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(Appendix : Corrections to previous author's paper)

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Abstract

As well known (*e.g.* [4]) every graph is isomorphic to the line graph of a hypergraph. In this note, for any graph G with maxclique partition, we shall characterize the hypergraph H whose line graph $L(H)$ is isomorphic to G. We also consider the complete r-partite graphs($r \geq 3$) with maxclique partition.

Key words: graph, hypergraph, line graph, maxclique partition, complete r-partite graph.

1 Graphs with maxclique partition

In this note the terminology and notion concerning graphs and hypergraphs follow Chartrand and Lesniak [2] and Duchet [3] respectively unless otherwise stated. We assume always that any graphs and hypergraphs are finite, simple and connected. Let G be a graph. For any subgraph G' of G we denote by $V(G')$ and $E(G')$ the vertex set and the edge set of G' respectively. For any subset W of $V(G)$, $< W >$ is the subgraph of G induced by W. Any complete subgraph of G is called a *clique*, and especially it is called a *maxclique* if it is not properly contained in another cliques. Let $MC(G)$ be the set of maxcliques of G. A subfamily F of $MC(G)$ is called a *maxclique partition* of G if the family $\{E(Q); Q \in F\}$ is a partition of $E(G)$. In this case we may assume that

(1.1) $|V(Q)| \geq 2$ *for any* $Q \in F$.

Moreover, by contraction of edges, we may assume that

(1.2) *Any* $Q \in F$ *has at most one vertex which does not belong to another members in* F.

For brevity we say that G is an *MCP-graph*, denoted by the pair (G, F), if there exists a maxclique partition F of G with (1.1) and (1.2).

In what follows let (G, F) be an MCP-graph. Then we can define a hypergraph $\Psi(G, F) := (F, \mathbf{E})$ on F, which is called an MP-hypergraph of MCP-graph (G, F) for brevity. Here the hyperedge set $\mathbf{E} = \{\psi(v); v \in V(G)\}$, where $\psi(v)$ is the subset of F

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defined by
(1.3) $\psi(v) = \{Q \in F; v \in V(Q)\}$.
By virtue of (1.2) the map $\psi : V(G) \ni v \mapsto \psi(v) \in \mathbf{E}$ is bijective. So we note that the hyperedge set $\mathbf{E}$ is identified with $V(G)$. For any $Q \in F$ we put
(1.4) $H(Q) := \{\psi(v); Q \in \psi(v)\}$.
Then the next Lemma follows immediately from the fact that F is a maxclique partition of G with (1.1) and (1.2).

Lemma 1.1.

(1.5) *For any distinct* $u, v \in V(G)$, $|\psi(u) \cap \psi(v)| \leq 1$ *and* $|\psi(u) \cap \psi(v)| = 1$ *if and only if* u *and* v *are adjacent,*

(1.6) $\psi(u) \cap \psi(v) = \{Q\}$ *if and only if* $uv \in E(Q)$,

(1.7) *For any* $Q \in F$, *any hyperedge* $\psi(w)$ *belongs to* $H(Q)$ *if* $\psi(w) \cap \psi(v) \neq \emptyset$ *for all* $\psi(v) \in H(Q)$,

(1.8) $|H(Q)| \geq 2$ *for any* $Q \in F$. □

The assertion (1.5) implies that the map ψ is an isomorphism from G to the line graph $\Psi(G) := L(\Psi(G, F))$ of the MP-hypergraph $\Psi(G, F)$. Each $H(Q), Q \in F$, induces a maxclique $< H(Q) >$ of $\Psi(G)$ by (1.7). Moreover the family $\Psi(F) := \{< H(Q) >; Q \in F\}$ is a maxclique partition of $\Psi(G)$ by (1.6). Therefore, under these notation, we get the following

Theorem 1.2. *Let* (G, F) *be an* MCP*-graph.*

(1) $\Psi(G)$ *is an* MCP*-graph with the maxclique partition* $\Psi(F)$,

(2) $(\Psi(G), \Psi(F))$ *is isomorphic to* (G, F). □

2 Characterization of MP-hypergraphs

In this section we shall characterize the hypergraph H whose line graph $L(H)$ becomes an MCP-graph. Let $H = (X, E)$ be a simple and connected hypergraph with finite vertex set X and hyperedge set E. For any $x \in X$ we set
(2.1) $H(x) := \{e \in E; x \in e\}$.
A subfamily E' of E is said to be *intersecting* if $e_1 \cap e_2 \neq \emptyset$ for any $e_1, e_2 \in E'$. An intersecting family E' is said to be *maximal* if it is not properly contained in another intersecting family of E.

Definition 2.1. *Any hypergraph* $H = (X, E)$ *is called an* ML*-hypergraph if it satisfies the following conditions*:

(2.2) *Each hyperedge* e *is nonempty,*

(2.3) $|H(x)| \geq 2$ *for any* $x \in X$,

(2.4) $|e_1 \cap e_2| \leq 1$ *for any distinct* $e_1, e_2 \in E$,

(2.5) $H(x)$ *is a maximal intersecting subfamily of* E *for any* $x \in X$. □

We note that the next condition (2.6) follows from the condition (2.4):
(2.6) $|H(x) \cap H(y)| \leq 1$ *for any distinct* $x, y \in X$.
Obviously the MP-hypergraph $\Psi(G, F)$ of any MCP-graph (G, F) is an ML-hypergraph by Lemma 1.1.

Lemma 2.2. *The line graph* $L(H)$ *of any* ML*-hypergraph* $H = (X, E)$ *is an* MCP*-graph, and the family* $H(X) := \{< H(x) >; x \in X\}$ *is a maxclique partition of* $L(H)$*, where* $< H(x) >$ *is the subgraph of* $L(H)$ *induced by* $H(x)$.

Proof. Since $H(x)$ is an intersecting family in E, $< H(x) >$ is a clique of $L(H)$, and is maximal by (2.5). Let $e_1, e_2 \in E$ be adjacent in $L(H)$. Then by (2.4) there exists an unique $x_0 \in X$ such that $e_1 \cap e_2 = \{x_0\}$. So the edge e_1e_2 in $L(H)$ is in the unique maxclique $< H(x_0) >$. Hence $H(X)$ is a maxclique partition of $L(H)$. For any $x \in X$, $H(x)$ contains at most one singleton and the order of $< H(x) >$ is at least two by (2.3). Thus $H(X)$ satisfies (1.1) and (1.2). This completes the proof. □

Combining Theorem 1.2 and Lemma 2.2 we have

Theorem 2.3. *A hypergraph* $H = (X, E)$ *is an* MP*-hypergraph of any* MCP*-graph* (G, F) *if and only if it is an* ML*-hypergraph. In this case* G *is isomorphic to the line graph of* H. □

For any ML-hypergraph $H = (X, E)$ we consider the Helly condition:
(2.7) *Any intersecting family of* E *is contained in* $H(x)$ *for some* $x \in X$.
If H satisfies (2.7), it is seen easily that $MC(L(H)) = H(X)$. Hence we have

Theorem 2.4. *For any graph* $G, MC(G)$ *is a maxclique partition of* G *if and only if* G *is the line graph of any* ML*-hypergraph with the Helly condition* (2.7). □

3 ML-graphs

Any 2-uniform hypergraph is identified with a simple graph. So any 2-uniform ML-hypergraph is called an ML-graph. For any graph G, the conditions (2.2) and (2.4) hold trivially. The condition (2.3) corresponds to the condition $\delta(G) \geq 2$, where $\delta(G) = \min\{deg(v); v \in V(G)\}$. For any $v \in V(G)$, let $H(v)$ be the set of edges incident to v. Evidently $H(v)$ is a maximal intersecting family if $deg(v) > 2$. On the other hand let $deg(v) = 2$ and $N(v) = \{w, z\}$, where $N(v)$ is the neighborhood of v. Then $H(v)$ is maximal if and only if $wz \notin E(G)$, that is, $< \{v\} \cup N(v) >$ is the path P_3. Consequently we have

Theorem 3.1. *Any graph* G *is an* ML*-graph if and only if it satisfies the following two conditions*:

(1) $\delta(G) \geq 2$,

(2) *For any* $v \in V$ *with* $deg(v) = 2$, $< \{v\} \cup N(v) >$ *is the path* P_3. □

If a graph G with $\delta(G) \geq 2$ contains no triangles, then it is an ML-graph satisfying the Helly condition (2.7). So from Theorem 2.4 we have

Theorem 3.2. *Let G be any graph with $\delta(G) \geq 2$, and $L(G)$ be the line graph of G. If G is triangle-free, then $MC(L(G))$ is a maxclique partition of $L(G)$.* □

4 Complete r-partite graphs with maxclique partition

For any $r \geq 2$, let $G := K(n_1, n_2, \cdots, n_r)$ be the complete r-partite graph with partite sets $V_j, |V_j| = n_j (j = 1, 2, \cdots, r)$. For the case $r = 2$, each edge of G is a maxclique and G has the maxclique partition $\{e; e \in E(G)\}$.

Now assume that $r > 2$ and G has a maxclique partition $F = \{Q_j; j = 1, 2, \cdots, m\}$. We note that each Q_j is of order r. Let $s = \Sigma_{j=1}^{r} n_j$. For any $v \in V(G)$ we put

(4.1) $E_v = \{Q \in F; v \in V(Q)\}$.

Then for every partite set V_j, F is partitioned into the disjoint family $\{E_v; v \in V_j\}$, and $|E_v| = \frac{s-n_j}{r-1}$ for any $v \in V_j$. So we have $n_j(s - n_j) = m(r-1)$ for any $j = 1, 2, \cdots, r$. From these relations we conclude that $n := n_1 = n_2 = \cdots = n_r, m = n^2$, and $|E_v| = n$.

Lemma 4.1. *If $K(n_1, n_2, \cdots, n_r)$ has a maxclique partition $F = \{Q_j; j = 1, 2, \cdots, m\}$, then $n := n_1 = n_2 = \cdots = n_r$ and $m = n^2$.* □

For any fixed positive integers n, r, let us denote by $K(n; r)$ the complete r-partite graph such that each partite set $V_j, j = 1, 2, \cdots, r$, is an n-set. Evidently $K(1; r) = K_r$ and $K(n, 2)$ are MCP-graphs. So in what follows let $n > 1$ and $r > 2$. Suppose $K(n; r)$ has a maxclique partition F. Let $\Psi(K(n; r), F) = (F, \mathbf{E})$ be the MP-hypergraph of $(K(n; r), F)$, where the hyperedge set $\mathbf{E} = \{E_v; v \in V(K(n; r))\}$. For any $Q \in F$ we put

(4.2) $H(Q) := \{E_v; v \in V(Q)\}$.

Then the above discussions are summarized as follows.

Lemma 4.2. *$\Psi(K(n; r), F)$ has the following properties:*

(4.3) $|F| = n^2$,

(4.4) $|H(Q)| = r$ *for any* $Q \in F$,

(4.5) $|E_v| = n$ *for any* $v \in V(K(n; r))$,

(4.6) *For any partite set $V_j, P_j := \{E_v; v \in V_j\}$ is a partition of F and $|P_j| = n$,*

(4.7) *For any distinct partite sets V_j, V_k, $|E_v \cap E_w| = 1$ for any $(v, w) \in V_j \times V_k$,*

(4.8) *For any $Q \in F$ and $E_v \in \mathbf{E}$ with $Q \notin E_v$, there exists an unique $E_w \in H(Q)$ for which $E_v \cap E_w = \emptyset$.* □

Let Q, E_v be as in (4.8). Then there is an unique P_j containing E_v. By (4.6) there exists an unique $E_w \in P_j$ with $Q \in E_w$. Hence we have (4.8).

5 AF-hypergraphs

Let n, r be any fixed integers with $n > 1$ and $r > 2$. We shall characterize the MP-hypergraph of MCP-graph $(K(n; r), F)$.

Definition 5.1. *Any ML-hypergraph $H = (X, E)$ is called an AF-hypergraph, denoted by $H(X, E; n, r)$, if the following three conditions hold:*

(5.1) $|e| = n$ *for any* $e \in E$,

(5.2) $|H(x)| = r$ *for any* $x \in X$,

(5.3) *For any $x \in X$ and $e \in E$ with $x \notin e$, there exists an unique $e_0 \in H(x)$ such that* $e \cap e_0 = \emptyset$. □

In (5.3), any hyperedges in $H(x)$ except e_0 must intersect the hyperedge e. From this fact we have

(5.4) $r \leq n + 1$.

By virtue of (5.3) we can define an equivalence relation $\equiv$ in E as follows:

(5.5) *For any $e_1, e_2 \in E, e_1 \equiv e_2$ if $e_1 = e_2$ or $e_1 \cap e_2 = \emptyset$.*

We denote by $\hat{e}$ the equivalence class containing $e \in E$ and by $\hat{E}$ the quotient set of E with respect to $\equiv$. Then under the these notation the following Lemma is seen easily from (5.1)-(5.3).

Lemma 5.2.

(5.6) *For any $x \in X, H(x)$ is a representative system of $\hat{E}$ and $|\hat{E}| = r$,*

(5.7) *For any $e \in E, \hat{e}$ induces a partition of X, i.e., $X = \cup\{f; f \in \hat{e}\}$, and $|\hat{e}| = n$,*

(5.8) $|X| = n^2$. □

Let $\hat{E} = \{\hat{e_j}; j = 1, 2, \cdots, r\}$. Then for any j, k with $1 \leq j < k \leq r, e \cap f \neq \emptyset$ for any $(e, f) \in \hat{e_j} \times \hat{e_k}$. Therefore the line graph of $H(X, E; n, r)$ is isomorphic to the complete r-partite graph $K(n; r)$, with partite n-sets $\hat{e_j}, j = 1, 2, \cdots, r$. On the other hand, as noted in Lemma 4.2, the MP-hypergraph of $(K(n; r), F)$ is an AF-hypergraph. Hence we have

Theorem 5.3. *Any hypergraph H is an MP-hypergraph of $K(n; r)$ if and only if it is an AF-hypergrap $H(X, E; n, r)$.* □

FromTheorem 5.3 and Theorem 2.2 we have

Theorem 5.4. *The complete r-partite graph $K(n; r)$ has a maxclique partition if and only if there exists an AF-hypergraph $H(X, E; n, r)$.* □

We note that any AF-hypergraph $H(X, E; n, n + 1)$ satisfies the condition:

(5.9) $|H(x) \cap H(y)| = 1$ *for any distinct* $x, y \in X$.

From (5.3) and (5.9), any AF-hypergraph $H(X, E; n, n + 1)$ is identified with the finite Affine plane of order n.

Theorem 5.5. *$K(n; n + 1)$ has a maxclique partition for any $n = p^m$, where p is a prime and m is a positive integer.*

Proof. This follows from Theorem 5.4 and the fact (*e.g.* [1]) that there exists the finite Affine plane $H(X, E; n, n+1)$ of order n for $n = p^m$. □

Remark 5.6. For any integer $n \geq 2$, let p be the least prime divisor of n. Then we can construct an AF-hypergraph $H(X, E; n, p+1)$. Hence $K(n; p+1)$ has a maxclique partition. Especially $K(n; 3)$ has has a maxclique partition for any n, and so has $K(n; 4)$ for any odd n.

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Appendix

There are some errors in the author's paper :
On set representations and intersection numbers of some graphs, Rep. Fac. Sci. Kagoshima Univ., **33**(2000), 39-46.
We correct these errors as follows.

(1) In Lemma 3.2(3) and Theorem 3.3 the equal = must be replaced by $\leq$. So Theorem 3.3 gives an upper estimation of the intersection number of the complete r-partite graph. However this estimation is not so good. For example $i(K(m_1, m_2, m_3)) = m_1 m_2 < m_1(m_2 + m_3 - 1)$.

(2) In Lemma 4.4 the family of maxcliques $\{Q_j; j \in [n-1]\}$ is not $MC(G_n)$ but a minimal maxclique edge cover of G_n, where n is even.

(3) By the above correction, in Theorem 4.5 $i(G_n) = \theta_m(G_n)$ holds only for odd n.