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Shape of the Apollonian Packing in the Euclidean Plane

Yukinao ISOKAWA* and Koukichi SAKAI †

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Abstract

We study shapes of curvilinear triangles of the Apollonian packing in the Euclidean plane. For this purpose we introduce an appropriate dynamical system. By simulating this dynamical system numerically, we find that a fractal attractor appears and that this attractor has almost the same shape as that of the original packing. We give a mathematical justification for this finding with recourse to some properties of Möbius transformations.

Key words: Apollonian packing, Fractal attractor, Möbius transformation.

1 Introduction

Let us consider three circles which contact each other and a curvilinear triangle whose sides are made of these circles. Inside the curvilinear triangle, we inscribe an open disk which touches all of these circles. Then we have three new curvilinear triangles. Repeating this procedure indefinitely, we obtain the well-known Apollonian packing of disks (Figure 1). The Apollonian packing has a long history of investigation and nowadays many aspects of it have been revealed (see, *e.g.* [1], [2], [3], and [4]).

In this paper we propose a new type of problem about the Apollonian packing and solve it. We are concerned with "shape" of the packing, neglecting "size" of it. To state more clearly, we are concerned with shapes of curvilinear triangles which are made in the process of the disk packing. Our study is motivated by investigations on sequences of the pedal triangles of a triangle. It is known that these sequences enjoy the ergodic

* Department of Mathematics Education, Faculty of Education, Kagoshima University, Kagoshima 890-0065, Japan.

† Department of Mathematics and Computer Science, Faculty of Science, Kagoshima University, Kagoshima 890-0065, Japan.

property (*cf.* [5], [6]). Namely, beginning from "almost" any triangle, its pedal triangles may have "almost" all shapes. Returning to the Apollonian packing, we ask what shapes the curvilinear triangles may have ?

In order to study this problem, we first carry out a numerical simulation, which displays an emergence of a fractal structure as an attractor of a dynamical system induced by the Apollonian packing. To our surprise, the displayed attractor has "almost" the same shape as that of the original packing. To explain a meaning of this phenomenon briefly, curvilinear triangles of the Apollonian packing may have only exceptional shapes, and the probability that a curvilinear triangle with such an exceptional shape is formed at random is precisely equal to the measure of the residual set in the Apollonian packing, *i.e.*, equal to zero. Our purpose in this paper is to give a precise formulation for the above loose statement and prove it.

2 Formulation of our problem

Let T be a curvilinear triangle of which three sides have curvatures (*i.e.*, inverses of radii) α, β and γ. As we concentrate our attention on its shape , not on its size, we introduce a triple of three non-negative real numbers (x, y, z) where

$$x = \frac{\alpha}{\alpha+\beta+\gamma}, y = \frac{\beta}{\alpha+\beta+\gamma}, \text{ and } z = \frac{\alpha}{\alpha+\beta+\gamma} .$$

Let M be an equilateral triangle. Then, as $x+y+z=1$, a triple (x, y, z) can be regarded as barycentric coordinate of a point in M. Thus the shape of a curvilinear triangle T can be represented by a point in M.

Now we pack an open disk D into T so that it may contact all of three sides of T. Then we get three new curvilinear triangles T_1, T_2 and T_3. As is well-known (*cf.* p.15 of [7]), the curvature σ of D is given by

$$\sigma = \alpha+\beta+\gamma+2\sqrt{\alpha\beta+\beta\gamma+\gamma\alpha} .$$

Accordingly, the shapes of T_i $(i = 1, 2, 3)$ are represented by coordinates (x_i, y_i, z_i) $(i = 1, 2, 3)$, where

$$\begin{cases} (x_1, y_1, z_1) & = \left(\frac{\sigma}{\sigma+\beta+\gamma}, \frac{\beta}{\sigma+\beta+\gamma}, \frac{\gamma}{\sigma+\beta+\gamma}\right) , \\ (x_2, y_2, z_2) & = \left(\frac{\alpha}{\alpha+\sigma+\gamma}, \frac{\sigma}{\alpha+\sigma+\gamma}, \frac{\gamma}{\alpha+\sigma+\gamma}\right) , \\ (x_3, y_3, z_3) & = \left(\frac{\alpha}{\alpha+\beta+\sigma}, \frac{\beta}{\alpha+\beta+\sigma}, \frac{\sigma}{\alpha+\beta+\sigma}\right) . \end{cases}$$

At present we introduce three maps f_1, f_2 and f_3 from M to M by

$$(1) \qquad \begin{cases} f_1(x,y,z) = & \left(\frac{1+2t}{1+y+z+2t}, \frac{y}{1+y+z+2t}, \frac{z}{1+y+z+2t}\right), \\ f_2(x,y,z) = & \left(\frac{x}{1+z+x+2t}, \frac{1+2t}{1+z+x+2t}, \frac{z}{1+z+x+2t}\right), \\ f_3(x,y,z) = & \left(\frac{x}{1+x+y+2t}, \frac{y}{1+x+y+2t}, \frac{1+2t}{1+x+y+2t}\right), \end{cases}$$

where $t = \sqrt{xy + yz + zx}$. We can express the shapes (x_i, y_i, z_i) of T_i as $(x_i, y_i, z_i) = f_i(x, y, z)$ $(i = 1, 2, 3)$.

Returning to the Apollonian packing, as a result of infinite repetitions of the disk packing process, we get a family of curvilinear triangles. The shapes of these curvilinear triangles can be represented by iterated images of an initial point (x, y, z) by f_i $(i = 1, 2, 3)$,

$$f_{i_n} \cdots f_{i_2} f_{i_1}(x, y, z) \ .$$

So we are faced with a dynamical system on M induced by three maps f_1, f_2 and f_3.

Now we can state our problem precisely: what kind of attractor A does this dynamical system have ?, that is, what is the shape of A ? Here a word "attractor" means, for a given (x, y, z), the set of all limit points of appropriate subsequences of iterated images $f_{i_n} \cdots f_{i_2} f_{i_1}(x, y, z)$ as n tends to the infinity.

At this point we perform a numerical simulation to display the attractor A (Figure 2). In this figure we can see a remarkable structure, which, roughly speaking, is the same as the original Apollonian packing. The attractor A seems to consist of a family of closed curves and their limit points although these closed curves are not circles (with only the incircle of M being an exception).

3 Main result

In this section we investigate the structure of the attractor A closely. For this purpose we introduce one more map, which will turn to be a key tool to study the problem. Suppose that the side of an equilateral triangle M has the unit length and the centroid of M lies at the origin of the Euclidean plane. Consider a curvilinear triangle $\mathcal{T}$ which is defined by three circles $\mathcal{K}_i$ $(i = 1, 2, 3)$ with center at K_i and with equal radius $a = (2 + \sqrt{3})/2$ (Figure 3), where

$$K_1 = \left(0, -\frac{2a}{\sqrt{3}}\right), K_2 = \left(a, \frac{a}{\sqrt{3}}\right), \text{ and } K_3 = \left(-a, \frac{a}{\sqrt{3}}\right) .$$

Note that a curvilinear triangle $\mathcal{T}$ circumscribes M. Now we introduce a real-valued function ϕ of a variable R,

$$\rho = \phi(R) = a \cdot \frac{R}{1+\sqrt{1-3R^2}} \; . \tag{2}$$

And define a map Φ from M to $\mathcal{T}$ by

$$(\xi, \eta) = \Phi(X, Y) = \left(\frac{X}{R}\rho, \frac{X}{R}\rho\right) = \left(\frac{X}{R}\phi(R), \frac{X}{R}\phi(R)\right) \; , \tag{3}$$

where both (X, Y) and (ξ, η) have to be understood as Cartesian coordinates and $R = \sqrt{X^2+Y^2}$. Then we can easily verify the following lemma.

Lemma 1 *The map Φ is a homeomorphism from M to $\mathcal{T}$.*

Now we define three transformations g_i $(i = 1, 2, 3)$ in $\mathcal{T}$ by $g_i = \Phi \circ f_i \circ \Phi^{-1}$. The next lemma shows that g_i's are geometrically much simpler than f_i's. To state the lemma, we introduce three circles $\mathcal{L}_i$ $(i = 1, 2, 3)$ with center L_i and radius a, where

$$L_1 = (0, b), L_2 = \left(-\frac{\sqrt{3}}{2}b, -\frac{1}{2}b\right), \text{ and } L_3 = \left(\frac{\sqrt{3}}{2}b, --\frac{1}{2}b\right) \; ,$$

with $b = \frac{3+\sqrt{3}}{3}a$. Furthermore, we introduce three lines l_i $(i = 1, 2, 3)$: a line l_1 which passes through two points K_2 and K_3, a line l_2 which passes through points K_3 and K_1, and a line l_3 which passes through points K_1 and K_2.

Lemma 2 *Every g_i $(i = 1, 2, 3)$ is a Möbius transformation which is the composition of an inversion with respect to circle $\mathcal{L}_i$ and a reflection with respect to axis l_i, with the inversion being performed first and the reflection next.*

Proof We prove the lemma only for g_1, for the assertion for g_2 and g_3 can be established in similar ways. Consider a point P in M with barycentric coordinate (x, y, z), and denote its Cartesian coordinate by (X, Y). Since the three vertices of an equilateral triangle M have Cartesian coordinates $(0, \frac{\sqrt{3}}{3})$, $(-\frac{1}{2}, -\frac{\sqrt{3}}{6})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{6})$, we have

$$\begin{cases} X &= \frac{1}{2}\,(z-y) \\ Y &= \frac{\sqrt{3}}{6}\,(2x-y-z) \; . \end{cases} \tag{4}$$

To put $R = \sqrt{X^2+Y^2}$, it can be easily checked that

$$xy + yz + zx = \frac{1}{3} - R^2 \; .$$

Now we consider an image P_1 of P by the map f_1, and suppose that it has a barycentric coordinate (x_1, y_1, z_1) and a Cartesian coordinate (X_1, Y_1). Then, using (1) and (4), we can deduce

$$\begin{cases} X_1 &= \frac{3X}{5-2\sqrt{3}Y+2\sqrt{3}\sqrt{1-3R^2}} \\ Y_1 &= \frac{\frac{2\sqrt{3}}{3}+Y+2\sqrt{1-3R^2}}{5-2\sqrt{3}Y+2\sqrt{3}\sqrt{1-3R^2}} \end{cases} \tag{5}$$

We put $R_1 = \sqrt{X_1^2 + Y_1^2}$ and $\rho_1 = \phi(R_1)$. Then (5) yields

$$\sqrt{1 - 3R_1^2} = \frac{2\sqrt{3} - 6Y + 3\sqrt{1 - 3R^2}}{5 - 2\sqrt{3}Y + 2\sqrt{3}\sqrt{1 - 3R^2}} ,$$

which in turn, being substituted into (2), gives

$$\frac{\rho_1}{R_1} = a \cdot \frac{5 - 2\sqrt{3}Y + 2\sqrt{3}\sqrt{1 - 3R^2}}{(5 + 2\sqrt{3}) - (6 + 2\sqrt{3})Y + (3 + 2\sqrt{3})\sqrt{1 - 3R^2}} . \tag{6}$$

Furthermore, from (2), we can deduce

$$R = \frac{2a\rho}{a^2 + 3\rho^2} \text{ and } \sqrt{1 - 3R^2} = \frac{a^2 - 3\rho^2}{a^2 + 3\rho^2} . \tag{7}$$

Now let Q and Q_1 be images of P and P_1 by Φ respectively, and suppose that they have Cartesian coordinates (ξ, η) and (ξ_1, η_1) respectively. Then, combining (3), (5), (6), and (7), we obtain

$$\xi_1 = \frac{\rho_1}{R_1} X_1 = \frac{a^2 \xi}{\xi^2 + (\eta - b)^2} . \tag{8}$$

Similarly

$$\eta_1 = \frac{\rho_1}{R_1} Y_1 = \frac{2a}{\sqrt{3}} - \left[\frac{a^2(\eta - b)}{\xi^2 + (\eta - b)^2} + b \right] . \tag{9}$$

Therefore both the expression (8) and (9) establish the assertion of the lemma . □

Let C be the incircle of M, and let us define a family of closed curves

$$C_{i_1 i_2 \cdots i_n} = f_{i_n} \cdots f_{i_2} f_{i_1}(C)$$

for $n = 1, 2, \ldots$ and $i_1, i_2, \cdots \in \{1, 2, 3\}$. A direct calculation shows that all C_i $(i = 1, 2, 3)$ are ellipses, but for $n \geq 2$, closed curves $C_{i_1 i_2 \cdots i_n}$ are so complicated that they seem to be intractable. Thus, instead of $C_{i_1 i_2 \cdots i_n}$, we will consider their images by the map Φ, $\mathcal{C}_{i_1 i_2 \cdots i_n} = \Phi(C_{i_1 i_2 \cdots i_n})$. Then it can be easily verified that $\mathcal{C} = \Phi(C) = C$. Moreover, by the definition of g_i's, we have

$$\begin{aligned} \mathcal{C}_{i_1 i_2 \cdots i_n} &= \Phi \circ (f_{i_n} \cdots f_{i_2} f_{i_1})\,(C) \\ &= (g_{i_n} \cdots g_{i_2} g_{i_1}) \circ \Phi(C) \\ &= (g_{i_n} \cdots g_{i_2} g_{i_1}) \circ \mathcal{C} . \end{aligned}$$

Consequently, since any Möbius transformation transforms circles into circles with some obvious exceptions, all closed curves $\mathcal{C}_{i_1 i_2 \cdots i_n}$ are really circles.

Now let us consider the closure of

$$\bigcup_{n=0}^{\infty} \bigcup_{i_1,i_2,\cdots,i_n \in \{1,2,3\}} \mathcal{C}_{i_1 i_2 \cdots i_n}$$

and denote it by $\mathcal{A}$. Here we adopt a convention that $\mathcal{C}_{i_1 i_2 \cdots i_n}$ indicates $\mathcal{C}$ if the length of indices n equals zero. Following Chapter 18 of [1], we call $\mathcal{A}$ an Apollonian gasket. Let $\mathcal{D}_{i_1 i_2 \cdots i_n}$ be the interior of circle $\mathcal{C}_{i_1 i_2 \cdots i_n}$ and define open curvilinear triangles

$$\mathcal{T}_{i_1 i_2 \cdots i_n} = g_{i_n} \cdots g_{i_2} g_{i_1} \mathcal{T}^o \ ,$$

where $\mathcal{T}^o$ means the interior of $\mathcal{T}$ and the above convention is again adopted. Now we can show the following fact.

Lemma 3

(a)

$$\overline{\mathcal{T}_{i_1 i_2 \cdots i_n}} = \mathcal{D}_{i_1 i_2 \cdots i_n} \cup \left(\bigcup_{j=1}^{3} \overline{\mathcal{T}_{i_1 i_2 \cdots i_n j}} \right)$$

and all pairs among $\mathcal{D}_{i_1 i_2 \cdots i_n}$ *and* $\mathcal{T}_{i_1 i_2 \cdots i_n j}$ $(j = 1, 2, 3)$ *are disjoint.*

(b)

$$\mathcal{A} = \mathcal{T} \backslash \left(\bigcup_{n=0}^{\infty} \bigcup_{i_1,i_2,\cdots,i_n \in \{1,2,3\}} \mathcal{D}_{i_1 i_2 \cdots i_n} \right)$$

(c) *Circles* $\mathcal{C}_{i_1 i_2 \cdots i_n}$ *contact with the circle* $\mathcal{C}$ *if and only if no* i_k *for* $k = 1, 2, \ldots, n-1$ *coincides with* i_n*, that is,* $i_n \cap \{i_1, i_2, \ldots, i_{n-1}\} = \emptyset$.

Proof Since the initial curvilinear triangle consists of the open disk $\mathcal{D}$ and three curvilinear triangles $\mathcal{T}_i$ $(i = 1, 2, 3)$, iterated applications of g_i's establish the assertion (a) immediately.

Now we put

$$\mathcal{A}' = \mathcal{T} \backslash \left(\bigcup_{n=0}^{\infty} \bigcup_{i_1 i_2 \cdots i_n \in \{1,2,3\}} \mathcal{D}_{i_1, i_2, \cdots, i_n} \right) \ .$$

Since it is obvious that all $\mathcal{C}_{i_1 i_2 \cdots i_n} \subset \mathcal{A}'$ and $\mathcal{A}'$ is closed, we have $\mathcal{A} \subset \mathcal{A}'$. To prove the converse, consider any point Q which belongs to $\mathcal{A}'$. Since Q is not contained in any disk $\mathcal{D}_{i_1 i_2 \cdots i_n}$, with the aid of (a), we may choose a sequence $\{i_n : n = 1, 2, \ldots\}$ such that $Q \in \overline{\mathcal{T}_{i_1 i_2 \cdots i_n}}$. Since diameters of $\overline{\mathcal{T}_{i_1 i_2 \cdots i_n}}$ tend to zero as n tends to the infinity, we see that Q belongs to a limit set of

$$\mathcal{C}_{i_1} \cup \mathcal{C}_{i_1 i_2} \cup \mathcal{C}_{i_1 i_2 i_3} \cup \cdots \ .$$

So that the assertion (b) is confirmed.

Finally we will prove (c) only for case that the last index i_n equals 1. For that case it suffices to show that $\mathcal{C}_{i_1 i_2 \cdots i_n}$ contact with the circle $\mathcal{K}_1$ if and only if all i_k $(k = 1, 2, \ldots, n)$ equals either 2 or 3, because $\mathcal{C} = g_1(\mathcal{K}_1)$. We prove "if" part by the induction on n. Suppose the assertion is true for $n-1$, that is, suppose that every $\mathcal{C}_{i_1 i_2 \cdots i_{n-1}}$ with i_k $(k = 1, 2, \ldots, n-1)$ being equal to either 2 or 3 contact with $\mathcal{K}_1$. Then, since $\mathcal{K}_1$ is invariant under both g_2 and g_3, both $\mathcal{C}_{i_1 i_2 \cdots i_{n-1} 2}$ and $\mathcal{C}_{i_1 i_2 \cdots i_{n-1} 3}$ also contact with $\mathcal{K}_1$. Thus the "if" part is shown.

Now we will prove "only if" part of (c). To suppose the contrary, we may assume that there exists a circle $\mathcal{D}_{i_1 i_2 \cdots i_n}$ with some i_k being equal to 1 contacts with $\mathcal{K}_1$. Then, since $\mathcal{D}_{i_1 i_2 \cdots i_k} \subset \mathcal{T}_1$, we have

$$\mathcal{D}_{i_1 i_2 \cdots i_n} \subset \mathcal{T}_{1 i_{k+1} \cdots i_n} \subset \bigcup_{j=1}^{3} \mathcal{T}_{1j} .$$

Thus the circle $\mathcal{D}_{i_1 i_2 \cdots i_n}$ never contacts with $\mathcal{K}_1$, which completes the proof of the lemma. $\square$

Now we synthesize the previous lemmas to obtain the following theorem.

Theorem 1 *The attractor A coincides with the image of the Apollonian gasket $\mathcal{A}$ by the inverse Φ^{-1}, namely, $A = \Phi^{-1}(\mathcal{A})$.*

Proof First we show $A \subset \Phi^{-1}(\mathcal{A})$. Assuming the contrary, we consider a sequence of points $\{P_n : n = 1, 2, \ldots\}$ such that every P_n is an iterated image of a point P_0 by f_i's and it converges to a point $\tilde{P}$ outside $\Phi^{-1}(\mathcal{A})$. Transforming these points by Φ, we have a sequence of points $\{Q_n : n = 1, 2, \ldots\}$ such that every Q_n is an iterated image of a point Q_0 by g_i's and it converges to a point $\tilde{Q}$ outside $\mathcal{A}$. Since the point $\tilde{Q}$ lies outside $\mathcal{A}$, by the property (b) of Lemma 3, it lies in a certain open disk $\mathcal{D}_{i_1 i_2 \cdots i_n}$. Thus, for sufficiently large n, all Q_n lie in the same disk. On the other hand, these points belong to curvilinear triangles which are made by more than n iterated applications of g_i's. This contradicts to the property (a) of Lemma 3. Thus we have $A \subset \Phi^{-1}(\mathcal{A})$.

Next we show $A \supset \Phi^{-1}(\mathcal{A})$. Since the attractor A is closed, it suffices to show that $A \supset C_{i_1 i_2 \cdots i_n}$ for all $C_{i_1 i_2 \cdots i_n}$. Moreover, since A is invariant under f_i's, it is sufficient to prove that $A \supset C$, or equivalently, $\Phi(A) \supset \mathcal{C}$. Now the property (c) of Lemma 3 tells that the circle $\mathcal{C}$ is surrounded by an infinitely many circles $\mathcal{C}_{i_1 i_2 \cdots i_n}$. Moreover it is obvious that diameters of these circles tend to zero as n tends to the infinity. So that any point on $\mathcal{C}$ can be a limit point of these surrounding circles. Thus we have completed the proof of the theorem. $\square$

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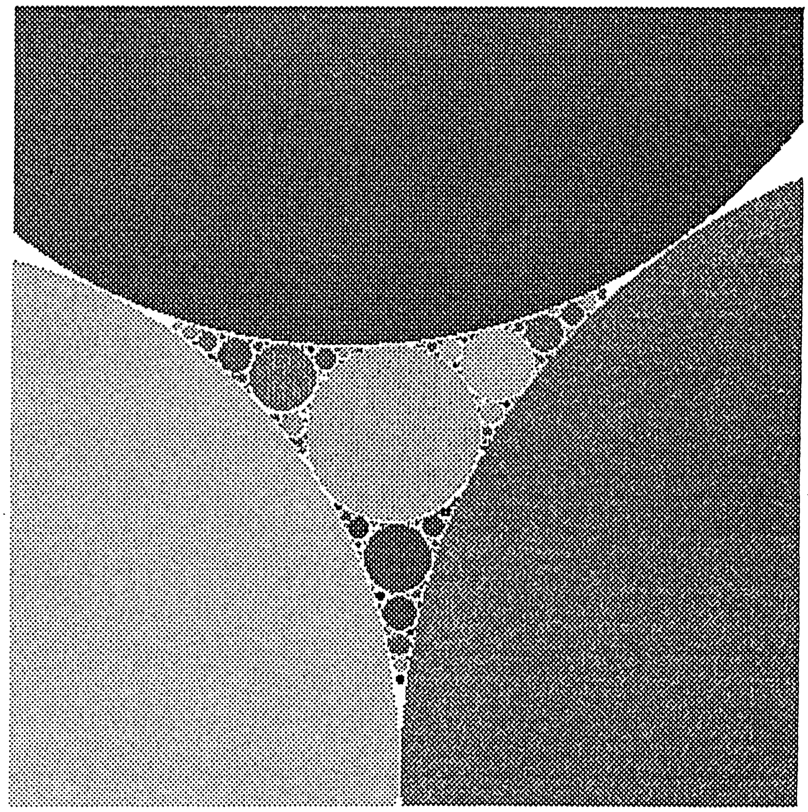

Figure 1

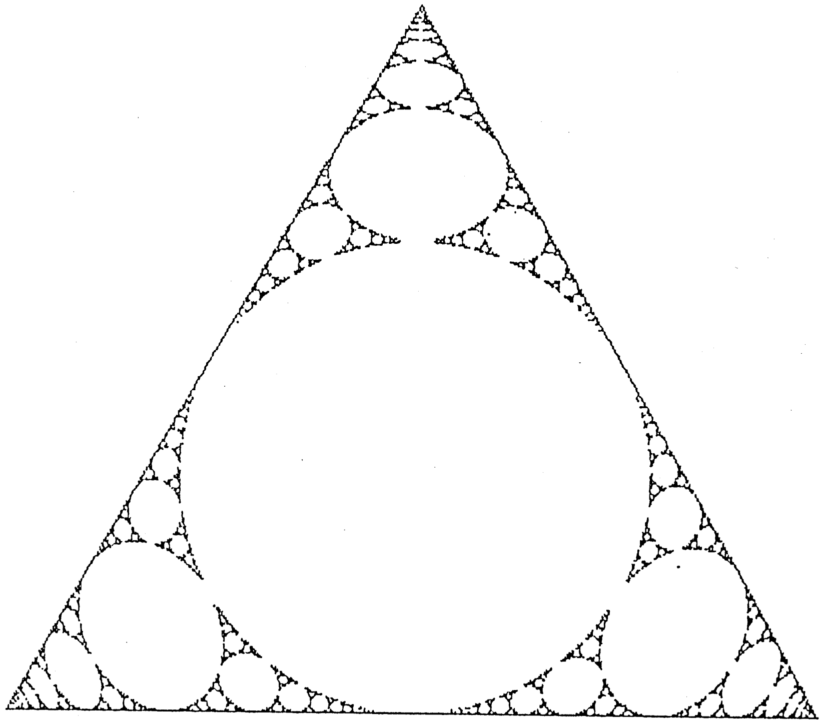

Figure 2

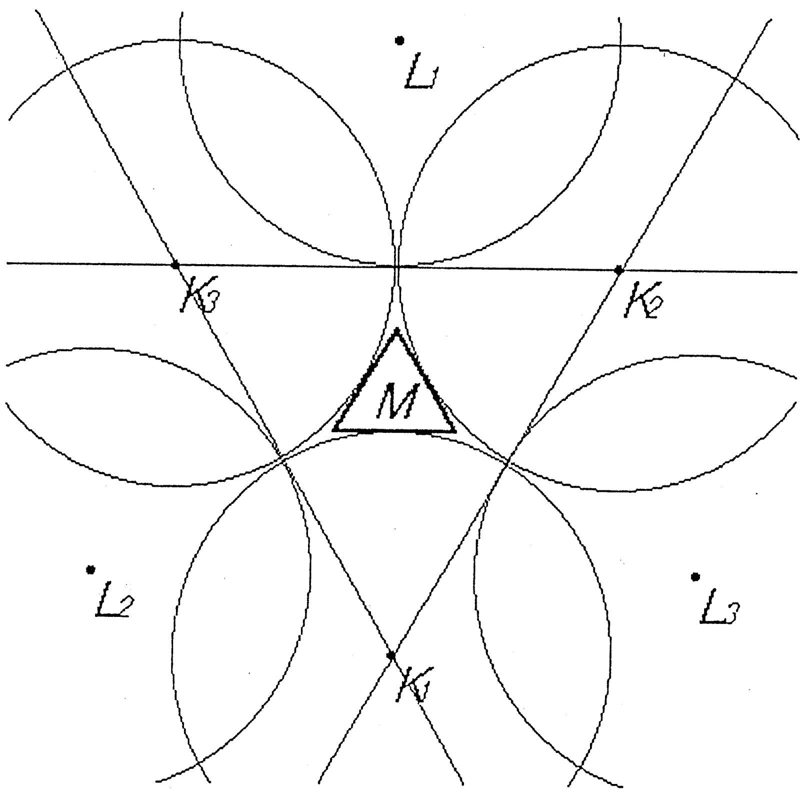

Figure 3