Rep. Fac. Sci. Kagoshima Univ., (Math. Phys. Chem.) No. 2, p. 1-15, 1969

ON FINITE TOPOLOGICAL SPACES II

By

Mitsunobu SHIRAKI

(Received September 30, 1969)

§ 1. Introduction.

In this paper we shall investigate several algebraic properties of topogenous matrices of finite T_0-spaces which we have introduced and studied in our previous paper [1]. In § 2 we shall define an algebra of functions on a finite T_0-space and characterize the topogenous matrix of the space as a certain transformation on this algebra. In § 3 we shall introduce topological invariants which we call the eigen values and the eigen spaces of a finite T_0-space. These invariants seem to be powerful to study the classification problem of finite T_0-spaces. In § 4 we shall give some simple examples.

§ 2. Algebras on finite T_0-spaces.

Let (X, τ) be a finite T_0-space on a set $X=\{a_1, a_2, \cdots, a_n\}$ and U_i be the minimal basic neighborhood of $a_i \in X$.

Then the topology τ of X corresponds to a matrix $A=[a_{ij}]$ such that

$$(1) \qquad \begin{aligned} a_{ij}&=1 \qquad \text{for } a_j \in U_i, \\ a_{ij}&=0 \qquad \text{otherwise,} \end{aligned}$$

which we call the T_0-topogenous matrix of (X, τ).

Now let φ_i be the characteristic function χ_{U_i} of U_i in X, and let ψ_i be the characteristic function χ_{a_i} of $\{a_i\}$ in X. Then we obviously have

$$(2) \qquad \varphi_i=\sum\{\psi_j \mid a_j \in U_i\} \quad (i=1, 2, \cdots, n),$$

and we can note

$$(3) \qquad \varphi_i=\sum a_{ij}\psi_j \quad (i=1, 2, \cdots, n),$$

where

$$\begin{aligned} a_{ij}&=1 \qquad \text{for } a_j \in U_i, \\ a_{ij}&=0 \qquad \text{otherwise.} \end{aligned}$$

Hence let $\varphi=\begin{bmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_n\end{bmatrix}$ and $\psi=\begin{bmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_n\end{bmatrix}$, then we have

$$\varphi = A\psi \tag{4}$$

where A is the topogenous matrix $[a_{ij}]$.

We call $\varphi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix}$ a basis of the space (X, τ).

In particular $\psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_n \end{bmatrix}$ is a basis of the discrete space (X, δ).

Let φ_1 and φ_2 be two bases of the spaces (X, τ_1) and (X, τ_2) respectively. Then we have $\tau_1 = \tau_2$ if and only if φ_2 is a permutation of φ_1.

Next, on the set $\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ we define a binary operation by the multiplication of real function. Then we have clearly

$$\varphi_i \varphi_j = \bigvee \{\varphi_k \mid a_k \in U_i \cap U_j\}, \tag{5}$$

where the symbol $\bigvee$ denotes the supremum. In a basis ψ of the discrete space, the following is evident.

$$\begin{aligned} \psi_i \psi_j &= 0 \qquad \text{if } i \neq j, \\ \psi_i \psi_j &= \psi_i \qquad \text{if } i = j. \end{aligned} \tag{6}$$

LEMMA 1. *Let φ be a basis of a finite T_0-space (X, τ). Then*

$$\varphi_i \varphi_j = \sum \alpha_k \varphi_k, \tag{7}$$

where α_k are integers, and the summands $\alpha_k \varphi_k$ are defined for such k that $a_k \in U_i \cap U_j$.

PROOF. We can find a suitable basis φ of (X, τ) such that $\varphi = A\psi$, where A is a triangular topogenous matrix. Since the diagonal elements of A are 1, we have *det* $|A| = 1$, and the inverse matrix A^{-1} of A is also a triangular matrix whose elements are integers, and we have

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix} = A^{-1} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix}.$$

Therefore ψ_m is described as

$$\psi_m = \sum \{\gamma_p \varphi_p \mid a_p \in U_m\}, \tag{8}$$

where γ_p are integers. On the other hand, $\varphi_i = \sum \{\psi_k \mid a_k \in U_i\}$ and $\varphi_j = \sum \{\psi_l \mid a_l \in U_j\}$ imply

$$\varphi_i \varphi_j = \sum \{\psi_m \mid a_m \in U_i \cap U_j\}. \tag{9}$$

From (8) and (9) we have

(10) $$\varphi_i\varphi_j=\sum\{\alpha_p\varphi_p \mid a_p \in U_i\cap U_j\}.$$

By Lemma 1 we obtain

THEOREM 1. *Let* $\varphi=\begin{bmatrix}\varphi_1\\ \vdots\\ \varphi_n\end{bmatrix}$ *be a basis of a finite* T_0*-space,* $R(\varphi)$ *be the set* $\{\sum\alpha_i\varphi_i \mid \alpha_i :$ *integer*$\}$, *and define algebraic operations in* $R(\varphi)$ *as follows* :

(11) $$\left(\sum_{i=1}^{n}\alpha_i\varphi_i\right)+\left(\sum_{i=1}^{n}\beta_i\varphi_i\right)=\sum_{i=1}^{n}(\alpha_i+\beta_i)\varphi_i.$$

(12) $$\gamma\left(\sum_{i=1}^{n}\alpha_i\varphi_i\right)=\sum_{i=1}^{n}(\gamma\alpha_i)\varphi_i.$$

(13) $$\left(\sum_{i=1}^{n}\alpha_i\varphi_i\right)\left(\sum_{j=1}^{n}\beta_j\varphi_j\right)=\sum(\alpha_i\beta_j)\ (\varphi_i\varphi_j).$$

Then $R(\varphi)$ *is an algebra over the ring* J *of rational integers.*

If φ is a basis of the space (X, τ), then $\varphi=\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ represents simultaneously the basis of the algebra $R(\varphi)$. The correspondence $\psi\rightarrow\varphi=A\psi$ induces a ring isomorphism of the algebra $R(\varphi)$ onto $R(\psi)$.

A continuous mapping of a finite T_0-space to another finite T_0-space induces in a natural manner a homomorphism between the above defined function algebras.

THEOREM 2. *Let* $h=\begin{bmatrix}h_1\\ \vdots\\ h_n\end{bmatrix}$ *and* $g=\begin{bmatrix}g_1\\ \vdots\\ g_m\end{bmatrix}$ *be bases of finite* T_0*-spaces* (X, τ) *and* (Y, σ) *respectively, and let* f *be a continuous mapping of* (X, τ) *into* (Y, σ). *Then* f *induces a homomorphism* $f_* : R(g)\rightarrow R(h)$.

PROOF. Let $X=\{a_1, a_2, \cdots, a_n\}$ and $Y=\{b_1, b_2, \cdots, b_m\}$, and let $\{V_1, V_2, \cdots, V_m\}$ be the minimal basic neighborhood system of (Y, σ).

First, define a mapping f_*: $\{g_1, g_2, \cdots, g_m\}\rightarrow R(h)$ as follows : $f_*(g_i)$ is the characteristic function of $f^{-1}(V_i)$ in X. In an analogous argument which we have used in the proof of Lemma 1, we obtain

$$f_*(g_i)=\sum\{\gamma_k h_k \mid f(a_k) \in V_i\},$$

where γ_k are integers, then $f_*(g_i)$ belongs to $R(h)$ and the mapping f_* is well-defined.

Second, we extend the mapping f_* to a mapping on $R(h)$ which we denote by the same letter f_* as follows :

(14) $$f_*(\sum\alpha_i g_i)=\sum\alpha_i f_*(g_i),$$

where α_i are integers. We shall prove

$$f_*(g_i g_j)=f_*(g_i)\, f_*(g_j).$$

Since $f_*(g_i)=\chi_{f^{-1}(V_i)}$, we have

(15) $$f_*(g_i)\,f_*(g_j)=\chi_{f^{-1}(V_i)}\chi_{f^{-1}(V_j)}=\chi_{f^{-1}(V_i)\cap f^{-1}(V_j)}=\chi_{f^{-1}(V_i\cap V_j)}.$$

From (7) and (14) we have

(16) $$f_*(g_ig_j)=f_*(\sum_l\{\alpha_l g_l\,|\,b_l\in V_i\cap V_j\})=\sum\{\alpha_l\chi_{f^{-1}(V_l)}\,|\,b_l\in V_i\cap V_j\}.$$

On the other hand,

$$\chi_{V_i\cap V_j}=g_ig_j=\sum_l\{\alpha_l g_l\,|\,b_l\in V_i\cap V_j\}=\sum_l\{\alpha_l\chi_{V_l}\,|\,b_l\in V_i\cap V_j\}.$$

Let a_k be an element of $f^{-1}(V_i\cap V_j)$. Then

$$\chi_{V_i\cap V_j}\,(f(a_k))=1,$$

and

(17) $$\sum_l\{\alpha_l\chi_{V_l}(f(a_k))\,|\,b_l\in V_i\cap V_j\}=1.$$

Since $f(a_k)\in V_l$ implies $\chi_{f^{-1}(V_l)}(a_k)=1$, we have

(18) $$\sum_l\{\alpha_l\chi_{f^{-1}(V_l)}(a_k)\,|\,b_l\in V_i\cap V_j\}=1.$$

If $a_k\notin f^{-1}(V_i\cap V_j)$, then in a similar calculation we have

(19) $$\sum_l\{\alpha_l\chi_{f^{-1}(V_l)}(a_k)\,|\,b_l\in V_i\cap V_j\}=0.$$

Therefore,

(20) $$\sum_l\{\alpha_l\chi_{f^{-1}(V_l)}\,|\,b_l\in V_i\cap V_j\}=\chi_{f^{-1}(V_i\cap V_j)}.$$

From (15), (16) and (20), we have

$$f_*(g_ig_j)=f_*(g_i)f_*(g_j),$$

and

$$\begin{aligned}f_*((\sum\alpha_ig_i)(\sum\beta_jg_j))&=f_*(\sum(\alpha_i\beta_j)\,(g_ig_j))\\&=\sum(\alpha_i\beta_j)f_*(g_ig_j)\\&=\sum(\alpha_i\beta_j)f_*(g_i)f_*(g_j)\\&=(\sum\alpha_if_*(g_i))\,(\sum\beta_jf_*(g_j))\\&=f_*(\sum\alpha_ig_i)f_*(\sum\beta_jg_j).\end{aligned}$$

Thus $f_*:R(g)\rightarrow R(h)$ is a ring homomorphism.

LEMMA 2. *Under the condition of Theorem* 2, *let U be any open set of space* (Y, σ). *Then*

$$f_*(\chi_U)=\chi_{f^{-1}(U)}.$$

PROOF. We note χ_U in the form

$$\begin{aligned}\chi_U&=\vee\{g_k \mid b_k \in U\}\\ &=\sum\{\beta_k g_k \mid b_k \in U\}\\ &=\sum\{\beta_k \chi_{V_k} \mid b_k \in U\},\end{aligned}$$

where β_k are integers. Then

$$\begin{aligned}f_*(\chi_U)&=f_*(\sum\{\beta_k g_k \mid b_k \in U\})\\ &=\sum\{\beta_k f_*(g_k) \mid b_k \in U\}\\ &=\sum\{\beta_k \chi_{f^{-1}(V_k)} \mid b_k \in U\}.\end{aligned}$$

Let $a_l \in f^{-1}(U)$, and take a V_k such that $f(a_l) \in V_k \subset U$. Then we have $\chi_{V_k}(f(a_l))=1$, and it follows from $\chi_U(f(a_l))=1$ that

$$\sum_k\{\beta_k \chi_{V_k}(f(a_l)) \mid b_k \in U\}=1.$$

Since $\chi_{f^{-1}(V_k)}(a_l)=\chi_{V_k}(f(a_l))$, we have

$$\sum_k\{\beta_k \chi_{f^{-1}(V_k)}(a_l) \mid b_k \in U\}=1.$$

In a similar way, $a_l \notin f^{-1}(U)$ implies

$$\sum_k\{\beta_k \chi_{f^{-1}(V_k)}(a_l) \mid b_k \in U\}=0.$$

Hence

$$f_*(\chi_U)=\sum_k\{\beta_k \chi_{f^{-1}(V_k)} \mid b_k \in U\}=\chi_{f^{-1}(U)}.$$

THEOREM 3. *Let f be a continuous mapping of a finite* T_0-*space* (X, τ) *into a finite* T_0-*space* (Y, σ), *and let t be a continuous mapping of* (Y, σ) *into a finite* T_0-*space* (Z, η). *Also, let* φ, *h and g be bases of the spaces* (X, τ), (Y, σ) *and* (Z, η) *respectively. Then we have*

$$(t\circ f)_*=f_*\circ t_*.$$

PROOF. Let $g=\begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix}$ and let $\{V_1, V_2, \cdots, V_m\}$ be the minimal basic neighborhood system of (Z, η). Then we need only to prove the following

$$(t\circ f)_*(g_i)=(f_*\circ t_*)\,(g_i)\ (i=1, 2, \cdots, m).$$

From the definition of the induced homomorphism,

$$(t\circ f)_*(g_i)=\chi_{(t\circ f)^{-1}(V_i)},$$

and

$$f_*(t_*(g_i))=f_*(\chi_{t^{-1}(V_i)}).$$

By Lemma 2, we have

$$f_*(\chi_{t^{-1}(V_i)})=\chi_{f^{-1}(t^{-1}(V_i))}.$$

Therefore

$$(t\circ f)_*(g_i)=(f_*\circ t_*)(g_i).$$

THEOREM 4. *Let f be a homeomorphism of a finite T_0-space (X, τ) onto a finite T_0-space (Y, σ). Then the induced homomorphism f_* is an isomorphism.*

PROOF. Let φ and h be bases of the spaces (X, τ) and (Y, σ) respectively. We remark that, if $i: X\to X$ is the identity mapping, the induced homomorphism $i_*: R(\varphi)\to R(\varphi)$ is also the identity automorphism.

If f is the homeomorphism in the Theorem, then $f\circ f^{-1}$ and $f^{-1}\circ f$ are the identity mappings, and from Theorem 3,

$$(f\circ f^{-1})_*=f_*^{-1}\circ f_*, \qquad (f^{-1}\circ f)_*=f_*\circ f_*^{-1}.$$

Then $f_*^{-1}\circ f_*$ and $f_*\circ f_*^{-1}$ are both identity automorphisms. Therefore f_* is an isomorphism.

§ 3. Eigen values in finite T_0-spaces.

In [1] we have defined that two (n, n) matrices A and B are equivalent and noted as $A\sim B$ when there exists a permutation matrix P such that $B=P'AP$.

THEOREM 5. *Let A and B be two topogenous matrices, Then A is equivalent to B if and only if AA' is equivalent to BB'.*

PROOF. Suppose A is equivalent to B. Then by the above definition there exists a permutation matrix P such that $B=P'AP$, and

$$BB'=(P'AP)(P'AP)'=P'APP'AP.$$

Since a permutation matrix is orthogonal, we have $PP'=E$, and

$$BB'=P'(AA')P.$$

Thus

$$BB'\sim AA'.$$

The sufficiency of this theorem follows from the next three lemmas.

LEMMA 3. *Let A and B be two triangular T_0-topogenous matrices. If $AA' = BB'$, then $A=B$.*

PROOF. For two triangular T_0-topogenous matrices $A=[a_{ij}]$ and $B=[b_{ij}]$, suppose $AA'=BB'=[c_{ij}]$. Since A is a triangular T_0-topogenous matrix, A has the following form :

$$a_{ij}=1 \text{ or } 0,$$

$$a_{ii}=1 \ (i=1, 2, \cdots, n),$$

$$a_{ij}=0 \text{ for } i<j.$$

Therefore we have

$$c_{1i}=\sum_{k=1}^{n} a_{1k}a_{ik}=a_{i1}.$$

Similarily,

$$c_{1i}=\sum_{k=1}^{n} b_{1k}b_{ik}=b_{i1},$$

and

$$a_{i1}=b_{i1} \ (i=1, 2, \cdots, n).$$

Then the first column of A is equal to that of B.

Next assume that the j th column of A is equal to the j th column of B for $j=1, 2, \cdots, k-1$. Then for $l\geq k$, we have

$$c_{kl}=\sum_{j=1}^{k-1} a_{kj}a_{lj}+a_{lk},$$

$$c_{kl}=\sum_{j=1}^{k-1} b_{kj}b_{lj}+b_{lk}.$$

Since $a_{kj}=b_{kj}$ and $a_{lj}=b_{lj}$, we have

$$a_{lk}=b_{lk}.$$

If $l<k$, then we also have $a_{lk}=b_{lk}=0$. Hence the k th columns of A and B are equal.

Thus by induction we have $A=B$.

If $A=[a_{ij}]$ is a (n, n) T_0-topogenous matrix, then A determines a finite T_0-topological space (see [1]). In the following we represent the underlying set by $X=\{a_1, a_2, \cdots, a_n\}$, and the corresponding minimal basic neighborhood system by $\boldsymbol{B}=\{U_1, U_2, \cdots, U_n\}$.

LEMMA 4. *Let A be a T_0-topogenous matrix. Then $AA'=[c_{ij}]$ has the following properties.*

(1) *AA' is symmetric and its determinant $|AA'|$ is* 1.

(2) *c_{ij} is the number of elements which are contained in $U_i \cap U_j$, where U_i and U_j are the minimal basic neighborhoods of a_i and a_j respectively.*

PROOF. (1) is obvious.
Let $A=[a_{ij}]$, then we have

$$a_{ik}a_{jk}=1 \Leftrightarrow a_{ik}=a_{jk}=1,$$
$$\Leftrightarrow a_k \in U_i \text{ and } a_k \in U_j.$$

Therefore $c_{ij}=\sum_{k=1}^{n} a_{ik}a_{jk}$ is the number of elements a_k which are contained in $U_i \cap U_j$.

LEMMA 5. *Let A be a triangular T_0-topogenous matrix and B be a non-triangular T_0-topogenous matrix. Then $AA' \neq BB'$.*

PROOF. Assume that $A=[a_{ij}]$ is a triangular T_0-topogenous matrix, and let $p<q$. If $a_{pq}=1$, then $a_{pq}=0$ since A is a triangular matrix. Hence we have

$$a_p \in U_q, \qquad a_q \notin U_p.$$

It follows from Lemma 3 that

$$c_{pq}=c_{pp}<c_{qq}.$$

If $a_{qp}=0$, then $a_{pq}=0$ since A is a triangular matrix. Hence we have

$$a_p \notin U_q, \qquad a_q \notin U_p.$$

It follows that

$$c_{pq}<c_{pp}, \qquad c_{pq}<c_{qq}.$$

Therefore to prove Lemma 4, it suffices to prove that if A is not triangular, then for $AA'=[c_{ij}]$ there exists a pair (p, q), $p<q$, such that $C_{pp}>C_{qq}$ and $C_{pq}=C_{qq}$.
Since A is not triangular, there exists a pair (p, q) such that $p<q$ and

$$a_{pp}=1, \qquad a_{pq}=1,$$
$$a_{qp}=0, \qquad a_{qq}=1,$$

in other words,

$$a_q \in U_p, \qquad a_p \notin U_q.$$

Hence we have

$$U_q \subset U_p, \qquad U_q \neq U_p,$$

it follows from Lemma 3 that

$$c_{pq}=c_{qq}<c_{pp}.$$

Proof of the sufficiency of Theorem 5.

Assume $BB' \sim AA'$. We take a triangular T_0-topogenous matrix C which is equivalent to B. Then we have

$$CC' \sim BB' \sim AA'.$$

Then there is a permutation matrix P such that

$$CC' = P(AA')P = (PAP')(PAP')'.$$

Since C is triangular, by Lemma 5, PAP' must be triangular, and by Lemma 3, we have

$$C = PAP'.$$

Therefore A is equivalent to C and to B.

Now we shall define important topological invariants of a finite T_0-space.

Definition 1. Let A be a topogenous matrix of a finite T_0-space X. Then the characteristic polynomial, the eigen values, the eigen spaces and the eigen vectors of the matrix AA' are said to be the *characteristic polynomial*, the *eigen values*, the *eigen spaces* and the *eigen vectors* of the space X, respectively.

Example. Consider the following finite T_0-space. The set is $X = \{a_1, a_2, a_3\}$, and the family of minimal basic neighborhoods are $U_1 = \{a_1\}$, $U_2 = \{a_2\}$, $U_3 = \{a_1, a_2, a_3\}$. The triangular T_0-topogenous matrix of this space is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Therefore

$$AA' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

The characteristic polynomial $P(x)$ of the space X is

$$P(x) = |xE - AA'| = x^3 - 5x^2 + 5x - 1,$$

and the eigen values of the space X are

$$1, \qquad 2-\sqrt{3}, \qquad 2+\sqrt{3}.$$

The following important theorem is an immediate consequence of the above definition.

THEOREM 6. *A finite T_0-space is characterized completely by two topological invariants, the eigen values and the eigen vectors, of the space.*

THEOREM 7. *The eigen values of a finite T_0-space are positive. If the space has a rational eigen value, it must be* 1.

PROOF. Let A be a topogenous matrix of a finite T_0-space X. Then AA' is a positive Hermitian matrix. Hence its eigen values are positive.

Since $|AA'|=1$, the characteristic polynomial of the space has the form

$$P(x)=x^n-(T_r(AA'))x^{n-1}+\cdots+(-1)^n,$$

where the coefficients are integers. Therefore, if $P(x)$ has a rational root, it must be 1 or -1.

For the product of finite T_0-spaces, we have the following theorem.

THEOREM 8. *For finite T_0-spaces X, Y, let M, N ; $P_1(x)$, $P_2(x)$ and $(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $(\mu_1, \mu_2, \cdots, \mu_m)$ be the topogenous matrices, the characteristic polynomials and the eigen values of X and Y, respectively. And let L and $P(x)$ be the topogenous matrix and the characteristic polynomial of the product space $X\times Y$, respectively. Then*

(1) *LL' is equivalent to the direct product of MM' and NN', that is $LL'\sim(MM')\times(NN')$.*

(2) $P(x)=\Pi\{(x-\lambda_i\mu_j)\,|\,i=1, 2, \cdots, n\,;\, j=1, 2, \cdots, m\}$.

PROOF. First, as we have proved in [1], the topogenous matrix of the product space $X\times Y$ is equivalent to the direct product of the topogenous matrices of X and Y. Hence

$$LL'\sim(M\times N)\,(M\times N)'.$$

Since $(M\times N)\,(M\times N)'=(M\times N)\,(M'\times N')=(MM')\times(NN')$, we have

$$LL'\sim(MM')\times(NN').$$

Next, we consider orthogonal matrices C_1 and C_2 such that

$$MM'=C_1S_1C_1^{-1}, \qquad NN'=C_2S_2C_2^{-1}.$$

S_1 is a diagonal matrix whose diagonal elements $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigen values of MM', and S_2 is a diagonal matrix whose diagonal elements $\mu_1, \mu_2, \cdots, \mu_m$ are eigen values of NN'. Therefore

$$\begin{aligned} MM'\times NN' &= (C_1S_1C_1^{-1})\times(C_2S_2C_2^{-1}) \\ &= (C_1\times C_2)\,(S_1\times S_2)\,(C_1^{-1}\times C_2^{-1}) \\ &= (C_1\times C_2)\,(S_1\times S_2)\,(C_1\times C_2)^{-1}. \end{aligned}$$

Since C_1, C_2 are orthogonal matrices, $C_1\times C_2$ is also orthogonal. And $S_1\times S_2$ is a diagonal matrix whose diagonal elements are $\lambda_1\mu_1$, $\lambda_1\mu_2$, $\cdots$, $\lambda_1\mu_m$, $\cdots$, $\lambda_n\mu_1$, $\lambda_n\mu_2$, $\cdots$, $\lambda_n\mu_m$ which are the eigen values of the product space. Therefore we have

$$P(x)=\Pi\{(x-\lambda_i\mu_j)\,|\,i=1, 2, \cdots, n; j=1, 2, \cdots, m\}.$$

In general, the characteristic polynomials of matrices AB and BA are equal. Especially, so are those of AA' and $A'A$.

From this, it follows that

THEOREM 9. *Any finite T_0-space and its dual space have the same eigen values.*

REMARK. The concept of the eigen values of spaces seems to be powerful to classify finite T_0-spaces. We do not know any different two finite T_0-spaces with the same eigen values except in the case that one is the dual of the other.

Let X be a finite partially ordered set and a be an element of X. Then, by the *length* $l[a]$ of a, we mean the maximum of all the lengths i of the chains $a_0<a_1<\cdots<a_i=a$ in X.

THEOREM 10. *Let X be a finite T_0-space, and assume that there exist distinct two points a_i and a_j of X such that*

(1) $l[a_i]=l[a_j]$.

(2) *If a_k is a point of X such that $a_i\neq a_k\neq a_j$, then $a_k>a_i$ is equivalent to $a_k>a_j$ and also $a_k<a_i$ is equivalent to $a_k<a_j$.*

Then 1 *is an eigen value of X.*

PROOF. Let A be the topogenous matrix of X and let $AA'=[c_{kl}]$. We have already seen that c_{kl} is the number of the points which are contained in the intersection $U_k\cap U_l$ of the minimal basic neighborhoods U_k of a_k and U_l of a_l.

From the condition (2), it is easy to calculate that if $i\neq k\neq j$, then

$$c_{ik}=c_{ki}=c_{kj}=c_{jk},$$

and

$$c_{ii}=c_{jj}.$$

On the other hand clearly we have $c_{ij}\leq c_{ii}$. Also $l[a_i]=l[a_j]$ implies $a_j\notin U_i$. Now if $a_l\in U_j$ and $l\neq j$, then $a_l\leq a_j$, and from the assumption of the theorem we have $a_l\leq a_i$. Therefore $a_l\in U_i$. From this we can prove

$$c_{ij}=c_{ii}-1.$$

From the above discussion, the i th row and the j th row of the matrix $AA'-E$ have the same components. Hence the characteristic polynomial $P(x)=|xE-AA'|$ has an eigen value 1.

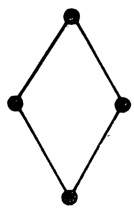
§ 4. Examples.

Finally we shall mention the scheme of all T_0-spaces consisting of four elements, and the associated partially ordered sets, topogenous matrices A, AA' and characteristic polynomials $P(x)$.

(1) 

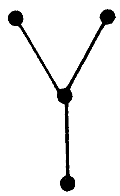
$$A=\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&1&1&1\\1&2&2&2\\1&2&3&3\\1&2&3&4\end{pmatrix}$$

$$\begin{aligned}P_1(x)&=x^4-10x^3+15x^2-7x+1\\&=(x-1)\,(x^3-9x^2+6x-1).\end{aligned}$$

(2) 

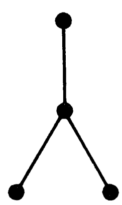
$$A=\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&1&1&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&1&1&1\\1&2&1&2\\1&1&2&2\\1&2&2&4\end{pmatrix}$$

$$\begin{aligned}P_2(x)&=x^4-9x^3+16x^2-9x+1\\&=(x-1)^2(x^2-7x+1).\end{aligned}$$

(3) 

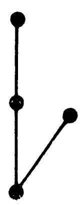
$$A=\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&0&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&1&1&1\\1&2&2&2\\1&2&3&2\\1&2&2&3\end{pmatrix}$$

$$\begin{aligned}P_3(x)&=x^4-9x^3+14x^2-7x+1\\&=(x-1)\,(x^3-8x^2+6x-1).\end{aligned}$$

(4) 

$$A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&3&3\\1&1&3&4\end{pmatrix}$$

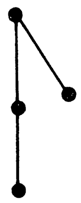
$$P_4(x)=P_3(x).$$

(5) 

$$A=\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&0&1&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&1&1&1\\1&2&1&1\\1&1&2&2\\1&1&2&3\end{pmatrix}$$

$$P_5(x)=x^4-8x^3+14x^2-7x+1.$$

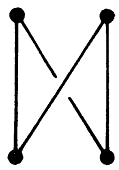
(6)



$$A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\1&1&1&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&0&1&1\\0&1&0&1\\1&0&2&2\\1&1&2&4\end{pmatrix}$$

$P_6(x)=P_5(x)$.

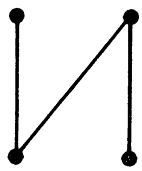
(7)



$$A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\1&1&0&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&3&2\\1&1&2&3\end{pmatrix}$$

$$P_7(x)=x^4-8x^3+14x^2-8x+1$$
$$=(x-1)^2(x^2-6x+1).$$

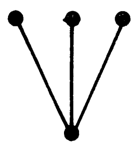
(8)



$$A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\1&1&0&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&0&1&1\\0&1&0&1\\1&0&2&1\\1&1&1&3\end{pmatrix}$$

$P_8(x)=x^4-7x^3+13x^2-7x+1$.

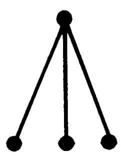
(9)



$$A=\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&0&0&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&1&1&1\\1&2&1&1\\1&1&2&1\\1&1&1&2\end{pmatrix}$$

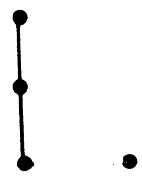
$$P_9(x)=x^4-7x^3+12x^2-7x+1$$
$$=(x-1)\,(x^2-5x+1).$$

(10)



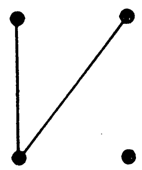
$$A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&1&1&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\\1&1&1&4\end{pmatrix}$$

$P_{10}(x)=P_9(x)$.

(11) 

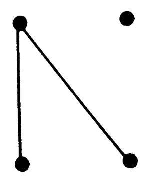
$$A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&1&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&0&0&0\\0&1&1&1\\0&1&2&2\\0&1&2&3\end{pmatrix}$$

$$P_{11}(x)=x^4-7x^3+11x^2-6x+1$$
$$=(x-1)\,(x^3-6x^2+5x-1).$$

(12) 

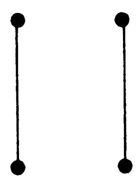
$$A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&0&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&0&0&0\\0&1&1&1\\0&1&2&1\\0&1&1&2\end{pmatrix}$$

$$P_{12}(x)=x^4-6x^3+10x^2-6x+1$$
$$=(x-1)^2(x^2-4x+1).$$

(13) 

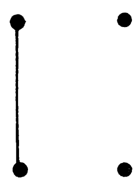
$$A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1&1&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&1\\0&1&1&3\end{pmatrix}$$

$$P_{13}(x)=P_{12}(x).$$

(14) 

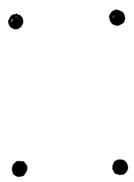
$$A=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&1&0&0\\1&2&0&0\\0&0&1&1\\0&0&1&2\end{pmatrix}$$

$$P_{14}(x)=x^4-6x^3+11x^2-6x+1$$
$$=(x^2-3x+1)^2$$

(15) 

$$A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&1&2\end{pmatrix}$$

$$P_{15}(x)=x^4-5x^3+8x^2-5x+1$$
$$=(x-1)^2(x^2-3x+1).$$

(16) 

$$A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\qquad AA'=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$\begin{aligned}P_{16}(x)&=x^4-4x^3+6x^2-4x+1\\&=(x-1)^4.\end{aligned}$$

Reference

[1] M. Shiraki : On finite topological spaces. Reports of the Faculty of Science Kagoshima Univ. No. **1** (1968) 1-8.