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Yôichi KOSHIBA*

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We shall consider below an result on the topological fields and the concept of compact open topology. We mean by linear topology on linear space k^m, i.e. of product space of n copies of a topological field k.

Theorem *Let k be a topological field, $M(n,k)$ be $n \times n$ matrix space with entry in k.$M(n,k)$ is regarded as a family of continuous maps from linaer space k^n into k^n itself. Then the compact open topology on $M(n,k)$ as family of maps, coincides to the linear topology on $M(n,k)$, i.e. isomorphic to k^{n^2}.*

Proof When we regard $M(n,k)$ as a family of maps, $M(n,k)$ has also the simple topology(=topology of pointwise convergence). We shall compare the simple topology, and the linear topology, the compact open topology through three steps. □

Lemma 1 *The simple topology on the space $M(n,k)$ is stronger than the linear topology on the space $M(n,k)$.*

We can assume $n = 2$ without loss of generality.
Let $\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \Longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be convergence in simple topology on $M(n,k)$.
For any point $(x,y) \in k^2$,

$$\lim_{n\to\infty} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

holds good. This limit is the convergence in linear topology on k^2. If we put

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then we obtain $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} c_n = c$. Similarily by putting

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

* Department of Mathematics and Computer Science, Faculty of Science, Kagoshima University, Kagoshima, 890-0065, Japan. e-mail:koshiba@sci.kagoshima-u.ac.jp

then we obtain $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} d_n = d$. Therefore in the linear topology,
$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \Longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ □

Remark Generally, the compact open topology is stronger than the simple topology. Since the set of a single point is compact, from the definition of compact open topology, this topology is stronger than simple topology. □

Lemma 2 *The linear topology on $M(n,k)$ is stronger than the compact open topology on M(n,k) as a family of maps.*

Proof By checking following elementary equations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$
$$X = ax + by$$
$$Y = cx + dy$$

variable $\begin{pmatrix} X \\ Y \end{pmatrix}$ is continuous in variables $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix}$. Because variable X, Y are continuous function form, i.e. plus and multiplications of variable, a, b, c, d, x, y.

Let X, Y be topological spaces.

$$Y^X \times X \Longrightarrow Y$$

$$(f, x) \xrightarrow{\varphi} f(x) \in Y$$

$$f \in Y^X \qquad x \in X .$$

Above diagramme, we call the map φ evaluation map. In our case, $X = Y = k^n$, the evaluation map is linear transformation from k^n into k^n by matrix of $M(n,k)$. Generally, if the evaluation map is continuous for a subset S of Y^X, then the topology of a familys of maps(really in our case, the linear topology on the space $M(n,k)$) is stronger than the compact open topology on the S. □

We now return the proof of our Theorem. From Lemma 1, Remark, Lemma 2, we have

$$\text{Linear Topology}=\text{Simple Topology}=\text{Compact Open topology}$$

for topologies on space $M(n,k)$. We have verified the theorem. □

Especially, let k be the real number field, the complex number field, or a local field i.e. complete field with a non-archimedian valuation individually, in matrix space $M(n,k)$, linear topology (if k is real, its a Euclidean topology) coincides to compact open topology as maps $k^n \to k^n$. Replacing $M(n,k)$ to $m \times n$ matrix space $M(m,n;k)$, similar result holds clearly.

**We add an result on the evaluation map. We give an elementary example of non continuous evaluation map.

Example Let X,Y be closed interval $[0,1]$ with ordinary Euclidean topology. In mapping space Y^X, we define the points $f_n \in Y^X$, point f $\in Y^X$ as follows:

$$f_n(x) = x^n \quad n = 1, 2, \cdots$$

and

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1. \end{cases} .$$

The space Y^X is Hausdorff. In the simple topology on space Y^X, points sequence f_n converges to point f.

$$\lim_{n \to \infty} f_n = f.$$

If we define the sequence $\{x_n\}$ in X by $x_n = 1 - \dfrac{1}{n}$, then $\lim_{n \to \infty} x_n = 1$. For point sequence $(f_n, x_n) \in Y^X \times X$,

$$\lim_{n \to \infty} (f_n, x_n) = (f, 1).$$

If the evaluation map φ is continuous, the equation

$$\lim_{n \to \infty} \varphi((f_n, x_n)) = \varphi((f, 1))$$

must be hold.

$$\varphi((f, 1)) = f(1) = 1,$$

$$\varphi((f_n, x_n)) = f_n(x_n) = \left(1 - \frac{1}{n}\right)^n .$$

But from elementary analysis,

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}.$$

Therefore this evaluation map can not be continuous.

References

[1] J. L. Kelly, General Topology, Graduate Texts in Mathematics 27, Springer-Verlag, 1955, Especialy Chapter 7.