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The purpose of this note is to show that any compact Hausdorff space is represented as the inverse limit space of an inverse limit system in which each space is a compact subspace of a finite dimensional Euclidean cube.

Let X be a compact Hausdroff space and let $I=[0, 1]$ be the closed interval with the usual topology. Suppose that $C(X)=\{\varphi_\mu: \mu \in M\}$ is the family of all continuous mappings from X to I. Now we consider a family $D=\{\{\varphi_\mu\}_{\mu\in\alpha}: \alpha$ is a finite subset of $M\}$. And we define an order relation $<$ in D by saying $f<g$, where $f=\{\varphi_\mu\}_{\mu\in\alpha}$ and $g=\{\varphi_\mu\}_{\mu\in\beta}$, if $\alpha\subset\beta$. Then $(D, <)$ is a directed set. Because, given $f=\{\varphi_\mu\}_{\mu\in\alpha}\in D$ and $g=\{\varphi_\mu\}_{\mu\in\beta}\in D$, we take $h=\{\varphi_\mu\}_{\mu\in\alpha\cup\beta}$. Of course h is in D (Such the h is denoted by $f\vee g$). Then we have obviously $h>f$ and $h>g$, hence $(D, <)$ is a directed set.

For each $\varphi_\mu\in C(X)$, setting $H_\mu=\varphi_\mu(X)$, H_μ is a compact subset of I. And for each $f=\{\varphi_\mu\}_{\mu\in\alpha}\in D$, we define a mapping $f: X\to\Pi\{H_\mu: \mu\in\alpha\}$ by

$$f(x)=\{\varphi_\mu(x) : \mu\in\alpha\},$$

and set $X_f=f(X)$. Then f is continuous and X_f is a compact subspace of a finite dimensional Euclidean space.

Next, we consider the family $\{X_f: f\in D\}$. For each $f, g\in D$ with $f<g$, a mapping π_{fg}: $X_g\to X_f$ is defined by

$$\pi_{fg}g(x) = f(x).$$

Then π_{fg} has the following properties:

(1) π_{fg} is well defined.

(2) π_{fg} is continuous onto.

(3) π_{ff} is identity.

(4) if $f<g<h$, then $\pi_{fg}\pi_{gh}=\pi_{fh}$.

In fact, suppose $f<g$, where $f=\{\varphi_\mu\}_{\mu\in\alpha}$ and $g=\{\varphi_\mu\}_{\mu\in\beta}$. If $g(x)=g(y)$, then $\varphi_\mu(x)=\varphi_\mu(y)$ for $\mu\in\beta$. Since $f<g$, we have $\alpha\subset\beta$, so that $\varphi_\mu(x)=\varphi_\mu(y)$ for $\mu\in\alpha$. Hence $f(x)=f(y)$, and we have (1). (2) is evident since π_{fg} is a projection of the product space onto its factor space. (3) and (4) follow immediately from the definition of the mapping π_{fg}. Therefore we can conclude that the family $\{X_f, \pi_{fg}\}$ is an inverse limit system over the directed set D.

Moreover, since X_f is a non empty compact Hausdorff space, the inverse limit space

X_∞ of the inverse limit system $\{X_f, \pi_{fg}\}$ is non empty compact Hausdroff [1].

Now, the evalution mapping $e: X \to \Pi\{X_f \colon f \in D\}$ is continuous [2]. And $e(x) \in X_\infty$ since $e(x) = \{f(x) : f \in D\}$ and $\pi_{hf} f(x) = h(x)$ whenever $h < f$.

The mapping e is injective. To prove this, suppose that x and y are two distinct points of X. Since X is a compact Hausdorff space, there exists a mapping $\varphi_\nu \in C(X)$ such that

$$\varphi_\nu(x) = 0 \text{ and } \varphi_\nu(y) = 1.$$

Take $h = \{\varphi_\mu\}$ consisting of only one element φ_μ. Then h is a member of D and $h(x) \neq h(y)$. Thus $e(x) \neq e(y)$, so that e is injective.

Next, we shall show that $e(X)$ is dense in X_∞. For this, it is sufficient to prove that every open neighborhood of any point of X_∞ contains a point of $e(X)$. Let $\{x_f\} \in X_\infty$, and suppose that $\Pi\{U_f \colon f \in D\}$ is an arbitrary open neighborhood of $\{x_f\}$, where each U_f is an open neighborhood of x_f in X_f, and $U_f = X_f$ for all but a finite number of $f \in D$. Let the finite elements of D be $\{g, \cdots, h\}$, and take $\psi = g \vee \cdots \vee h$. Then there exists a $y \in X$ such that $x_\psi = \psi(y) \in X_\psi$, since $x_\psi \in X_\psi$ and $X_\psi = \psi(X)$. When considering $\{f(y) \colon f \in D\} \in X_\infty$,

$$\pi_{g\psi}\psi(y) = g(y), \cdots, \quad \pi_{h\psi}\psi(y) = h(y),$$

and since $\psi(y) = x_\psi$ and for $l < \psi$ $\pi_{l\psi} x_\psi = x_l$, we have

$$g(y) = x_g, \cdots, \ h(y) = x_h.$$

It follows that $\{f(y)\} \in \Pi\{U_f \colon f \in D\}$. This proves that $e(X)^- = X_\infty$.

Since X is a compact Hausdorff space and e is a continuous mapping, $e(X)$ also is a compact Hausdroff space. Moreover since X_∞ is Hausdroff, $e(X)$ is closed in X_∞. Consequentely,

$$e(X) = e(X)^- = X_\infty,$$

and therefore e is homemorphism.

Thus we have established the following theorem.

THEOREM. *Every compact Hausdroff space is homeomorphic to the inverse limit space of an inverse limit system in which each space is a compact subspace of a finite dimensional Euclidean space.*

References

[1] S. EILENBERG and N. STEENROD: Foundations of Algebraic Topology. Princeton University Press, Princeton, 1952.

[2] J.L. KELLEY: General Topology. Van Nostrand, Princeton, 1955.