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Rep. Fac. Sci., Kagoshima Univ.,
No. 31, pp. 1 ~ 7 (1998)

On Graphs Having Exact Three Vertices with the Same Degree

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(Received August 28, 1998)

Abstract

This work is a continuation of the author's papers [3] and [4]. The purpose of this paper is to investigate the construction of all graphs which have exact three vertices with the same degree. Since the degree sequences of such graphs are determined completely in [4] , our work is how to realize the graphs with such degree sequences. The results in this paper are stated almost without rigorous proof, because these are obtained by paraphrase from the statements concerning degree sequences in [4].

Key words: Graph, Graph having exact three vertices with the same degree.

1 Preliminary

In this paper we use freely the terminology and notation concerning graphs in G.Chartrand and L.Lesniak [1]. For any positive integer n and non-negative integer m with $m < n$, we use the following notation:

$$[n] := \{1, 2, 3, \cdots, n\}, \quad [m, n] := \{m, m+1, \cdots, n\}.$$

Let G be a graph. $V(G), E(G)$ and $n(G)$ are the set of vertices, the set of edges of G and the order of G respectively. The degree sequence $d(G)$ of G is the non-increasing $n(G)$-sequence of degree of vertices of G, and the maximum degree and the minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$ respectively. The symbols K_n and N_n are the complete graph and the empty graph of order n respectively, and $K_{m,n}$ is the complete bipartite graph.

For any graph G we define the four kinds of operations p, q, z and c as follows:

$p(G)$ = *the join of* G *and* N_1,

$q(G)$ = *the graph which is obtained from the disjoint union of* $p(p(G))$ *and* K_2 *adding the new edge* (u, v), *where* u *the vertex of* $p(p(G))$ *with the highest degree and* v *one vertex of* K_2,

$z(G)$ = *the disjoint union of* G *and* N_1,

$c(G)$ = *the complement of* G.

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Evidently these operations are invertible, namely, for any graphs G and H, $p(G) \cong p(H)$ if and only if $G \cong H$, and the analogy holds for the operations q, z and c. For any set $\mathcal{F}$ of graphs, we put $p(\mathcal{F}) = \{p(G); G \in \mathcal{F}\}$ and $q(\mathcal{F})$, $z(\mathcal{F})$ and $c(\mathcal{F})$ are defined similarly.

For any graph G of order n with $d(G) = \{s_1, s_2, \ldots, s_{n-1}, s_n\}$ we have

(1.1) $d(p(G)) = \{n, s_1 + 1, s_2 + 1, \ldots, s_{n-1} + 1, s_n + 1\}$

(1.2) $d(q(G)) = \{n + 2, n + 1, s_1 + 2, s_2 + 2, \ldots, s_{n-1} + 2, s_n + 2, 2, 1\}$

(1.3) $d(z(G)) = \{s_1, s_2, \ldots, s_{n-1}, s_n, 0\}$

(1.4) $d(c(G)) = \{n - 1 - s_n, n - 1 - s_{n-1}, \ldots, n - 1 - s_2, n - 1 - s_1\}$.

The operations p, q, z, c for any graphs and the above relations play essential role in the subsequent sections.

Let G be a graph, and a, b, c, d be four different vertices of G such that $(a, b), (c, d) \in E(G)$ and $(a, c), (b, d) \notin E(G)$. In this case $s = \{(a, b), (c, d)\}$ is called a switching of G, and the switching operation for G is to make the graph $s(G) = G - (a, b) - (c, d) + (a, c) + (b, d)$. Obviously G and $s(G)$ have the same degree sequence. The next theorem stated in [2] teach us how to get all graphs with the same degree sequence.

Switching Theorem *Let G be a graph. Then any graph H with $d(H) = d(G)$ is obtained by a finite sequences of switching operations for G.*

2 Classification and Γ_n for $n = 3, 4, 5, 6$

Let us denote by Γ the class of graphs which have exact three vertices with the same degree and mutually different degree for the other vertices, and for any integer $n \geq 3$ denote by Γ_n the set of all graphs of order n in Γ. The degree sequence $\{s_1, s_2, \cdots, s_n\}$ of any graph G in Γ_n is given in the following form: for some $k \in [n - 2]$

(2.1) $n - 1 \geq s_1 > s_2 > \ldots > s_{k-1} > s_k = s_{k+1} = s_{k+2} > s_{k+3} > \cdots > s_n \geq 0$,

where $s_1 = \Delta(G)$ and $s_n = \delta(G)$.

For any fixed integers s_1 and s_n with $n - 1 \geq s_1 > s_n \geq 0$, $\Gamma_n(s_1, s_n)$ be the set of graphs G in Γ_n with $(\Delta(G), \delta(G)) = (s_1, s_n)$. For the sake of brevity any sequence $\{s_1, s_2, \cdots, s_n\}$ of non-negative integers with the property (2.1) is denoted by $s_n(s_1, s_n; s_k)$. If $s_1 - s_n = n - 3$, then the sequence $s_n(s_1, s_n; k)$ for any integer $k \in [s_n, s_1]$ is uniquely determined by s_1, s_n and k. In this case, $\Gamma_n(s_1, s_n; k)$ is the set of graphs G with the same degree sequence $d(G) = s_n(s_1, s_n; k)$. The next theorem is an immediate consequence of the classification of degree sequences with (2.1) noted in Lemma 1.1 in [4].

Theorem 2.1 *For any integer $n \geq 3, \Gamma_n$ is partitioned into the following five subclasses:*
$\Gamma_n = \Gamma_n(n - 1, 2) \cup \Gamma_n(n - 1, 1) \cup \Gamma_n(n - 2, 1) \cup \Gamma_n(n - 2, 0) \cup \Gamma_n(n - 3, 0)$. □

In order to clarify inductively the structure of Γ_n, we begin on the graph lists of Γ_n for $n = 3, 4, 5, 6$. Theorems 2.2-2.5 follow from Lemmas 1.2-1.5 in [4] and the **Graph cards** given at pp.19-24 in [5].

Theorem 2.2 *Γ_3 consists of two graphs as follows:*

(1) $\Gamma_3(2,2) = \{K_3\}$

(2) $\Gamma_3(0,0) = \{N_3\} = \{c(K_3)\}$

(3) $\Gamma_3(2,1) = \Gamma_3(1,1) = \Gamma_3(1,0) = empty.$ □

Theorem 2.3 *Γ_4 consists of two graphs as follows:*

(1) $\Gamma_4(3,1) = \{K_{1,3}\} = \{p(N_3)\}$

(2) $\Gamma_4(2,0) = \{z(K_3)\}$

(3) $\Gamma_4(3,2) = \Gamma_4(2,1) = \Gamma_4(1,0) = empty.$ □

Theorem 2.4 *Γ_5 consists of six graphs as follows:*

(1) $\Gamma_5(4,2) = empty$

(2) $\Gamma_5(4,1) = \{p(z(K_3))\}$, *(graph no.45 in the Graph cards)*

(3) $\Gamma_5(3,1) = \cup_{k=1}^{3}\Gamma_5(3,1;k)$, *where*

1) $\Gamma_5(3,1;1) = \{G\}$, *where G is the graph no.30 in the Graph cards*

2) $\Gamma_5(3,1;2) = \{H, c(H)\}$, *where H is the graph no.36 in the Graph cards*

3) $\Gamma_5(3,1;3) = c(\Gamma_5(3,1;1))$

(4) $\Gamma_5(3,0) = \{z(K_{1,3})\}$

(5) $\Gamma_5(2,0) = empty.$ □

Theorem 2.5 *Γ_6 consists of twelve graphs as follows:*

(1) $\Gamma_6(5,2) = p(\Gamma_5(3,1))$

(2) $\Gamma_6(5,1) = p(\Gamma_5(3,0))$

(3) $\Gamma_6(4,1) = \cup_{k=1}^{3}\Gamma_6(4,1;k)$, *where*

1) $\Gamma_6(4,1;1) = \{G\}$, *where G is the graph no.93 in the Graph cards.*

2) $\Gamma_6(4,1;2) = \{G_j; j = 1,2,3,4\}$,*where $G_j (j = 1,2,3,4)$ is the graph no.118, 119, 120, 121 in the Graph cards respectively.*

3) $\Gamma_6(4,1;3) = c(\Gamma_6(4,1;2))$

4) $\Gamma_6(4,1;4) = c(\Gamma_6(4,1;1))$

(4) $\Gamma_6(4,0) = z(\Gamma_5(4,1))$

(5) $\Gamma_6(3,0) = z(\Gamma_5(3,1)).$ □

3 Inductive construction of Γ_n

In this section we shall consider the inductive construction of Γ_n from Γ_m with $m < n$. The folloing Lemmas 3.1- 3.4 are consequences of the results stated in the section 3 of [4]. At first from (1.4) we get

Lemma 3.1 *For any integer $n \geq 3, \Gamma_n$ is closed under the operation c. More precisely we have*

(1) $c(\Gamma_n(n-1,2)) = \Gamma_n(n-3,0)$

(2) $c(\Gamma_n(n-1,1)) = \Gamma_n(n-2,0)$

(3) $c(\Gamma_n(n-2,1)) = \Gamma_n(n-2,1)$

(4) $c(\Gamma_n(n-2,1;k)) = \Gamma_n(n-2,1;n-1-k)$, *where* $k \in [n-2]$. □

The four classes in Γ_n except $\Gamma_n(n-2,1)$ are constructed from the subclasses of Γ_{n-1} by the operations p and z. The next is concluded from (1.1) and (1.3).

Lemma 3.2 *For any integer $n \geq 4$, we have*

(1) $\Gamma_n(n-1,2) = p(\Gamma_{n-1}(n-3,1))$

(2) $\Gamma_n(n-1,1) = p(\Gamma_{n-1}(n-3,0)) \cup p(\Gamma_{n-1}(n-4,0))$

(3) $\Gamma_n(n-2,0) = z(\Gamma_{n-1}(n-2,1)) \cup z(\Gamma_{n-1}(n-2,2))$

(4) $\Gamma_n(n-3,0) = z(\Gamma_{n-1}(n-3,1))$. □

Let us consider the subclass $\Gamma_n(n-2,1)$. The next lemma is an immediate consequence from Theorem 2.11 in [4].

Lemma 3.3 *For any positive integer m, we have:*

(1) $\Gamma_{4m-1}(4m-3,1)$ *and* $\Gamma_{4m}(4m-2,1)$ *are empty*

(2) $\Gamma_{4m+1}(4m-1,1)$ *and* $\Gamma_{4m+2}(4m,1)$ *are partitioned into the following subclasses:*

1) $\Gamma_{4m+1}(4m-1,1) = \cup_{k=m}^{3m} \Gamma_{4m+1}(4m-1,1;k)$

2) $\Gamma_{4m+2}(4m,1) = \cup_{k=m}^{3m+1} \Gamma_{4m+2}(4m,1;k)$. □

Let G be in $\Gamma_{n-4}(n-6,1;k-2)$. Then $d(q(G)) = s_n(n-2,1;k)$ from (1.2). Hence by the operation q for $\Gamma_{n-4}(n-6,1)$ we get some subclasses of $\Gamma_n(n-2,1)$.

Lemma 3.4 *Let m be any integer with $m > 1$. Then for $n = 4m+1$ and $n = 4m+2$ we have*

$$q(\Gamma_{n-4}(n-6,1)) \subset \Gamma_n(n-2,1)$$

More precisely we have

(1) $q(\Gamma_{4m-3}(4m-5,1;k-2)) \subset \Gamma_{4m+1}(4m-1,1;k)$ *for any* $k \in [m+1, 3m-1]$

(2) $q(\Gamma_{4m-2}(4m-4,1;k-2)) \subset \Gamma_{4m+2}(4m,1;k)$ *for any* $k \in [m+1, 3m]$. □

A graph in $\Gamma_n(n-2,1;k)$ is called a representative of $\Gamma_n(n-2,1;k)$, a representative system $R(\Gamma_n(n-2,1))$ of $\Gamma_n(n-2,1)$ is the collection of each representative of $\Gamma_n(n-2,1;k)$ for all $k \in [n-2]$. Let $n = 4m+1$. By virtue of Lemma 3.4 we can get inductively a representative of $\Gamma_n(n-2,1;k)$ for any $k \in [m+1, 3m-1]$. Since $\Gamma_n(n-2,1;3m) = c(\Gamma_n(n-2,1;m))$, it remains to give a representative of $\Gamma_n(n-2,1;m)$. The next Lemma is due to *the l-procedure* stated in [2, p.119].

Lemma 3.5 *For any positive integer m, the following graph $G_{4m+1}(m)$[resp.$G_{4m+2}(m)$] is a representative of $\Gamma_{4m+1}(4m-1,1;m)$[resp.$\Gamma_{4m+2}(4m,1;m)$]:*

(1) $G = G_{4m+1}(m)$

$V(G) = [4m+1]$,
$E(G) = E_1 \cup E_2$ *consists of* $4m^2$ *edges* (j,k) *given in the form:*
$E_1 = \{(j,k); j \in [m], k \in [j+1, 4m+1-j]\}$ *and*
$E_2 = \{(j,k); j \in [m+1, 2m], k \in [j+1, 4m-j] \cup \{2m+1+j\}\}$.

(2) $G = G_{4m+2}(m)$

$V(G) = [4m+2]$,
$E(G) = E_1 \cup E_2$ *consists of* $4m^2+2m$ *edges* (j,k) *given in the form:*
$E_1 = \{(j,k); j \in [m], k \in [j+1, 4m+2-j]\}$ *and*
$E_2 = \{(j,k); j \in [m+1, 2m], k \in [j+1, 4m+1-j] \cup \{2m+2+j\}\}$. □

From the above lemmas and the switching theorem, Γ_n is inductively constructed from Γ_m with $m < n$.

Theorem 3.6 *For any integer $n > 3$, we have:*

(1) $\Gamma_n(n-1,2) = p(\Gamma_{n-1}(n-3,1))$

(2) $\Gamma_n(n-1,1) = p(\Gamma_{n-1}(n-3,0)) \cup p(\Gamma_{n-1}(n-4,0))$

(3) $\Gamma_n(n-2,1)$ *is empty for* $n = 0,3 \pmod 4$

(4) *for* $n = 4m+1$ *and* $n = 4m+2, m > 1$, *a representative system* $R(\Gamma_n(n-2,1))$ *is given by*
$R(\Gamma_n(n-2,1)) = q(R(\Gamma_{n-4}(n-6,1)) \cup \{G_n(m), c(G_n(m)\}$,
where $G_n(m)$ *is the graph given in Lemma 3.5*

(5) *for* $n = 4m+1$ *and* $n = 4m+2, m > 1, \Gamma_n(n-2,1)$ *is the set of all graphs obtained by the finite sequences of switching operations for any graphs in* $R(\Gamma_n(n-2,1))$

(6) $\Gamma_n(n-2,0) = z(\Gamma_{n-1}(n-2,1)) \cup z(\Gamma_{n-1}(n-2,2))$

(7) $\Gamma_n(n-3,0) = z(\Gamma_{n-1}(n-3,1))$. □

By virtue of Theorem 3.6 we get a characterization of the class Γ by the operations p, q, c and switchings.

Theorem 3.7 *The class Γ is the smallest class of graphs among any classes Λ of graphs with the properties* (1)-(6) :

(1) Λ *does not contain any graphs G with* $n(G) < 3$

(2) Λ *is closed under the operation* c

(3) $p(G) \in \Lambda$ *for any graph* $G \in \Lambda$ *with* $\Delta(G) \leq n(G) - 2$

(4) $q(G)$ *and* $s(G)$ *belong to* Λ *for any graph* $G \in \Lambda$ *with* $\Delta(G) = n(G) - 2$ *and* $\delta(G) = 1$, *and for any switching s of G*

(5) *the graphs* $G_n(m)$ *for* $n = 4m+1, 4m+2, m \geq 2$, *given in Lemma 3.5, belong to* Λ

(6) $K_3, \Gamma_5(3,1;1), \Gamma_5(3,1;2), \Gamma_6(4,1;1)$ *and* $\Gamma_6(4,1;2)$ *are contained in* Λ. □

4 Estimation of cardinal number of Γ_n

In this section we shall give a lower estimation of the cardinal number T_n of Γ_n. For any integer $n \geq 3$, let a_n, b_n, c_n, d_n and e_n be the cardinal number of $\Gamma_n(n-1,2), \Gamma_n(n-1,1), \Gamma_n(n-2,1), \Gamma_n(n-2,0)$ and $\Gamma_n(n-3,0)$ respectively. From Theorems 2.2-2.5 and 3.6 we have:

$c_n = 0$ for any $n = 0,3 \pmod 4$, $c_5 = 4$ and $c_6 = 10$,

$a_n = e_n = c_{n-1}$,

$b_n = d_n = b_{n-1} + c_{n-2}$ and $b_4 = b_5 = b_6 = 1$.

From the above relations T_n is expressed only in terms of c_k as follows:

$$T_n = c_n + 2\sum_{k=3}^{n-1} c_k + 2.$$

More precisely we have

Theorem 4.1 *For any integer $m \geq 3$, we have:*

(0) $T_3 = T_4 = 2, T_5 = 6, T_6 = 20, T_7 = T_8 = 30$

(1) $T_9 = c_9 + 30, T_{10} = c_{10} + 2c_9 + 30$

(2) $T_{4m-1} = T_{4m} = 2\sum_{k=2}^{m-1}(c_{4k+1} + c_{4k+2}) + 30$

(3) $T_{4m+1} = c_{4m+1} + 2\sum_{k=2}^{m-1}(c_{4k+1} + c_{4k+2}) + 30$

(4) $T_{4m+2} = c_{4m+2} + 2c_{4m+1} + 2\sum_{k=2}^{m-1}(c_{4k+1} + c_{4k+2}) + 30$. □

For any positive integer m, let $g_m(k)$ and $h_m(k)$ be the cardinal number of $\Gamma_{4m+1}(4m+1,1;k)$ and $\Gamma_{4m+2}(4m,1;k)$ respectively. Then from Theorems 2.4-2.5 we have

$g_1(1) = g_1(3) = 1, g_1(2) = 2$,

$h_1(1) = h_1(4) = 1, h_1(2) = h_1(3) = 4$.

Let $G = G_{4m+1}(4m-1,1;m)$ be the graph given in Lemma 3.5. Then we see that the graphs obtained by the following switching operations $s_{j,k}$ and $s_{m+j,k}$ for G are mutually non-isomorphic:

$s_{j,k} = \{(4m+1,2m),(j,k)\}$, where $j \in [m], k \in [2m+1,3m+1] \cup [3m+3,4m+1-j]$,

$s_{m+j,k} = \{(4m+1,2m),(m+j,k)\}$, where $j \in [m-1], k \in [2m+1,3m-j] \cup \{3m+j+1\}$.

The number of the above switchings is equal to $2m^2 - 1$. So from the switching theorem we have $g_m(m) \geq 2m^2$. Similarly for a representative G of $\Gamma_{4m+1}(4m-1,1;k)$ or $\Gamma_{4m+2}(4m,1;k)$, counting the number of the switching operations for G which induce mutually non-isomorphic graphs, we get a lower estimations for $g_m(k)$ and $h_m(k)$ as follows.

Theorem 4.2 *For any integer $m \geq 2$, we have:*

$$g_m(k) = g_m(4m-k) \geq \begin{cases} 2m^2 & \textit{for} \quad k = m \\ 8m^2 - 6m & \textit{for} \quad k \in [m+1, 2m-1] \\ 8m^2 - 6m + 1 & \textit{for} \quad k = 2m, \end{cases}$$

$$h_m(k) = h_m(4m+1-k) \geq \begin{cases} 2m^2 & \textit{for} \quad k = m \\ 8m^2 - 2m - 1 & \textit{for} \quad k \in [m+1, 2m]. \end{cases}$$

Consequently we have

$c_{4m+1} = \sum_{k=m}^{3m} g_m(k) \geq 16m^3 - 16m^2 + 6m + 1,$

$c_{4m+2} = \sum_{k=m}^{3m+1} h_m(k) \geq 16m^3 - 2m.$ □

Combining with Theorems 4.1 and 4.2 we get a lower estimation of T_n.

Theorem 4.3 *For any integer $m \geq 3$, we have*

(0) $T_9 \geq 107$ *and* $T_{10} \geq 198$

(1) $T_{4m-1} = T_{4m} \geq \sum_{k=2}^{m-1}(32m^3 - 16m^2 + 4m + 1) + 30$

(2) $T_{4m+1} \geq 16m^3 - 16m^2 + 6m + 2\sum_{k=2}^{m-1}(32m^3 - 16m^2 + 4m + 1) + 31$

(3) $T_{4m+2} \geq 48m^3 - 32m^2 + 10m + 2\sum_{k=2}^{m-1}(32m^3 - 16m^2 + 4m + 1) + 32.$ □

Our final goal is to express exactly T_n by means of n, but this is unsolved.

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