t))) = s_{4m-3}(4m-5,1;t-2),1,1$. Obviously $h(h(s(t)))$ is graphical if and only if so is $s_{4m-3}(4m-5,1;t-2)$. Since $3(m-1) \geq t-2 \geq m-1$, $s_{4m-3}(4m-5,1;t-2)$ is graphical by the inductive hypothesis, and hence so are $h(h(s(t)))$ and $s(t)$ by Lemma 2.1. From Lemma 2.9, $s(m)$ is graphical and so is $s(3m) = c(s(m))$. Similarly we have (2). □

By virtue of Lemmas 2.7-2.10, $GS_n(n-2,1)$ is characterized explicitly as follows.

Theorem 2.11 *Let m be any positive integer. Then we have*

(1) *$GS_{4m-1}(4m-3,1)$ and $GS_{4m}(4m-2,1)$ are empty.*

(2) *$GS_{4m+1}(4m-1,1) = \{s_{4m+1}(4m-1,1;t);\ t \in [m,3m]\}$.*

(3) *$GS_{4m+2}(4m,1) = \{s_{4m+2}(4m,1;t);\ t \in [m,3m+1]\}$.*

3 Enumeration of $GS(n,3)$

Let n be any positive integer with $n \geq 3$, and let a_n, b_n, c_n, d_n, e_n and g_n be the cardinal number of $GS_n(n-1,2), GS_n(n-1,1), GS_n(n-2,1), GS_n(n-2,0), GS_n(n-3,0)$ and $GS(n,3)$ respectively. The next two lemmas are immediate consequences from Theorems 2.5, 2.6 and 2.11.

Lemma 3.1
(1) $a_n = e_n = c_{n-1}$ *and* $b_n = d_n = a_{n-1} + b_{n-1}$ *for any positive integher* $n \geq 4$.
(2) $a_3 = 1$, $b_3 = c_3 = d_3 = 0$ *and* $e_3 = 1$.

Lemma 3.2 *For any positive integer* m, *we have*

$$a_{n+1} = c_n = \begin{cases} 0 & if\ \ n = 4m-1, 4m \\ 2m+1 & if\ \ n = 4m+1 \\ 2m+2 & if\ \ n = 4m+2. \end{cases}$$

The next follows from Lemmas 3.1 and 3.2.

Lemma 3.3 *For any positive integer* m, *we have*

$$b_{4m} = b_{4m+1} = b_{4m+2} = 2m^2 + m - 2,$$

$$b_{4m-1} = \begin{cases} 2m^2 - m - 2 & if\ \ m \geq 2 \\ 0 & if\ \ m = 1. \end{cases}$$

Since $g_n = 2(a_n + b_n) + c_n$ by Lemma 3.1, we conclude the next theorem from Lemmas 3.2 and 3.3.

Theorem 3.4 *Let* m *be any positive integer. The cardinal number* g_n *of* $GS(n,3)$ *is expressed in the following form:*

$$g_n = \begin{cases} 4m^2 + 2m - 4 & if\ \ n = 4m-1, 4m \\ 4m^2 + 4m - 3 & if\ \ n = 4m+1 \\ 4m^2 + 8m & if\ \ n = 4m+2. \end{cases}$$

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