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1. Summary and Introduction

The purpose of this paper is to construct an infinite class of 2–designs which includes Alltop's designs [1] as a special case in a sense. We follow Alltop's method for constructing 2–designs. So, to complete the argument, we quote Alltop's method of constructing t–designs.

For a finite set Ω let $\sum_k(\Omega)$ denote the class of k–subsets of Ω where k–subset means any set of k elements of Ω. By a t–(v, k, λ) design we mean a pair $(\Omega, \mathscr{D})$, where $|\Omega|=v$, $\mathscr{D}\subset\sum_k(\Omega)$ and each t–subset of Ω is contained in exactly λ members of $\mathscr{D}$. S_v, the group of the permutations of Ω, acts in a natural way on $\sum_k(\Omega)$, $\sum_j(\sum_k(\Omega))$, etc. Alltop's method for constructing t–designs is to start with a group G, acting on Ω, and let $\mathscr{D}$ be any proper orbit under G in $\sum_k(\Omega)$. G decomposes $\sum_t(\Omega)$ into m orbits $Q_1, \cdots\cdots, \frac{m}{Q_m}$. By the t–proportionality vector of G on Ω we mean the m–tuple $(|Q_1|, \cdots\cdots, |Q_m|)$. Now let Δ be any member of $\sum_k(\Omega)$, $t<k<v-1$. We call $(u_1, \cdots\cdots, u_m)$ the t–proportionality vector of Δ, where u_i is the number of members of Q_i contained in Δ. Let $\mathscr{D}$ be the orbit of Δ under G. The number λ_i of members of $\mathscr{D}$ containing any $\Gamma_i\in Q_i$ is $u_i|\mathscr{D}|/|Q_i|$. If $\lambda_1=\lambda_2=\cdots\cdots=\lambda_m$, then $(\Omega, \mathscr{D})$ is a t–(v, k, λ).

2. 2–designs from a wreath product group

Let $v=mn$ and $\Omega=\{\alpha_1, \cdots, \alpha_n, \beta_1 \cdots \beta_n, \gamma_1\cdots\cdot \gamma_n, \cdots\cdot\}$, and let $G=S_m \wr S_n$. Although G is not doubly transitive on Ω, G decomposes $\sum_2(\Omega)$ into only 2 orbits. Let Q_1 be the orbit of $\{\alpha_i, \beta_j\}$ and Q_2 the orbit of $\{\alpha_i, \beta_i\}$. The 2–proportionality vector of G on Ω is $\left(\binom{mn}{2}-\binom{m}{2}n, \binom{m}{2}n\right)$. For $ms\leqq k<n$ let

$$\Delta=\{\alpha_1, \cdots, \alpha_{k-(m-1)s}, \beta_1 \cdots \beta_s, \gamma_1 \cdots \gamma_s, \cdots\}.$$

For Δ, $$u_1=\binom{k}{2}s-\binom{m}{2}s \quad \text{and} \quad u_2=\binom{m}{2}s.$$

The orbit of Δ under G yields a 2–design if

$$\frac{u_2}{u_1}=\frac{\binom{m}{2}n}{\binom{mn}{2}-\binom{m}{2}n}. \tag{1}$$

(1) will hold provided $ms(v-1)=k(k-1)$.
The number of blocks will be

$$b=m^{k-ms}\binom{n}{s}\binom{n-s}{k-ms}$$

and

$$\lambda=\frac{b\binom{m}{2}s}{\binom{m}{2}n}=\frac{bs}{n}.$$

Thus we have the following theorem.

THEOREM. *If* $v-1\left|\frac{k(k-1)}{m}\right.$, $m|v$ *and* $\frac{k(k-1)}{v-1}\leqq k<\frac{v}{m}$, *then* 2–(v, k, λ) *design exists such that* $\lambda=\frac{bs}{n}$,

$$b=m^{k-ms}\binom{n}{s}\binom{n-s}{k-ms} \quad \textit{and} \quad n=\frac{v}{m}.$$

REMARK. In the above theorem if we put $m=2$, then 2–(v, k, λ) design reduces to Alltop's design in a sense.

For example, $k=12$, $m=3$, $v=45$, $s=1$ and $\lambda=3^9\cdot\binom{14}{5}$ satisfy the conditions of the theorem. So 2–$(45, 12, \lambda)$ design exists where $\lambda=3^9\cdot\binom{14}{5}$.

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Reference

[1] W.O. Alltop, Some 3-designs and a 4-design, J. Comb. Theory **11** (1971), 190–195.