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# FINITE $T_0$-SPACES AND SIMPLICIAL STRUCTURES

By

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## § 1. Introduction.

We consider the problem to represent faithfully a given finite $T_0$-space by a set of simplexes. To represent the space one may naturally consider the nerve of the minimal basic neighborhood system of the finite $T_0$-space. But such the nerve representation is not sufficient to characterize the following simple spaces. In a given set $X=\{a_1, a_2, a_3\}$ we consider two topologies whose minimal basic neighborhoods are

$$\boldsymbol{B}_1=\{\{a_1\}, \{a_1, a_2\}, \{a_1, a_2, a_3\}\},$$

$$\boldsymbol{B}_2=\{\{a_1\}, \{a_1, a_2\}, \{a_1, a_3\}\}.$$

Then these topological spaces have the same nerve representations, that is, there exists the same simplicial complex consisting of a triangle and all faces of the triangle.

On the other hand, M. C. McCord [1] has had some interesting results for homology and homotopy properties of finite spaces, and for a given finite $T_0$-space he has constructed a simplicial complex taking each totally ordered subset as a simplex, where he has used that the finite $T_0$-space is equivalent to a partially ordered set.

However this simplicial complex is not sufficient to characterize the given finite $T_0$-space. For instance, in a set $X=\{a_1, a_2, a_3, a_4\}$, we consider two distinct topologies whose minimal basic neighborhood systems are

$$\boldsymbol{B}_1=\{\{a_1\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_2, a_3, a_4\}\},$$

$$\boldsymbol{B}_2=\{\{a_1\}, \{a_2\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_3, a_4\}\}.$$

Then these spaces have the same McCord's simplicial complexes.

In this note, we shall introduce the concept of a partially simplicial complex which consists of open simplexes. Such a complex characterizes completely a finite $T_0$-space.

## § 2. Partially simplicial complexes.

Two (geometric) open simplexes $\sigma_1$ and $\sigma_2$ in the Euclidean space $R^m$ are said to be *properly joined* if

$$\bar{\sigma}_1 \cap \bar{\sigma}_2 = \bar{\sigma}_3,$$

where $\bar{\sigma}_i$ is the closure of $\sigma_i$, and $\sigma_3$ is the common face of $\sigma_1$ and $\sigma_2$.

DEFINITION 1. A set $K$ of (geometric) open simplexes is said to be a *quasi simplicial complex* if any two simplexes of $K$ are properly joined.

DEFINITION 2. Let $K_1$ and $K_2$ be two quasi simplicial complexes. Then a mapping $\varphi: K_1 \to K_2$ is said to be *quasi simplicial* if

$$\sigma < \tau \Rightarrow \varphi(\sigma) < \varphi(\tau) \qquad \text{for} \quad \sigma, \tau \in K,$$

where the symbol $<$ is the face relation.

DEFINITION 3. Let $K$ be a star-finite quasi complex. Then we consider a space which is homeomorphic to the subspace $\cup\{\sigma \mid \sigma \in K\}$ of the Euclidean space. Such a space is denoted by $|K|$, and is called the *quasi polytope* of $K$.

DEFINITION 4. A finite quasi simplicial complex $K$ is said to be a *n-partially simplicial complex* if

(1) If $\sigma$, $\tau \in K$ and $\sigma \cap \tau$ is a $k$-simplex, then the set $\{\rho \in K \mid \rho < \sigma \cap \tau\}$ has $k+1$ elements.

(2) $\{v^0 \mid v^0 < \tau, \tau \in K\}$ has $n$ elements, where $v^0$ is a 0-simplex.

EXAMPLE. We set $\sigma^0 = <a_1>$, $\sigma^1_1 = <a_1 a_2>$, $\sigma^1_2 = <a_1 a_3>$, $\sigma^2 = <a_1 a_2 a_3 a_4>$. Then

$$K = \{\sigma^0, \sigma^1_1, \sigma^1_2, \sigma^2\}$$

is a 4-partially simplicial complex.

DEFINITION 5. Let $K_1$ and $K_2$ be two partially simplicial complexes. Then a mapping $f: K_1 \to K_2$ is said to be an *isomorphism* if the following are satisfied:

(1) $f$ is bijective.

(2) $f$ and $f^{-1}$ are quasi simplicial.

If such an isomorphism exists between $K_1$ and $K_2$, then $K_1$ and $K_2$ are said to be *isomrophic*, and we denote by $K_1 \approx K_2$.

The following is evident.

THEOREM 1. *Let $K_1$ and $K_2$ be two partially simplicial complexes, then*

$$K_1 \approx K_2 \Rightarrow |K_1| \simeq |K_2|.$$

## § 3. Simplicial presentation.

Let $(X, \mathcal{U})$ be a finite topological space. We define an order relation $\leq$ in $X$ by saying $x \leq y$ when $U_x \subset U_y$ (where $U_x$ and $U_y$ are minimal basic neighborhoods of $x$ and $y$ respectively). Then $(X, \leq)$ is a quasi ordered set.

We have mentioned the following lemmas in [2].

LEMMA 1. *A finite topological space $(X, \mathfrak{U})$ is a $T_0$-space if and only if $(X, \leq)$ is a partially ordered set.*

LEMMA 2. *Suppose $(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ are finite $T_0$-spaces with the associated partially ordered sets $(X, \leq)$ and $(Y, \leq)$ respectively. Then a mapping $f$: $(X, \mathfrak{U}) \to (Y, \mathfrak{V})$ is continuous if and only if $f$: $(X, \leq) \to (Y, \leq)$ is order-preserving, $f$ is a homeomorphism if and only if $f$ is an order-isomorphism of $(X, \leq)$ onto $(Y, \leq)$.*

We shall now verify the following theorem.

THEOREM 2. *Let $(X, \tau)$ be a finite $T_0$-space. Then there exists a partially simplicial complex $K$ with the following properties:*

(1) *There exists a bijective correspondence that assigns to each point of $X$ a simplex of $K$.*

(2) *$K$ is a topological invariant of $(X, \tau)$.*

*Conversely, for each partially simplicial complex $K$ there exists a finite $T_0$-space $(K, \mathfrak{U})$, whose induced partially simplicial complex $L$ is isomorphic to $K$. If $K$ and $L$ are isomorphic partially simplicial complexes, then the corresponding $T_0$-spaces $(K, \mathfrak{U})$ and $(L, \mathfrak{V})$ are homeomorphic.*

PROOF. Let $(X, \tau)$ be a finite $T_0$-space such that $X = \{a_1, a_2, \cdots, a_n\}$ and let $(X, \leq)$ be a partially ordered set which is induced from $(X, \tau)$. We consider the $(n-1)$-simplex $\sigma^{n-1} = \langle a_1 a_2 \cdots a_n \rangle$, and denote the closure of the simplex $\sigma^{n-1}$ by $K(\sigma^{n-1})$.

We define a mapping $g$: $X \to K(\sigma^{n-1})$ as follows: For $a_i \in X$, let $\sigma_i$ be the face of $\sigma^{n-1}$ whose vertices are $\{a_k \in X \mid a_k \geq a_i\}$. Then we put

$$g(a_i) = \sigma_i \in K(\sigma^{n-1}).$$

Setting $K = g(X)$, we shall verify that $K$ is a $n$-partially simplicial complex, i.e., $K$ satisfies (1) and (2) of Definition 4.

From the above definition we immediately find that $K$ is a quasi simplicial complex.

We shall show that $K$ satisfies (1) of Definition 4. Suppose that $\sigma_i, \sigma_j \in K$ and $\rho = \sigma_i \cap \sigma_j$, where $\sigma_i = \langle a_{i_1} \cdots a_{i_h} \rangle$, $\sigma_j = \langle a_{j_1} \cdots a_{j_l} \rangle$, $\rho = \langle a_{r_1} \cdots a_{r_k} \rangle$. Then $\{a_{r_1}, \cdots, a_{r_k}\} = \{a_{i_1}, \cdots, a_{i_h}\} \cap \{a_{j_1}, \cdots, a_{j_l}\}$. Now set

$$\sigma_q = g(a_q) \qquad (q = r_1, \cdots, r_k).$$

If $a_m$ is a vertix of $\sigma_q$, then we have $a_q \leq a_m$. Since $a_q$ is a vertex of $\sigma_i$, we have $a_i \leq a_q$. Thus we have $a_i \leq a_m$. Hence $a_m$ is a vertex of $\sigma_i$. An analogous argument shows that $a_m$ is a vertex of $\sigma_j$. So $a_m$ is a vertex of $\rho = \sigma_i \cap \sigma_j$. Thus

$$\sigma_q < \rho \qquad (q = r_1, \cdots, r_k).$$

If $r_t \neq r_s$, then since $a_{r_t} \neq a_{r_s}$, from the definition of $\sigma_i$ we have either $a_{r_t} \not< \sigma_{r_s}$ or $a_{r_s} \not< \sigma_{r_t}$. So $\sigma_{r_t} \neq \sigma_{r_s}$. Hence $\rho$ has distinct $k$ faces $\sigma_{r_1}, \sigma_{r_2}, \cdots, \sigma_{r_k}$.

Now, from the construction of $K$ it is clear that $K$ satisfies (2) of Definition 4.

We remark that $a_i \leq a_j$ holds in $(X, \leq)$ if and only if $g(a_i) > g(a_j)$ holds in $K$.

We have to show that $K$ is a topological invariant. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two homeomorphic finite $T_0$-spaces, and let $K$ and $L$ be two partially simplicial complexes corresponding to $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ as above respectively. We show that $K \approx L$.

Consider the partially ordered sets $(X, \leq)$ and $(Y, \leq)$ associated with $(X, \mathcal{U})$ and $(Y, \mathcal{V})$, respectively. Then by Lemma 2, there exists a bijective mapping $f\colon (X, \leq) \to (Y, \leq)$ such that $x_1 \leq x_2 \Leftrightarrow f(x_1) \leq f(x_2)$. Then we have

$$g(x_1) > g(x_2) \Leftrightarrow g(f(x_1)) > g(f(x_2)).$$

This implies that $K \approx L$.

We shall prove the latter assertions of Theorem 2. Let $K$ be a partially simplicial complex, and define an ordering $\leq$ between elements of $K$ as follows:

$$\sigma \leq \tau \Leftrightarrow \sigma > \tau, \qquad \text{for} \quad \sigma, \tau \in K.$$

Then $(K, \leq)$ is a partially ordered set whose ordering defines a $T_0$-topology $\mathcal{U}$ of $K$. Let $L$ be the partially simplicial complex which is defined by $(K, \mathcal{U})$. As we have already verified, there exists a bijection $g\colon (K, \mathcal{U}) \to L$ such that

$$\sigma > \tau \Leftrightarrow \sigma \leq \tau \Leftrightarrow g(\sigma) > g(\tau).$$

Hence we have $K \approx L$.

Finally, if $h\colon K \to L$ is an isomorphism of a partially simplicial complex $K$ to a partially simplicial complex $L$, then for $\sigma, \tau \in K$,

$$\sigma > \tau \Leftrightarrow h(\sigma) > h(\tau),$$

and

$$\sigma \leq \tau \Leftrightarrow h(\sigma) \leq h(\tau).$$

Therefore $(K, \mathcal{U})$ and $(L, \mathcal{V})$ are homeomorphic.

Thus the proof of Theorem 2 is complete.

DEFINITION 6. Let $(X, \tau)$ be a finite $T_0$-space, whose associated partial order is denoted by $\leq$. A partially simplicial complex $K$ is said to be a *simplicial presentation* of $(X, \tau)$ if there exists a mapping $f\colon K \to X$ such that

(1) $f$ is bijective.

(2) $\sigma < \tau \Leftrightarrow f(\sigma) \geq f(\tau)$.

The mapping $f$ is called a *simplicial presentation mapping*.

From this definition, $f=g^{-1}: K\rightarrow X$ in the proof of Theorem 2 is a simplicial presentation mapping, and $K$ is a simplicial presentation of $X$.

For the following two lemmas we reffer McCord [1].

LEMMA 3. *Let $P$ be a continuous mapping of a topological space $X$ to another topological space $Y$, and $\mathcal{U}$ be a basis-like open cover of $Y$ satisfying the following condition: for each $U\in\mathcal{U}$, the restriction $P|P^{-1}(U): P^{-1}(U)\rightarrow U$ is a weak homotopy equivalence. Then $P$ itself is a weak homotopy equivalence.*

LEMMA 4. *Let $X$ be a finite $T_0$-space, and let $U_i$ be the minimal basic neighborhood of $a_i$, and let $\boldsymbol{B}$ be the minimal basic neighborhood system. Then*

(1) *$\boldsymbol{B}$ is a basis-like open cover of $X$.*

(2) *$U_i$ is contractible to a point $a_i$.*

We now obtain the following lemma.

LEMMA 5. *Let $(X, \tau)$ be a finite $T_0$-space, $f: K\rightarrow X$ be a simplicial presentation mapping, and $K$ be a simplicial presentation of $(X, \tau)$. Then, for each $a_i\in X$, $|f^{-1}(U_i)|$ is a contractible open set of $|K|$, where $U_i$ is the minimal basic neighborhood of $a_i$.*

PROOF. For each $a_i\in X$, $a_i$ is the maximum element of $U_i$, and $\sigma_i=f^{-1}(a_i)$ is an open simplex such that $\{a_k\in X|a_k\geq a_i\}$ are its vertices. Since we have $a_j\in U_i\Leftrightarrow a_j\leq a_i\Leftrightarrow f^{-1}(a_j)>f^{-1}(a_i)$, $f^{-1}(a_i)$ is the common face of all simplexes $f^{-1}(a_j)$ such that $a_j\in U_i$. $T_i=\{a_k\in X|\ \exists\, a_j\leq a_i,\ a_k\geq a_j\}$ is the set of vertices of $f^{-1}(U_i)$ and $V_i=\{a_k\in X|a_k\geq a_i\}$ is the set of vertices of $f^{-1}(a_i)$, so we have $T_i\supset V_i$.

Now, suppose that

$$x=\sum\{x_k a_k|a_k\in T_i\}$$

is the barycentric coordinates of $x$ with respect to $T_i$. Set

$$\alpha(x)=\sum\{x_k|a_k\in V_i\}$$

and

$$\varphi(x)=\sum\left\{\frac{1}{\alpha(x)}x_k a_k|a_k\in V_i\right\}.$$

Then $\varphi(x)\in f^{-1}(a_i)\ (=\sigma_i)$. If $x\in\sigma_i$, then $\alpha(x)=1$ and $\varphi(x)=x$. Hence $\varphi: |f^{-1}(U_i)|\rightarrow|\sigma_i|$ is a retraction.

Next, we define $H: |f^{-1}(U_i)|\times I\rightarrow|f^{-1}(U_i)|$ by

$$H(x, t)=(1-t)x+t\varphi(x).$$

Since $x\in|f^{-1}(U_i)|$, there is a $a_k\in U_i$ such that $x\in|f^{-1}(a_k)|$. Then from $x\in\sigma_k$ and $\sigma_i<\sigma_k$, we have $H(x, t)\in|f^{-1}(U_i)|$. And

$$H(x, 0)=x,$$

$$H(x, 1)=\varphi(x),$$

$$H(x, t)=x \qquad \text{if} \quad x \in |\sigma_i|.$$

Thus $|\sigma_i|$ is the strong deformation retract of $|f^{-1}(U_i)|$. Since the open simplex $\sigma_i$ is contractible to its barycenter $b(\sigma_i)$, the proof of Lemma 3 is complete.

Let $X$ be a finite $T_0$-space, $K$ be a simplicial presentation of $X$, and let $f\colon K\rightarrow X$ be the presentation mapping. We use the same symbol $f$ to represent the following mapping $f\colon |K|\rightarrow X$. For $x \in |K|$, there is an unique simplex $\sigma_i \in K$ such that $x \in |\sigma_i|$. Then we set

$$f(x)=f(\sigma_i).$$

$f$ is continuous. To show this, let $f(x)=f(\sigma_i)=a_i$, and let $W$ be any open neighborhood of $a_i$. If $U_i$ is the minimal neighborhood of $a_i$, then $U_i\subset W$. Now the open star of $\sigma_i$:

$$\mathrm{St}(\sigma_i)=\cup\{\sigma_k \in K\,|\,\sigma_k>\sigma_i\}$$

is an open set of $|K|$, hence it is an open neighborhood of $x$. If $y \in \mathrm{St}(\sigma_i)$, there exists an unique open simplex $\sigma_k$ such that $y \in \sigma_k$ and $\sigma_k>\sigma_i$. Then

$$f(y)=f(\sigma_k), \qquad f(\sigma_k)\leq f(\sigma_i)=a_i.$$

Hence $f(y)\leq a_i$, that is, $f(y) \in U_i\subset W$. Thus

$$f(\mathrm{St}(\sigma_i)) \subset W.$$

The next theorem follows immediately from Lemmas 3, 4, and 5.

THEOREM 3. *Let $X$ be a finite $T_0$-space, and let $f\colon K\rightarrow X$ be the simplicial presentation mapping of the simplicial presentation $K$ of $X$ to $X$, then $f$ induces the weak homotopy equivalence $f\colon |K|\rightarrow X$.*

LEMMA 6. *If $K$ is a quasi complex, then*

$$N=\{<b(\sigma_0)b(\sigma_1)\cdots b(\sigma_k)>|\,\sigma_0<\sigma_1<\cdots<\sigma_k,\ \sigma_i \in K\}$$

*is a simplicial complex, where $b(\sigma_k)$ is the barycenter of $\sigma_k$.*

PROOF. From the definition of $N$, it is clear that $s>\tau$ and $s \in N$ imply $\tau \in N$, and for $s_1$, $s_2 \in N$ we have

$$s_1=<b(\sigma_1)b(\sigma_2)\cdots b(\sigma_i)>,\ \sigma_1<\sigma_2<\cdots<\sigma_i,\ \sigma_k \in N,$$

$$s_2=<b(\tau_1)b(\tau_2)\cdots b(\tau_j)>,\ \tau_1<\tau_2<\cdots<\tau_j,\ \tau_k \in N.$$

Also, if $\eta_1<\eta_2<\cdots<\eta_l$ and $\{\sigma_1, \sigma_2, \cdots, \sigma_i\}\cap\{\tau_1, \tau_2, \cdots, \tau_j\}=\{\eta_1, \eta_2, \cdots, \eta_l\}$, then

$<b(\eta_1)b(\eta_2)\cdots b(\eta_l)> \in N$. Since any two simplexes of $K$ are properly joined,

$$s_1 \cap s_2 = <b(\eta_1)b(\eta_2)\cdots b(\eta_l)> \in N.$$

Hence $s_1 \cap s_2$ is the common face of $s_1$ and $s_2$. Thus $N$ is a simplicial complex.

$N$ is said to be the *nucleus* of the quasi simplicial complex $K$.

Let $K$ be a quasi simplicial complex. We consider the simplicial complex

$$\mathrm{Cl}\, K = \{s \mid \exists\, \sigma \in K : s < \sigma\}$$

which is induced by $K$, and let $\mathrm{Cl}\, K_1$ be the first barycentric subdivision of $\mathrm{Cl}\, K$. Then

$$K_1 = \{\sigma \in \mathrm{Cl}\, K_1 \mid \sigma \subset |K|\}$$

is called the *first barycentric subdivision* of $K$.

Lemma 7. *Let $K$ be a partially simplicial complex, and let $N$ be the nucleus of $K$. Then $|N|$ is a strong deformation retract of $|K|$.*

Proof. Let $K_1$ be the first barycentric subdivision of $K$, and let

$$B = \{b_i \mid b_i = b(\sigma_i), \quad i = 1, 2, \cdots, n\}$$

be the set of vertices of $N$. Then

$$\mathrm{St}(b_i) = \cup \{\sigma \in K_1 \mid b_i < \sigma\}$$

is defined as the *open star* of $b_i$ in $K_1$.

First, we remark that

$$|K| = \cup \{\mathrm{St}(b_i) \mid b_i \in B\}.$$

For, if $x \in |K|$, then since $|K| = |K_1|$, there exist unique simplexes $\sigma \in K$ and $\sigma_1 \in K_1$ such that $x \in \sigma$ and $x \in \sigma_1$. Let $b(\sigma) = b_i \in B$. Since $K_1$ is the first barycentric subdivision of $K$, and since $\sigma_1 \subset \sigma$, we have $\sigma_1 \subset \mathrm{St}(b_i)$ and $x \in \sigma_1 \subset \mathrm{St}(b_i) \subset \cup \{\mathrm{St}(b_i) \mid b_i \in B\}$. Hence $|K| = \cup \{\mathrm{St}(b_i) \mid b_i \in B\}$.

Second, we remark that $N$ is a full subcomplex of $K_1$. For, if $<b_1 b_2 \cdots b_p> \in K_1$ ($b_i \in B$), then $\sigma_1 < \sigma_2 < \cdots < \sigma_p$ and $b_i = b(\sigma_i)$. Hence $<b_1 b_2 \cdots b_p> \in N$ follows from the definition of $N$.

We shall now prove that $|N|$ is a strong deformation retract of $|K|$.

Let $x \in |K| = |K_1|$. There exists an unique open simplex $\sigma_1 \in K_1$ such that $x \in \sigma_1$. Then we set

$$\alpha(x) = \sum \{x(b_i) \mid b_i \in B\},$$

where $x(b_i)$ is the barycentric coordinate of $b_i$, and define

$$r(x) = \sum \left\{ \frac{1}{\alpha(x)} x(b_i) b_i \mid b_i \in B \right\}.$$

Then $r(x)$ belongs to a face $\tau$ of $\sigma_1$. Since vertices of $\tau$ belong to $B$, remarking that $N$ is a full subcomplex of $K_1$, we have $\tau \in N$ and $r(x) \in |N|$. On the other hand, if $x \in |N|$, then since $\alpha(x)=1$ we have $r(x)=x$. Thus $r: |K| \to |N|$ is a retraction. Next, we define a mapping $H: |K| \times I \to |K|$ by

$$H(x, t)=t \cdot r(x)+(1-t)x.$$

Then we have

$$H(x, 0)=x \qquad \text{for} \quad t=0,$$

$$H(x, 1)=r(x) \qquad \text{for} \quad t=1,$$

and

$$H(x, t)=x \qquad \text{for} \quad x \in |N|.$$

Hence $|N|$ is a strong deformation retract of $|K|$.

LEMMA 8. *Let $f: K \to L$ be a quasi simplicial mapping of a quasi simplicial complex $K$ to a quasi simplicial complex $L$. Then $f$ induces a simplicial mapping $f^*$ of the nucleus $N(K)$ of $K$ to the nucleus $N(L)$ of $L$.*

PROOF. $f^*: N(K) \to N(L)$ is defined by

$$f^*(b(\sigma_i))=b(f(\sigma_i)).$$

If $<b(\sigma_0)b(\sigma_1)\cdots b(\sigma_k)> \in N(K)$, where $\sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_k$, $\sigma_i \in K$, then

$$f^*(<b(\sigma_0)b(\sigma_1)\ldots b(\sigma_k)>)=\{b(f(\sigma_0), b(f(\sigma_1)), \cdots, b(f(\sigma_k))\}.$$

Since $f$ is quasi simplicial, we have $f(\sigma_0) \prec f(\sigma_1) \prec \cdots \prec f(\sigma_k)$. The simplex with vertices $\{b(f(\sigma_0)), b(f(\sigma_1)), \cdots, b(f(\sigma_k))\}$ is in $N(L)$. Thus $f^*$ is a simplicial mapping.

Let $K$ and $L$ be quasi simplicial complexes, and $g$ be a single valued transformation of the vertices of simplex of $K$ to the vertices of simplex of $L$. We call that $g$ is a *simplicial mapping* of $K$ to $L$, when for every simplex $\sigma=<a_1 a_2 \cdots a_p>$ of $K$, $<\{g(a_1), g(a_2), \cdots, g(a_p)\}>$ is a simplex of $L$.

By barycentric extension, we can extend this mapping $g$ to a continuous mapping, and again we call it the simplicial mapping of the quasi polytope $|K|$ to the quasi polytope $|L|$.

THEOREM 4. *Let $g$ be a continuous mapping of a finite $T_0$-space $X$ to a finite $T_0$-space $Y$, and let $f_X: K(X) \to X$ and $f_Y: K(Y) \to Y$ be two presentation mappings. If $K_1(X)$ and $K_1(Y)$ are the first barycentric subdivisions of $K(X)$ and $K(Y)$ respectively, then $g$ induces a simplicial mapping $g_1$ of $K_1(X)$ to $K_1(Y)$, and $f_Y \circ g_1 = g \circ f_X$ holds.*

PROOF. For $a_i \in X$, we set

$$S_i=\{b_k \in Y \mid b_k \geq g(a_i)\}.$$

Since $g$ is continuous, $a_j \geq a_i$ implies $S_j \subset S_i$. Let

$$T_i=\{b_k \in S_i \mid a_j > a_i \Rightarrow b_k \notin S_j\},$$

and define a mapping $g_1: K_1(X) \to K_1(Y)$ as follows: let $c$ be a vertex of a simplex in $K_1(X)$, and let

$$c=b(< a_{i_1}a_{i_2} \cdots a_{i_k} >),$$

where $a_{i_l} \in X (l=1, 2, \cdots, k)$. The simplex $<a_{i_1}a_{i_2} \cdots a_{i_k}>$ is a face of a certain simplex $f_X^{-1}(a_p) \in K(X)$. Then we have $a_{i_l} \geq a_p$ $(l=1, 2, \cdots, k)$. Since $g$ is continuous, we have $g(a_{i_l}) \geq g(a_p)$. Hence the simplex $< T_{i_l}>$ which is determined by $T_{i_l}$ is a face of $f_Y^{-1}(g(a_p))$. Therefore we have

$$<\cup\{T_{i_l} \mid l=1, 2, \cdots, k\}> \prec f_Y^{-1}(g(a_p)).$$

We now define $g_1$ by setting

$$g_1(c)=b(<\cup\{T_{i_l} \mid l=1, 2, \cdots, k\}>).$$

We show that $g_1$ is simplicial. Let $<c_1c_2 \cdots c_k> \in K_1(X)$, and let

$$c_i=b(\sigma_i) \qquad (i=1, 2, \cdots, k),$$

$$\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_k, \qquad \sigma_i=<a_{i_1}a_{i_2} \cdots a_{i_l}>.$$

Then

$$g_1(c_i)=b(\tau_i),$$

where $\tau_i=<\cup\{T_{i_h} \mid h=1, 2, \cdots, l\}>$, so

$$\sigma_i \prec \sigma_j \Rightarrow \tau_i \prec \tau_j.$$

Thus

$$< g_1(c_1) g_1(c_2) \cdots g_1(c_k)> \in K_1(Y).$$

Therefore $g_1$ is a simplicial mapping.

Finally we show that $f_Y \circ g_1 = g \circ f_X$. Suppose $x \in |K_1(X)|$. There exists an unique open simplex $\sigma \in K_1(X)$ such that $x \in \sigma$. From the definition of the first barycentric subdivision, $\sigma=<c_1c_2 \cdots c_p>$ is such that $c_i=b(\tau_i)$, $\tau_1 \prec \tau_2 \prec \cdots \prec \tau_p$, $\tau_p \in K(X)$, $\tau_p = f_X^{-1}(a_p)$. Since $g_1(c_p)=g_1(b(f_X^{-1}(a_p)))=b(f_Y^{-1}(g(a_p)))$,

$$g_1(x) \in <\{g_1(c_1), g_1(c_2), \cdots, g_1(c_p)\}> \subset f_Y^{-1}(g(a_p)).$$

Hence we have

$$f_Y(g_1(x))=f_Y(f_Y^{-1}(g(a_p)))=g(a_p).$$

On the other hand, since $<c_1c_2\cdots c_p>\subset\tau_p$, we have

$$f_X(x)=f_X(\sigma)=f_X(\tau_p)=a_p.$$

Then we have $g\circ f_X(x)=g(a_p)$, and $f_Y\circ g_1=g\circ f_X$.

THEOREM 5. *Let $g$ be a continuous mapping of a finite $T_0$-space $X$ to a finite $T_0$-space $Y$. Then $g$ induces a simplicial mapping $g_N$ of the nucleus $N(X)$ of $K(X)$ to the nucleus $N(Y)$ of $K(Y)$, and the following diagram is commutative, where $f_X$, $f_Y$, $\psi_X$, $\psi_Y$ are weak homotopy equivalences, and $r_X$, $r_Y$ are strong deformation retracts.*

$$\begin{tikzcd}
|K(X)| \arrow[dr, "r_X"] \arrow[ddr, "f_X"'] & & & |K(Y)| \arrow[dl, "r_Y"'] \arrow[ddl, "f_Y"] \\
& |N(X)| \arrow[r, "g_N"] \arrow[d, "\psi_X"] & |N(Y)| \arrow[d, "\psi_Y"'] & \\
& X \arrow[r, "g"'] & Y &
\end{tikzcd}$$

PROOF. Let $f_X\colon K(X)\to X$ and $f_Y\colon K(Y)\to Y$ be simplicial presentation mappings of $X$ and $Y$ respectively, and define $g_0\colon K(X)\to K(Y)$ by

$$g_0=f_Y^{-1}\circ g\circ f_X.$$

$g_0$ is a quasi simplicial mapping. Indeed, if $\sigma_i>\sigma_j$, then $f_X(\sigma_i)\leq f_X(\sigma_j)$. Since $g$ is a continuous mapping and hence an order-preserving mapping, we have $g\circ f_X(\sigma_i)\leq g\circ f_X(\sigma_j)$, and

$$f_Y^{-1}\circ g\circ f_X(\sigma_i)>f_Y^{-1}\circ g\circ f_X(\sigma_j).$$

Hence $g_0$ is a quasi simplicial mapping.

By Lemma 8, $g_0$ induces a simplicial mapping $g_N\colon N(X)\to N(Y)$. We define $\psi_X\colon |N(X)|\to X$ as follows: For each $x\in|N(X)|$ there is an unique open simplex $<b(\sigma_0)b(\sigma_1)\cdots b(\sigma_k)>\in N(K)$, $\sigma_0<\sigma_1<\cdots<\sigma_k$, $\sigma_i\in K(X)$ such that $x\in<b(\sigma_0)b(\sigma_1)\cdots b(\sigma_k)>$. Then we define

$$\psi_X(x)=f_X(\sigma_k).$$

Let $r_X$ be the retraction defined in the proof of Lemma 7. Then from the definition of $\psi_X$,

$$\psi_X\circ r_X=f_X.$$

Since $|N(X)|$ is a strong deformation retract of $|K(X)|$ and $f_X$ is a weak homotopy equivalence, $\psi_X$ is a weak homotopy equivalence. Now for each $x\in|N(X)|$ there is an unique open simplex $<b(\sigma_0)b(\sigma_1)\cdots b(\sigma_k)>\ni x$, $\sigma_0<\sigma_1<\cdots<\sigma_k$. Since $g_N(<b(\sigma_0)b(\sigma_1)\cdots b(\sigma_k)>)=<b(g_0(\sigma_0))b(g_0(\sigma_1))\cdots b(g_0(\sigma_k))>$,

$$g_N(x) \in \langle b(g_0(\sigma_0))b(g_0(\sigma_1))\cdots b(g_0(\sigma_k))\rangle.$$

Then we have

$$\psi_Y(g_N(x))=f_Y(g_0(\sigma_k))=g\circ f_X(\sigma_k).$$

Hence from $g\circ\psi_X(x)=g\circ f_X(\sigma_k)$, we have

$$\psi_Y\circ g_N=g\circ f_X.$$

## § 4. Partially simplicial complexes and induced finite $T_0$-spaces.

In Theorem 2, for every partially simplicial complex $K$, we have considered an equivalent finite $T_0$-space $(K, \mathcal{U})$ which has been constructed in the following way. For each $\sigma \in K$, put

$$V_\sigma=\{\tau \in K\,|\,\tau>\sigma\}.$$

Then $(K, \mathcal{U})$ is a finite $T_0$-space such that $V_\sigma$ is the minimal basic neighborhood of $\sigma$.

The following Lemma 9 and Theorem 6 are easily found.

Let $K$ be a partially simplicial complex, and $(K, \mathcal{U})$ be the corresponding finite $T_0$-space. Then clearly the identity mapping $i\colon K\to(K, \mathcal{U})$ is the simplicial presentation mapping. Hence we have the following lemma.

LEMMA 9. *The identity mapping $i\colon K\to(K, \mathcal{U})$ induces a weak homotopy equivalence $i$: $|K|\to(K, \mathcal{U})$.*

THEOREM 6. *Let $g$: $K\to L$ be a quasi simplicial mapping of a partially simplicial complex $K$ to a partially simplicial complex $L$. If $f_K$: $|K|\to(K, \mathcal{U})$ and $f_L$: $|L|\to(L, \mathcal{V})$ are two simplicial presentation mappings, then $g$ is a continuous mapping of $(K, \mathcal{U})$ to $(L, \mathcal{V})$, and $g$ has the following properties:*

(1) *let $K_1$ and $L_1$ be the first barycentric subdivisions of $K$ and $L$ respectively. Then $g$ induces a simplicial mapping $g_1$: $|K_1|\to|L_1|$ such that $f_L\circ g_1=g\circ f_K$.*

(2) *Let $N(K)$ and $N(L)$ be the nucleuses of $K$ and $L$ respectively, and let $\psi_K=f_K|N(K)$ and $\psi_L=f_L|N(L)$. Then $g$ induces a simplicial mapping $g_N$: $|N(K)|\to|N(L)|$ such that $\psi_L\circ g_N=g\circ\psi_K$.*

PROOF. If $\sigma$, $\tau \in (K, \mathcal{U})$ and $\sigma\leq\tau$, then we have $\sigma>\tau$ in $K$. Since $g$ is quasi simplicial, we have $g(\sigma)>g(\tau)$ in $L$ and $g(\sigma)\leq g(\tau)$ in $(L, \mathcal{V})$. Then by Lemma 2, $g$: $(K, \mathcal{U})\to(L, \mathcal{V})$ is continuous.

Thus (1) and (2) follow immediately from Theorem 4 and Theorem 5.

THEOREM 7. *Let $K$ be a $n$-partially simplicial complex, and let $\alpha_i$ be the number of $i$-simplexes of $K$. Then we have*

(1) $\alpha_0+\alpha_1+\cdots+\alpha_{n-1}=n.$

(2) $1 \leq \alpha_0 \leq n$.

(3) $0 \leq \alpha_i \leq n-i, \qquad (i=1, 2, \cdots, n-1)$.

PROOF. (1). We need to prove that a $n$-partially simplicial complex $K$ has just $n$ elements. Let $(K, \mathcal{U})$ be the finite $T_0$-space corresponding to $K$, and $L$ be the simplicial presentation of $(K, \mathcal{U})$ which has been constructed in the proof of Theorem 2, that is,

$$L = \{s(\sigma) \,|\, \sigma \in K\}$$

where $s(\sigma)$ is a simplex with vertices $\{\tau \in K \,|\, \tau > \sigma\}$. From the proof of Theorem 2, the number of vertices $\{\sigma \,|\, \sigma \in K\}$ of $L$ is equal to the number of simplexes of $L$. Then by the second assertion of Theorem 2, we have $K \approx L$. So $L$ has $n$ vertices. Therefore $K$ has $n$ elements.

(2). From (1), we have $\alpha_0 \leq n$. We prove that $\alpha_0 \geq 1$. Suppose $\sigma_0 \in K$ and $\sigma_0$ is a $k_0$-simplex. When $k_0=0$, we have $\alpha_0 \geq 1$. When $k_0>0$, from (1) of Definition 4, $\sigma_0$ has $k_0$ proper faces in $K$. We take such a face $\sigma_1$, and let $k_1$ be the dimension of $\sigma_1$. When $k_1=0$, we have $\alpha_0 \geq 1$. When $k_1>0$, $\sigma_1$ has $k_1$ proper faces in $K$. We repeat a similar process and find that $K$ has at least a 0-simplex. Thus (2) holds.

(3). Assume that $\alpha_i \geq n-i+1$ for some $i$. Any $i$-simplex in $K$ has $i$ proper faces in $K$. Hence the number of all simplexes in $K$ are not less than

$$(n-i+1)+i=n+1.$$

This contradicts (1).

## Reference

[1] M. C. McCORD: Singular homology groups and homotopy groups of finite topological spaces. Duke Math. J. Vol. **33**, (1966) 465-474.

[2] M. SHIRAKI: On finite topological spaces. Reports of the Faculty of Science Kagoshima Univ. No. **1** (1968) 1-8.