SPLINE INTERPOLATION AND TWO-SIDED APPROXIMATE METHODS FOR TWO-POINT BOUNDARY VALUE PROBLEMS

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SPLINE INTERPOLATION AND TWO-SIDED APPROXIMATE METHODS FOR TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract

In the present paper we consider the two-sided approximations by the use of spline functions. A selection of numerical results is presented in Tables 1–9.

1. Introduction

We shall consider here the two-sided approximations of the solution of the following nonlinear two-point boundary value problem:

$$x'' = f(t, x, x') \qquad (0 \leq t \leq 1) \tag{1}$$

with boundary conditions

$$a_0 x(0) - b_0 x'(0) = c_0, \tag{2}$$

$$a_1 x(1) + b_1 x'(1) = c_1, \tag{3}$$

where $f(t, x, y)$ is defined and sufficiently smooth in a region D of (t, x, y)-space intercepted by two planes $t=0$ and $t=1$.

We assume that the problem (1)–(3) has an isolated solution $\hat{x}(t)$ satisfying the internality condition

$U = \{(t, x, y) \mid |x - \hat{x}(t)| + |y - \hat{x}'(t)| \leq \delta, t \in [0,1]\} \subset D$ for some $\delta > 0$.

By the use of B-spline $Q_4(t)$, we shall consider the cubic spline function

$$x_k(t) = \sum \alpha_i Q_4(t/h - i) \qquad (nh = 1) \tag{4}$$

such that

$$x_k'' = P_k f(t, x_k, x_k') \qquad (0 \leq t \leq 1) \tag{5}$$

$$a_0 x_k(0) - b_0 x_k'(0) = c_0, \tag{6}$$

$$a_1 x_k(1) + b_1 x_k'(1) = c_1. \tag{7}$$

Here the operator $P_k (k=1, 2)$ is defined as follows:

(1) $$(P_1 f)(t) = \sum f_i L_i(t)$$

with the piecewise linear function $L_i(t)$ such that

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$$L_i(t_j) = L_i(jh) = \delta_{ij},$$

(2) $$(P_2 g)(t) = \sum \beta_i L_i(t)$$

such that the coefficient $\beta_i (i = 0, 1, \cdots, n)$ is determined by

$$(2\beta_0 + \beta_1)/6 = (2f_0 + f_1)/6$$

$$(\beta_{i+1} + 4\beta_i + \beta_{i-1})/6 = f_i \qquad (i = 1, 2, \cdots, n-1)$$

$$(2\beta_n + \beta_{n-1})/6 = (2f_n + f_{n-1})/6 .$$

In [4], we have proved the following asymptotic expansion:

$$e_k(t) = \hat{x}(t) - x_k(t; h) = (-1)^k h^2 \psi(t)/12 + o(h^2) \qquad (k = 1, 2)$$

where $x_k(t, h)$ is the solution of (5)–(7) and $\psi(t)$ is the solution of the variation equation of (1)–(3):

$$\psi'' = f_2(t, \hat{x}, \hat{x}')\psi + f_3(t, \hat{x}, \hat{x}')\psi' + \hat{x}^{(4)}(t)$$

subject to

$$a_0\psi(0) - b_0\psi'(0) = 0 ,$$

$$a_1\psi(1) + b_1\psi_1(1) = 0 .$$

Here $$f_k(x_1, x_2, x_3) = \frac{\partial f(x_1, x_2, x_3)}{\partial x_k} \qquad (k = 1, 2, 3) .$$

Section 2 describes the following asymptotic expansion:

$$e_k(t) = (-1)^k h^2 \psi(t)/12 + h^3 \lambda_k(t) + h^4 \psi_k(t) + o(h^4)$$

$$\text{for} \quad t = t_i \qquad (i = 0, 1, \cdots, n)$$

where $\lambda_1(t)=0$ and $\lambda_2(t)$ is given by the use of Green function $H(t, s)$ (for the definition of $H(t, s)$, see Remark):

$$\lambda_2(t) = -[H(t, 0)\hat{x}^{(4)}(0) + H(t, 1)\hat{x}^{(4)}(1)]/12 .$$

Thus we have the following theorem.

THEOREM 1. *Let $t=t_i$ $(i=0, 1, \cdots, n)$, we have:*

(1) *if* $\lambda_2(t) \neq 0$,

$$\hat{x}(t) - [x_1(t; h) + x_2(t; h)]/2 = h^3 \lambda_2(t)/2 + O(h^4)$$

$$\hat{x}(t) - [4x_2(t; h/2) - x_2(t; h)]/3 = -h^3 \lambda_2(t)/6 + O(h^4);$$

(2) *if* $\lambda_2(t) = 0$,

$$\hat{x}(t) - [x_1(t; h) + x_2(t; h)]/2 = h^4(\psi_1(t) + \psi_2(t))/2 + o(h^4)$$

$$\hat{x}(t) - [4x_1(t; h/2) - x_1(t; h) + 4x_2(t; h/2) - x_2(t; h)]/6$$
$$= -h^4(\psi_1(t) + \psi_2(t))/8 + o(h^4) .$$

COROLLARY 1. *Let $t=t_i$, then we have:*

$$\hat{x}(t) - [8x_2(t; h/2) - x_2(t; h) + x_1(t; h)]/8 = O(h^4)$$

$$\text{for} \quad \lambda_2(t) \neq 0 ;$$

$$\hat{x}(t)-[16x_1(t;h/2)-x_1(t;h)+16x_2(t;h/2)-x_2(t;h)]/30$$
$$= o(h^4) \qquad \text{for} \quad \lambda_2(t)=0\,.$$

If the value of the function $\lambda_2(t)$ is unknown, the following corollary 2 is available.

COROLLARY 2. *Let $t=t_i$, then we have*

$$\hat{x}(t)-[4x_1(t;h/2)-x_1(t;h)+4x_2(t;h/2)-x_2(t;h)]/6$$
$$= -h^3\lambda_2(t)/12-h^4(\psi_1(t)+\psi_2(t))/8+o(h^4)$$
$$\hat{x}(t)-[x_1(t;h)+x_2(t;h)]/2 = h^3\lambda_2(t)/2+h^4(\psi_1(t)+\psi_2(t))/2$$
$$+\ o(h^4)\,.$$

COROLLARY 3. *Let $t=t_i$, then we have*

$$\hat{x}(t)-[8x_1(t;h/2)-x_1(t;h)+8x_2(t;h/2)-x_2(t;h)]/14 = O(h^4)\,.$$

By the use of B-spline $Q_6(t)$, let us consider the quintic spline function of the form

$$x_k(t) = \sum \alpha_i Q_6(t/h-i)$$

so that

$$x_k'' = P_k f(t, x_k, x_k') \qquad (0 \le t \le 1) \tag{8}$$
$$a_0 x_k(0)-b_0 x_k'(0) = c_0\,, \tag{9}$$
$$a_1 x_k(1)+b_1 x_k'(1) = c_1\,. \tag{10}$$

Here the operator $P_k(k=3, 4)$ is defined as follows:

(3) $(P_3 g)(t)$ is a cubic spline function with the node t_i such that

$$(P_3 g)(t_i) = g(t_i) \qquad (i=0, 1, \cdots, n)$$
$$(P_3 g)'(t_i) = g'(t_i) \qquad (i=0, n)\,.$$

(4) $(P_4 g)(t)$ is a cubic spline function with the node t_i such that

$$(P_4 g, L_i) = \begin{cases} h(g_{i+1}+10g_i+g_{i-1})/12 & (i=1, 2, \cdots, n-1) \\ h(7g_0+3g_1)/20+h^2(3g_0'-2g_1')/60 & (i=0) \\ h(7g_n+3g_{n-1})/20-h^2(3g_n'-2g_{n-1}')/60 & (i=n) \end{cases}$$

and $\qquad (P_4 g)'(t_i) = g'(t_i) \qquad (i=0, n)\,,$

where for any $\varphi_1(t)$ and $\varphi_2(t) \in L^2[0,1]$, let us denote

$$\int \varphi_1(t)\varphi_2(t)\,dt \text{ by } (\varphi_1, \varphi_2).$$

In [5], we have proved the following asymptotic expansion:

$$e_k(t) = \hat{x}(t)-x_k(t;h) = d_k h^4\theta(t)+o(h^4) \qquad (k=3, 4)$$

with $d_3=1/720$ and $d_4=-1/240$.

Here $x_k(t;h)$ is the solution of (8)–(10) and $\theta(t)$ is the solution of the variation equation of (1)–(3) :

$$\theta'' = f_2(t, \hat{x}, \hat{x}')\theta+f_3(t, \hat{x}, \hat{x}')\theta'+\hat{x}^{(6)}(t)$$

subject to

$$a_0\theta(0)-b_0\theta'(0)=0\,,$$

$$a_1\theta(1)+b_1\theta'(1)=0\,.$$

In Section 3, we shall prove the following asymptotic expansion:

$$e_k(t)=d_k h^4\theta(t)+h^5\gamma_k(t)+h^6\theta_k(t)+o(h^6) \qquad \text{for} \quad t=t_i\,,$$

where $\gamma_3(t)=0$ and $\gamma_4(t)$ is given by

$$\gamma_4(t)=[H(t,0)\mathfrak{x}^{(6)}(0)+H(t,1)\mathfrak{x}^{(6)}(1)]/360\,.$$

Therefore we have the following theorem.

THEOREM 2. *Let* $t=t_i (i=0, 1, \cdots, n)$, *then we have*:

(3) *if* $\gamma_4(t)\neq 0$,

$$\mathfrak{x}(t)-[3x_3(t;h/2)+x_4(t;h/2)]/4=h^5\gamma_4(t)/128+O(h^6)$$

$$\mathfrak{x}(t)-[16x_4(t;h/2)-x_4(t;h)]/15=-h^5\gamma_4(t)/30+O(h^6)\;;$$

(4) *if* $\gamma_4(t)=0$,

$$\mathfrak{x}(t)-[3x_3(t;h/2)+x_4(t;h/2)]/4=h^6(3\theta_3(t)+\theta_4(t))/256+o(h^6)$$

$$\mathfrak{x}(t)-[3(16x_3(t;h/2)-x_3(t;h))+16x_4(t;h/2)-x_4(t;h)]/60$$

$$=-h^6(3\theta_3(t)+\theta_4(t))/80+o(h^6)\,.$$

If the value of $\gamma_4(t)$ is unknown, the following corollary is of much use.

COROLLARY. *For* $t=t_i$, *we have*:

$$\mathfrak{x}(t)-[3x_3(t;h)+x_4(t;h)]/4=h^5\gamma_4(t)/4+h^6(3\theta_3(t)+\theta_4(t))/4+o(h^6)$$

$$\mathfrak{x}(t)-[3(16x_3(t;h/2)-x_3(t;h))+16x_4(t;h/2)-x_4(t;h)]/60$$

$$=-h^5\gamma_4(t)/120-h^6(3\theta_3(t)+\theta_4(t))/80+o(h^6)\,.$$

2. Proof of Theorem 1

In what follows, we shall assume that $f(t)$ and $g(t)$ are sufficiently smooth. Before we proceed with analysis, we shall require the following lemmas 1–5.

LEMMA 1. *There exists the smooth function* $\mu(t)$ *such that*

$$I(t_k,(I-P_1)\,g)=-(h^2/12)\,I(t_k,g'')+h^4\mu(t_k)+O(h^5)\,,$$

where, for any continuous function $\varphi(t)$, *we shall denote*

$$\int H^{(m)}(t,s)\,\varphi(s)\,ds \text{ by } I_m(t;\varphi) \text{ and } I(t;\varphi)=I_0(t;\varphi)\,.$$

PROOF. Let $c_i=(t_i+t_{i+1})/2$, then we have

$$(I-P_1)\,g(t)=(1/2)(t-t_i)(t-t_{i+1})g''(c_i)+(1/6)(t-t_i)(t-c_i)(t-t_{i+1})\,g^{(3)}(c_i)$$

$$+(1/24)\{(t-c_i)^4-(h/2)^4\}\,g^{(4)}(c_i)+O(h^5)\,,$$

from which follows

$$I(t_k, (I-P_1)g) = -(h^3/12)\sum H(t_k, c_i)\, g''(c_i) - (h^5/480)\sum [H''(t_k, c_i)\, g''(c_i) + (2/3) H'(t_k, c_i)\, g^{(3)}(c_i) + H(t_k, c_i)\, g^{(4)}(c_i)] + O(h^5)\,,$$

where $H'(t, s)$ denote the differentiation with respect to s. By the means of the mid-point rule:

$$\int_a^b f(t)\,dt = (b-a) f((b+a)/2) + (1/24)(b-a)^2 [f'(t)]_a^b + \cdots\,,$$

we have

$$I(t_k; (I-P_1)g) = -(h^2/12)\, I(t_k; g'') - (h^4/480)\,[I_2(t_k; g'') + (2/3)\, I_1(t_k; g^{(3)}) + I(t_k; g^{(4)})] + (h^4/288)\, \mu_2(t_k) + O(h^5)\,,$$

where

$$\mu_m(t) = [(H(t, s)\, g^{(m)}(s))']_{t+}^1 + [(H(t, s)\, g^{(m)}(s))']_0^{t-}\,.$$

LEMMA 2. *There exists the smooth function* $\nu_{m,j}(t)$ *such that*

$$I(t_k; (I-P_1) f e_j^{(m)}) = h^4 \nu_{m,j}(t_k) + O(h^5) \qquad (m = 0, 1)\,.$$

PROOF. For the function $e_j(t)$, we have

$$e_j^{(m)}(t) = (-1)^j h^2 \psi^{(m)}(t)/12 + O(h^3) = d_j h^2 \psi^{(m)}(t) + O(h^3) \qquad (m = 0, 1)$$

$$e_j^{(m)}(t) = O(h^{4-m}) \qquad (m = 2, 3, 4)$$

$$e_1''(t_k) = d_1 h^2 \kappa''(t_k) + O(h^3) \qquad (\kappa(t) = \psi(t) - \mathfrak{X}''(t))$$

$$\{e_2''(t_{i+1}) + 4e_2''(t_i) + e_2''(t_{i-1})\}/6 = (h^2/6)\mathfrak{X}_i^{(4)} + d_2 h^2 \kappa''(t_i) + O(h^3)$$

$$\{2e_2''(t_0) + e_2''(t_1)\}/6 = d_2 h^2 \kappa''(t_0) + O(h^3)$$

$$\{2e_2''(t_n) + e_2''(t_{n-1})\}/6 = d_2 h^2 \kappa''(t_n) + O(h^3) \qquad ([4])\,.$$

From above, we shall show the following asymptotic explansion:

$$e_2''(t) = h^2\{d_2 \kappa''(t) + (1/3 - 1/\sqrt{3})\mathfrak{X}^{(4)}(t)/2\} + O(h^3) \qquad (t = 0, 1)\,.$$

Now we have

$$\begin{aligned} e_2''(0) &= 3d_2 h^2 (a_0 \kappa_0''/2 + a_1 \kappa_1'' + \cdots + a_n \kappa_n''/2) \\ &\quad + (h^2/2)(a_1 \mathfrak{X}_1^{(4)} + a_2 \mathfrak{X}_2^{(4)} + \cdots + a_{n-1} \mathfrak{X}_{n-1}^{(4)}) + O(h^3) \\ &= 3d_2 h^2 \textstyle\sum'' a_i \kappa_i'' + (h^2/2) \sum a_i \mathfrak{X}_i^{(4)} + O(h^3)\,, \end{aligned}$$

where a_i $(i = 0, 1, \cdots, n)$ satisfies:

$$a_0 + a_1/2 = 1\,, \qquad a_{i+1}/2 + 2a_i + a_{i-1}/2 = 0\,, \qquad a_n + a_{n-1}/2 = 0\,.$$

Let $\beta_i = a_i \kappa_i''$, then we have

$$\beta_0 + \beta_1/2 = \kappa_0'' + a_1 h \kappa_0^{(3)}/2 + O(h^2)$$

$$\beta_{i+1}/2 + 2\beta_i + \beta_{i-1}/2 = (a_{i+1} - a_{i-1})\, h \kappa_i^{(3)}/2 + O(h^2) \qquad (i = 1, 2, \cdots, n-1)$$

$$\beta_n + \beta_{n-1}/2 = -a_{n-1} h \kappa_n^{(3)}/2 + O(h^2)$$

from which follows

$$\sum'' a_i\kappa_i'' = \kappa_0''/3+O(h)\,, \qquad \sum a_i\mathfrak{x}_i^{(4)} = (1/3-1/\sqrt{3})\,\mathfrak{x}_0^{(4)}+O(h)\,.$$

For the case $m=0$, we have

$$I(t_k;(I-P_1)fe_j) = -(h^3/12)\sum H(t_k,c_i)f(c_i)\,e_j''(c_i)-(d_jh^4/12)\,I(t_k;f''\psi)$$
$$-(d_jh^4/6)\,I(t_k;f'\psi')-(h^4/480)\,I(t_k;f\mathfrak{x}^{(4)})+O(h^5)\,.$$

Thus we have only to show:

$$h^3\sum H(t_k,c_i)f(c_i)\,e_j''(c_i) = -(h^5/8)\sum H(t_k,c_i)f(c_i)\,\mathfrak{x}^{(4)}(c_i)$$
$$+h^3\sum H(t_k,t_i)f_ie_j''(t_i)+O(h^5)$$
$$=\begin{cases} d_1h^4I(t_k;f\kappa'')-(h^4/8)\,I(t_k;f\mathfrak{x}^4)+O(h^5) & (j=1)\\ d_2h^4I(t_k;f\kappa'')+(h^4/24)\,I(t_k;f\mathfrak{x}^4)+O(h^5) & (j=2)\,. \end{cases}$$

For the case $m=1$, we have

$$I(t_k;(I-P_1)fe_j') = -(h^3/12)\sum H(t_k,c_i)(fe_j')''(c_i)$$
$$-(h^5/720)\sum H'(t_k,c_i)f(c_i)\mathfrak{x}^{(4)}(c_i)-(h^5/480)\sum H(t_k,c_i)$$
$$\times[f(c_i)\,\mathfrak{x}^{(5)}(c_i)+4f'(c_i)\,\mathfrak{x}^{(4)}(c_i)]+O(h^5)\,,$$

where

$$h^3\sum H(t_k,c_i)f(c_i)\,e_j^{(3)}(c_i) = -(h^5/24)\sum H(t_k,c_i)f(c_i)\,\mathfrak{x}^{(5)}(c_i)$$
$$-h^3\sum (H(t_k,s)f(s))'(t_i)\,e_j''(t_i)+h^2[H(t_k,1)f(1)\,e_j(1)$$
$$-H(t_k,0)f(0)\,e_j''(0)]+O(h^5)\,.$$

Thus we have the desired result.

LEMMA 3. *There exists the smooth function $\tau(t)$ such that*

$$I(t_k;(P_1-P_2)\,g) = (h^2/6)\,I(t_k;g'')-(h^3/12)\,\eta_2(t_k)+h^4\tau(t_k)+O(h^5)\,.$$

PROOF. Taylor expansion gives us

$$I(t_k;(P_1-P_2)\,g) = \sum H(t_k,t_i)\int L_i(s)(P_1-P_2)\,g(s)\,ds$$
$$+(1/2)\sum H''(t_k,c_i)\int (s-t_i)(s-t_{i+1})(P_1-P_2)\,g(s)\,ds+O(h^5)\,.$$

By the means of the trapezoidal rule:

$$h\sum'' H(t_k,t_i)\,g_i'' = I(t_k;g'')+(h^2/12)\,\mu_2(t_k)+O(h^3)\,,$$

we have

$$\sum H(t_k,t_i)\int L_i(s)(P_1-P_2)\,g(s)\,ds = (h^3/6)\sum H(t_k,t_i)\,g_i''$$
$$+(h^5/72)\sum H(t_k,t_i)\,g_i^{(4)}+O(h^5) = (h^2/6)\,I(t_k;g'')+(h^4/72)\,\mu_2(t_k)$$
$$+(h^4/72)\,I(t_k;g^{(4)})-(h^3/12)\,\eta_2(t_k)+O(h^5)$$

with $\qquad \eta_m(t) = H(t,0)\,g^{(m)}(0)+(-1)^m\,H(t,1)\,g^{(m)}(1)\,.$

Since $(P_1-P_2)\,g(t) = \sum \xi_iL_i(t)$, we have

$$\sum H''(t_k, c_i)\int (s-t_i)(s-t_{i+1})(P_1-P_2)\, g(s)\, ds$$

$$= -(h^3/12)\sum H''(t_k, c_i)(\xi_i+\xi_{i+1}) = -(h^3/6)\sum H''(t_k,t_i)\xi_i + O(h^5)$$

$$= -(h^4/36)\, I_2(t_k; g'') + O(h^5)\,,$$

where the component ξ_i $(i=0, 1, \cdots, n)$ satisfies:

$$(2\xi_0+\xi_1)/6 = 0\,,$$

$$(\xi_{i+1}+4\xi_i+\xi_{i-1})/6 = h^2 g_i/6 + O(h^4) \qquad (i = 1, 2, \cdots, n-1)$$

$$(2\xi_n+\xi_{n-1})/6 = 0\,.$$

LEMMA 4. *There exists the smooth function* $\rho_m(t)$ *such that*

$$I(t_k; (P_1-P_2) f e_2^{(m)}) = h^4 \rho_m(t_k) + O(h^5) \qquad (m = 0, 1)\,.$$

PROOF. Since $\|(P_1-P_2) f e_2^{(m)}\| = \max|(P_1-P_2) f e_2^{(m)}(t)| = O(h^3)$,

$$\int H(t_k, s)(P_1-P_2) f e_2^{(m)}(s)\, ds = \sum H(t_k, t_i)\int L_i(s)(P_1-P_2) f e_2^{(m)}(s)\, ds + O(h^5)$$

$$= (h^3/12)\sum H(t_k, t_i)\{(f e_2^{(m)})''(t_i+) + (f e_2^{(m)})''(t_i-)\}$$

$$+ (h^4/36)\sum H(t_k, t_i)\{(f e_2^{(m)})^{(3)}(t_i+) - (f e_2^{(m)})^{(3)}(t_i-)\}$$

$$+ (h^5/144)\sum H(t_k, t_i)\{(f e_2^{(m)})^{(4)}(t_i+) + (f e_2^{(m)})^{(4)}(t_i-)\} + O(h^5)\,.$$

For the case $m=0$, we have only to show:

$$h^4 \sum H(t_k, t_i) f_i (e_2^{(3)}(t_i+) - e_2^{(3)}(t_i-))$$

$$= h^3 \sum H(t_k, t_i) f_i (e_2''(t_{i+1}) - 2e_2''(t_i) + e_2''(t_{i-1})) - h^5 \sum H(t_k, t_i) f_i \mathfrak{X}_i^{(4)} + O(h^5)$$

$$= h^5 \sum (H(t_k, s) f(s))''(t_i)\, e_2''(t_i) - h^4 I(t_k; f\mathfrak{X}^4) + O(h^5)$$

$$= -h^4 I(t_k; f\mathfrak{X}^{(4)}) + O(h^5)\,.$$

For the case $m=1$, we have only to show:

$$h^3 \sum H(t_k, t_i) f_i (e_2^{(3)}(t_i+) + e_2^{(3)}((t_i-))$$

$$= h^2 \sum H(t_k, t_i) f_i (e_2''(t_{i+1}) - e_2''(t_{i-1})) - (h^5/3)\sum H(t_k,t_i) f_i \mathfrak{X}_i^{(5)} + O(h^5)$$

$$= h^2 [H(t_k, 1) f(1)\, e_2''(1) - H(t_k, 0) f(0)\, e_2''(0)]$$

$$-2h^3 \sum (H(t_k, s) f(s))'(t_i)\, e_2''(t_i) - (h^4/3) I(t_k; f\mathfrak{X}^{(5)}) + O(h^5)\,.$$

Thus we have the desired result.

LEMMA 5. *For* $i=1, 2$, *we have*

$$I(t; P_i(f e_i^2)) = d_i^2 h^4 I(t; f\psi^2) + O(h^5)\,, \quad I(t; P_i(f e' e')) = d_i^2 h^4 I(t; f\psi'\psi') + O(h^5)\,,$$

$$I(t; P_i(f e_i e_i')) = d_i^2 h^4 I(t; f\psi\psi') + O(h^5)\,.$$

PROOF. By a simple calculation, we have

$$\|(I-P_i)\, g\| \le ch\|g'\| \qquad (i = 1, 2)$$

from which follows the desired result.

Combining these Lemmas gives us

$$e_j'' = f_2(t, \hat{x}, \hat{x}')e_j + f_3(t, \hat{x}, \hat{x}')\, e_j' + (I-P_j)(f_2 e_j + f_2 e_j')$$
$$- (I-P_j)\, \hat{x}'' + P_j(f_{22}e_j^2 + 2f_{23}e_je_j' + f_{33}e_j'e_j') + O(h^6)$$

subject to the homogeneous boundary conditions

$$a_0 e_j(0) - b_0 e_j'(0) = 0\,,$$
$$a_1 e_j(1) + b_1 e_j'(1) = 0\,.$$

Thus we bave

$$e_j(t) = \int H(t, s)(I-P_j)(f_2 e_j + f_3 e_j')\, ds - \int H(t, s)(I-P_j)\, \hat{x}''\, ds$$

$$+ \int H(t, s)\, P_j(f_{22}e_j^2 + 2f_{23}e_je_j' + f_{33}e_j'e_j')\, ds + O(h^6)\,,$$

from which follows Theorem 1.

3. Proof of Theorem 2

To prove Thorem 2, we have only to show the following lemmas 6–10.

LEMMA 6. *There exists the smooth function* $\bar{\mu}(t)$ *such that*

$$I(t_k; (I-P_3)\, g) = d_3 h^4\, I(t_k; g^{(4)}) + h^6\, \bar{\mu}(t_k) + O(h^7)\,.$$

PROOF. Let us rewrite in the form:

$$I(t_k; (I-P_3)\, g) = I(t_k; g-g_3) + I(t_k; g_3 - P_3 g)$$

where $g_3(t)$ is cubic on each subinterval $[t_i, t_{i+1}]$ such that

$$g_3(t_i) = g(t_i) \qquad \text{and} \qquad g_3''(t_i) = g''(t_i) \qquad\qquad (i = 0, 1, \cdots, n)\,.$$

By the means of Taylor series expansion, we have

$$I(t_k; g-g_3) = \sum H(t_k, c_i) \int (g-g_3)(s)\, ds + \sum \int \{H(t_k, s) - H(t_k, c_i)\}(g-g_3)(s)\, ds$$
$$= \sum H(t_k, c_i)\{(h^5/120)\, g^{(4)}(c_i) + (h^4/7!)\, g^{(6)}(c_i)\} + (h^7/7!) \sum H''(t_k, c_i)\, g^{(4)}(c_i)$$
$$+ O(h^8) = (h^4/120) \int H(t_k, s)\, g^{(4)}(s)\, ds + (h^6/7!) \int H(t_k, s)\, g^{(6)}(s)\, ds$$

$$+ (h^6/7!) \int H''(t_k, s)\, g^{(4)}(s)\, ds - (h^6/2880)\, \mu_4(t_k) + O(h^8)\,.$$

Since $(g_3 - P_3 g)(t_i) = 0$ $(i=0, 1, \cdots, n)$ we have

$$I(t_k; g_3 - P_3 g) = M_0 \int H(t_k, s)\, \phi_0(s)\, ds + M_1\Big(\int H(t_k, s)\, \phi_1(s)\, ds$$

$$+ \int H(t_k, s)\, \phi_0(s-h)\, ds\Big) + \cdots\,,$$

where the function $\phi_i(t)$ $(i=0, 1)$ is cubic such that

$$\phi_0(0) = \phi_0(h) = 0\,, \qquad \phi_0''(0) = 1 \qquad \text{and} \qquad \phi_0''(h) = 0$$
$$\phi_1(0) = \phi_1(h) = 0\,, \qquad \phi_1''(0) = 0 \qquad \text{and} \qquad \phi_1''(h) = 1;$$

the component $M_i=(g_3-P_3g)''(t_i)$ satisfies the following system of equations:

$$(2M_0+M_1)/6 = h^2g_0^{(4)}/24+7h^3g_0^{(5)}/360+O(h^4)$$
$$(M_{i+1}+M_i+M_{i-1})/6 = h^2g_i^{(4)}/12+h^4g_i^{(6)}/90+O(h^5) \qquad (i=1,2,\cdots,n-1)$$
$$(2M_n+M_{n-1})/6 = h^2g_n^{(4)}/24-7h^3g_n^{(5)}/360+O(h^4)\,.$$

Let $M_i=h^2g_i^{(4)}/12+\alpha_ih^3$, then we have

$$I(t_k;g_3-P_3g) = (h^2/12)\,[g_0^{(4)}\int H(t_k,s)\,\phi_0(s)\,ds+g_1^{(4)}\{\int H(t_k,s)\,\phi_1(s)\,ds$$
$$+\int H(t_k,s)\,\phi_0(s-h)\,ds\}+\cdots]+h^3\,[\alpha_0\int H(t_k,s)\,\phi_0(s)\,ds$$
$$+\alpha_1\{\int H(t_k,s)\,\phi_1(s)\,ds+\int H(t_k,s)\,\phi_0(s-h)\,ds\}+\cdots]\,,$$

where

$$\int H(t_k,s)\,\phi_0(s)\,ds = -(h^3/24)\,H(t_k,t_0)-(7h^4/360)\,H'(t_k,t_0)+O(h^5)$$

$$\int H(t_k,s)\,\phi_1(s)\,ds+\int H(t_k,s)\,\phi_0(s-h)\,ds$$
$$= -(h^3/12)\,H(t_k,t_1)-(h^5/90)\,H''(t_k,t_1)+O(h^7) \qquad (k\neq 1)$$

$$\int H(t_1,s)\,\phi_1(s)\,ds+\int H(t_1,s)\,\phi_0(s-h)\,ds$$
$$= -(h^3/12)\,H(t_1,t_1)-(7h^4/360)\,[H'(t_1,t_1+)-H'(t_1,t_1-)]+O(h^5)\,.$$

Thus we have

$$I(t_k;g_3-P_3g) = -(h^5/144)\,\textstyle\sum''\,H(t_k,t_i)\,g_i^{(4)}-(7h^7/1080)\,\sum H''(t_k,t_i)\,g_i^{(4)}$$
$$+(7h^6/360)\,\{[H'(t_k,s)\,g^{(4)}(s)]_{t_k+}^{1}+[H'(t_k,s)\,g^{(4)}(s)]_0^{t_k-}\}$$
$$-(h^6/12)\,\textstyle\sum''\,\alpha_iH(t_k,t_i)+O(h^7) = -(h^4/144)\,I(t_k;g^{(4)})-(h^6/1728)\,\mu_4(t_k)$$
$$-(h^6/1080)\,I_2(t_k;g^{(4)})+(7h^6/360)\{[H'(t_k,s)\,g^{(4)}(s)]_{t_k+}^{1}+[H'(t_k,s)\,g^{(4)}(s)]_0^{t_k-}\}$$
$$-(h^6/12)\,\textstyle\sum''\,\alpha_iH(t_k,t_i)+O(h^7)\,.$$

Here α_i $(i=0,1,\cdots,n)$ satisfies:

$$(2\alpha_0+\alpha_1)/6 = g_0^{(5)}/180+O(h)\,, \qquad (\alpha_{i+1}+4\alpha_i+\alpha_{i-1})/6 = -(h/360)\,g_i^{(6)}+O(h^2)$$
$$(2\alpha_n+\alpha_{n-1})/6 = -g_n^{(5)}/180+O(h)\,.$$

Let $\beta_i=\alpha_iH(t_k,t_i)$, then we have

$$(2\beta_0+\beta_1)/6 = g_0^{(5)}H(t_k,t_0)/180+O(h)$$
$$(\beta_{i+1}+4\beta_i+\beta_{i-1})/6 = -(h/360)\,H(t_k,t_i)\,g_i^{(6)}+hH'(t_k,t_i)(\alpha_{i+1}-\alpha_{i-1})+O(h^2)$$
$$(i\neq k)$$
$$(\beta_{k+1}+4\beta_k+\beta_{k-1})/6 = O(h)\,, \qquad (2\beta_n+\beta_{n-1})/6 = -g_n^{(5)}H(t_k,t_n)/180+O(h)$$

from which follows

$$\sum'' \beta_i = \eta_5(t_k)/180 - I(t_k\,;\, g^{(6)})/180 + O(h)\,.$$

Thus we have the desired result.

LEMMA 7. *There exists the smooth function* $\bar{v}_{m,j}(t)$ *such that*

$$I(t_k\,;\,(I-P_3)fe_j^{(m)}) = h^6\bar{v}_{m,j}(t_k) + O(h^7) \qquad (j=3,4\,;\ m=0,1)\,.$$

PROOF. For the error $e_j(t)$, we have

$$e_j^{(m)}(t) = d_j h^4 \theta^{(m)}(t) + O(h^5) \qquad (m=0,1)$$

$$e_j^{(m)}(t) = O(h^{6-m}) \qquad (m=2,3,4,5) \qquad ([5])\,,$$

from which follows

$$I(t_k\,; fe_j - (fe_j)_3) = \sum H(t_k, c_i)\{(h^5/120)(fe_j)^{(4)}(c_i) + (h^7/7\,!)(fe_j)^{(7)}(c_i)\}$$
$$+ (h^7/7\,!) \sum H''(t_k, c_i)(fe_j)^{(4)}(c_i) + O(h^7)\,.$$

Therefore we have only to show

$$h^5 \sum H(t_k, c_i) f(c_i)\, e_j^{(4)}(c_i) = h^5 \sum H(t_k, c_i) f(c_i)(e_j^{(4)}(t_{i+1}) + e_j^{(4)}(t_i))/2$$
$$-(h^7/8) \sum H(t_k, c_i) f(c_i)\, \mathfrak{x}^{(6)}(c_i) + O(h^7) = h^5 \sum H(t_k, t_i) f(t_i)\, e_j^{(4)}(t_i)$$
$$-(h^6/8)\, I(t_k\,; f\mathfrak{x}^{(6)}) + O(h^7)\,.$$

Let $\beta_i = H(t_k, t_i) f(t_i)\, e_j^{(4)}(t_i)$, then we have

$$(\beta_{i+1} + 4\beta_i + \beta_{i-1})/6 = H(t_k, t_i) f_i(e_j^{(4)}(t_{i+1}) + 4e_j^{(4)}(t_i) + e_j^{(4)}(t_{i-1}))/6 + O(h^3)$$
$$= H(t_k, t_i) f_i(e_j''(t_{i+1}) - 2e_j''(t_i) + e_j''(t_{i-1}))/h^2 + (h^2/12)\, H(t_k, t_i) f_i \mathfrak{x}_i^{(6)} + O(h^3)\,.$$

Thus we have

$$\beta_1 + \beta_2 + \cdots + \beta_{n-1} = \sum (H(t_k, s) f(s))''(t_i)\, e_j''(t_i) + (h/12)\, I(t_k\,; f\mathfrak{x}^{(6)}) + O(h^2)$$
$$= (h/12)\, I(t_k\,; f\mathfrak{x}^6) + O(h^2)\,.$$

Next we shall show

$$I(t_k\,;\,(fe_j)_3 - P_3(fe_j)) = -(h^4/144)\, I(t_k\,; f\mathfrak{x}^6) + O(h^7)\,.$$

Now we have

$$I(t_k\,;\,(fe_j)_3 - P_3(fe_j)) = -(h^3/12) \sum'' M_i H(t_k, t_i) + O(h^7)$$

where $M_i = (fe_j)_3''(t_i) - P_3''(fe_j)(t_i)$ satisfies:

$$(2M_0 + M_1)/6 = O(h^4)\,, \qquad (M_{i+1} + 4M_i + M_{i-1})/6 = (h^2/12) f_i e_j^{(4)}(t_i)$$
$$+ (7h^2/360) f_i \{e_j^{(4)}(t_{i+1}) - 2e_j^{(4)}(t_i) + e_j^{(4)}(t_{i-1})\} - (h^4/120) f_i \mathfrak{x}_i^{(6)} + O(h^5)$$
$$(i = 1, 2, \cdots, n-1)$$

$$(2M_n + M_{n-1})/6 = O(h^4)\,.$$

Let $\gamma_i = M_i H(t_k, t_i) = O(h^4)$, then we have

$$(\gamma_{i+1} + 4\gamma_i + \gamma_{i-1})/6 = (h^2/12)\, H(t_k, t_i) f_i e_j^{(4)}(t_i) - (h^4/120)\, H(t_k, t_i) f_i \mathfrak{x}_i^{(6)}$$

$$-(7h^2/360)\, H(t_k, t_i) f_i \{e_j^{(4)}(t_{i+1}) - 2e_j^{(4)}(t_i) + e_j^{(4)}(t_{i-1})\} + O(h^5)\,.$$

from which follows

$$\gamma_1 + \gamma_2 + \cdots + \gamma_{n-1} = (h^3/12)\, I(t_k; f\mathfrak{X}^{(6)}) + O(h^4)\,.$$

Now let us consider the case $m=1$.

$$I(t_k; (I-P_3)(fe_j')) = I(t_k; fe_j' - (fe_j')_3) + I(t_k; (fe_j')_3 - P_3(fe_j'))$$

where

$$I(t_k; fe_j' - (fe_j')_3) = \sum H(t_k, c_i)\{(h^5/5!)(fe_j')^{(4)}(c_i) + (h^7/7!)(fe_j')^{(6)}(c_i)\}$$
$$+ (h^7/7!) \sum H''(t_k, c_i)(fe_j')^{(4)}(c_i) + O(h^7)\,.$$

Since $\quad e_j^{(5)}(c_i) = (e_j^{(4)}(t_{i+1}) - e_j^{(4)}(t_i))/h - (h^2/24)\, \mathfrak{X}^{(7)}(c_i) + O(h^3)\,,$

we have only to show

$$h^4 \sum H(t_k, c_i) f(c_i)(e_j^{(4)}(t_{i+1}) - e_j^{(4)}(t_i)) = h^6 \zeta_j(t_k) + O(h^7)$$

for some smooth function $\zeta_j(t)$ $\qquad (j = 3, 4)$.

By a simple calculation, we have

$$h^4 \sum H(t_k, c_i) f(c_i)(e_j^{(4)})(t_{i+1}) - e_j^{(4)}(t_i)) = h^4 [H(t_k, 1) f(1)\, e_j^{(4)}(1) - H(t_k, 0) f(0)\, e_j^{(4)}(0)]$$
$$-h^5 \sum \{H'(t_k, t_i) f_i + H(t_k, t_i) f_i'\}\, e_j^{(4)}(t_i) + O(h^7)$$

from which we shall require the asymptotic expansion of the terms $e_j^{(4)}(0)$ and $e_j^{(4)}(1)$. By the means of the consistency relation, we have:

$$(e_j^{(4)}(t_{i+1}) + 4e_j^{(4)}(t_i) + e_j^{(4)}(t_{i-1}))/6 = (h^2/12)\, \mathfrak{X}_i^{(6)} + (1/h^2)(e_j''(t_{i+1}) - 2e_j''(t_i)$$
$$+ e_j''(t_{i-1})) + O(h^4)$$

$$(2e_j''(t_0) + e_j''(t_1))/6 = (h^2/24)\mathfrak{X}_0^6 + (e_j''(t_1) - e_j''(t_0))/h - e_j^{(3)}(t_0)/h + O(h^4)\,, \cdots\cdots$$

from which follows

$$e_j^{(4)}(0) = (h^2/4) \sum{}'' a_i \mathfrak{X}_i^{(6)} + (3/h^2)\{(e_j''(t_1) - e_j''(t_0))a_0 + (e_j''(t_n) - e_j''(t_{n-1}))\, a_n$$
$$+ \sum a_i (e_j''(t_{i+1}) - 2e_j''(t_i) + e_j''(t_{i-1})\} + (3/h)(a_n e_j^{(3)}(t_n) - a_0 e_j^{(3)}(t_0)) + O(h^5)\,.$$

Since $\quad (P_j f)'(t_i) = f'(t_i)$ $(i = 0, n)$, we have $e_j^{(3)}(t_i) = O(h^4)\,.$

Thus we have only to show the asymptotic expansion of the term:

$$e_j''(t_0) - 3 \sum{}'' a_i e_j''(t_i)\,.$$

For $j=3$, it follows from the definition of the operator P_3 that

$$e_3''(t_i) = d_3 h^4 p''(t_i) + O(h^5) \qquad (p(t) = \theta(t) - \mathfrak{X}^{(4)}(t))\,.$$

Since $\quad a_i = (2/\sqrt{3})(-0.5)^i\, [(1+\sigma)^{n-i} + (1-\sigma)^{n-i}]/[(1+\sigma)^n - (1-\sigma)^n] \qquad (\sigma = \sqrt{3}/2)\,,$

we have

$$\sum |a_i| < 4 \qquad \text{and} \qquad |a_i| < 1/2^{i-1}\,.$$

Thus we have

$$\sum{}'' a_i e_3''(t_i) = d_3 h^4 p''(0)/3 + O(h^5)\,.$$

For $j=4$, let us consider the following relationships between the values and its derivatives for the qunitic spline function $\phi(t)$:

$$(\phi_{i+2}''+26\phi_{i+1}''+66\phi_i''+26\phi_{i-1}''+\phi_{i-2}'')/120 = [(\phi_{i+2}-2\phi_{i+1}+\phi_i)+4(\phi_{i+1}-2\phi_i+\phi_{i-1}) + (\phi_i-2\phi_{i-1}+\phi_{i-2})]/6h \qquad (i=2, 3,\cdots, n-2)$$

$$(-138\phi_0''+2124\phi_1''+1206\phi_2''+48\phi_3'')/120 = [8(\phi_3-2\phi_2+\phi_1)+25(\phi_2-2\phi_1+\phi_0)]/h^2 - [(\phi_1-\phi_0)/h^2-\phi_0'/h]$$

$$h\phi_0''' = [10(\phi_3-2\phi_2+\phi_1)+35(\phi_2-2\phi_1+\phi_0)]/h^2-(57\phi_0''+324\phi_1''+153\phi_2''+6\phi_3'')/12, \cdots\cdot$$

from which follow

$$(e_{i+2}''+26e_{i+1}''+66e_i''+26e_{i-1}''+e_{i-2}'')/120 = -(h^4/180)\mathfrak{X}_i^{(6)}+d_4h^4p_i''+O(h^5) \qquad (i=2, 3,\cdots, n-2)$$

$$(-138e_0''+2124e_1''+1206e_2''+48e_3'')/120 = -(11h^4/180)\mathfrak{X}_0^{(6)}+27d_4h^4p_0''+O(h^5)$$

$$(57e_0''+324e_1''+153e_2''+6e_3'')/12 = -(h^4/4)\mathfrak{X}_0^{(6)}+45d_4h^4p_0''+O(h^5),\cdots\cdots$$

with $\qquad e_i'' = e_4''(t_i) \qquad (i=0, 1,\cdots, n).$

Thus we have

$$e_i'' = -(h^4/180)\mathfrak{X}_i^{(6)}+d_4h^4p_i''+h^4\xi_i+O(h^5)\,,$$

where ξ_i ($i=0, 1,\cdots, n$) satisfies the following system of equations:

$$\xi_{i+2}+26\xi_{i+1}+66\xi_i+26\xi_{i-1}+\xi_{i-2} = 0$$

$$-23\xi_0+354\xi_1+201\xi_2+8\xi_3 = -(2/3)\mathfrak{X}^{(6)}(0)$$

$$19\xi_0+108\xi_1+51\xi_2+2\xi_3 = 0\,,\ \cdots\cdots.$$

Here ξ_i is represented in the form:

$$\xi_i = A\alpha^i+B/\alpha^i+C\beta^i+D/\beta^i \qquad (i=0, 1,\cdots, n)$$

where α, $\beta(\alpha<\beta<-1)$ are the roots of the equation:

$$t^4+26t^3+66t^2+26t+1 = 0\,.$$

From above, we have

$$\begin{bmatrix} p(\alpha)/\alpha^n & p(\beta)/\beta^n \\ q(\alpha)/\alpha^n & q(\beta)/\beta^n \end{bmatrix}\begin{bmatrix} B \\ D \end{bmatrix}+\begin{bmatrix} p(1/\alpha)\alpha^n & p(1/\beta)\beta^n \\ q(1/\alpha)\alpha^n & q(1/\beta)\beta^n \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix}=\mathfrak{X}^6(1)\begin{bmatrix} 2/9 \\ -17/3 \end{bmatrix}$$

$$\begin{bmatrix} p(1/\alpha) & p(1/\beta) \\ q(1/\alpha) & q(1/\beta) \end{bmatrix}\begin{bmatrix} B \\ D \end{bmatrix}+\begin{bmatrix} p(\alpha) & p(\beta) \\ q(\alpha) & q(\beta) \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix}=\mathfrak{X}^6(0)\begin{bmatrix} 2/9 \\ -17/3 \end{bmatrix}$$

with $p(t)=t^2+26y+33$ and $q(t)=t^3-609t-832$.
Doing these calculations gives

$$A = O(1/\alpha^n)\,, \qquad B = b+O(h)\,, \qquad C = O(1/\beta^n) \text{ and } D = d+O(h)$$

for some constants b and d independent of h. By the means of the explicit representation of a_i, we have the asymptotic expansions of the terms: $e_4''(t_0)$ and $\sum'' a_i e_4''(t_i)$. Next we shall consider

$$I(t_k; (fe_j')-P_3(fe_j')) = -(h^3/24)\{M_0 H\{t_k, t_0)+2M_1 H(t_k, t_1)+\cdots\}+O(h^7)$$

where $M_i=(fe_j')''(t_i)-P_3''(fe_j')(t_i)$ satisfies:

$$(2M_0+M_1)/6 = (h/24)f_0(e_j^{(4)}(t_1)-e_j^{(4)}(t_0))-(h^3/720)f_0\mathfrak{x}_0^{(6)}+O(h^4)$$

$$(M_{i+1}+4M_i+M_{i-1})/6 = (h/24)f_i(e_j^{(4)}(t_{i+1})-e_j^{(4)}(t_{i-1}))+(h^2/3)f_i'e_j^{(4)}(t_i)$$
$$+(7h^2/72)f_i'\{e_j^{(4)}(t_{i+1})-2e_j^{(4)}(t_i)+e_j^{(4)}(t_{i-1})\}-(h^4/360)(f_i\mathfrak{x}_i^{(7)}+11f_i'\mathfrak{x}_i^{(6)})+O(h^5),$$

Let $\beta_i=M_i H(t_k, t_i)$, we have

$$(2\beta_0+\beta_1)/6 = -(h^3/720)\,H(t_k, t_0)f_0\mathfrak{x}_0^{(6)}+O(h^4)$$

$$(\beta_{i+1}+4\beta_i+\beta_{i-1})/6 = H(t_k, t_i)\,[(h/24)f_i(e_j^{(4)}(t_{i+1})-e_j^{(4)}(t_{i-1}))$$
$$+(h^2/3)f_i'e_j^{(4)}(t_i)+(7h^2/72)f_i'\{e_j^{(4)}(t_{i+1})-2e_4^{(4)}(t_i)+e_j^{(4)}(t_{i-1})\}]$$
$$-(h^4/360)\,H(t_k, t_i)(f_i\mathfrak{x}_i^{(7)}+11f_i'\mathfrak{x}_i^{(6)})+O(h^5) \qquad (i \neq k)\ \cdots\cdots.$$

Thus we have

$$\sum''\beta_i = -(h^2/12)\sum (H(t_k,s)f(s))'(t_i)\,e_j^{(4)}(t_i)+(h^2/3)\sum H(t_k, t_i)f_i'e_j^{(4)}(t_i)$$
$$-(h^3/720)\,[H(t_k, 0)\,\mathfrak{x}^{(6)}(0)+H(t_k, 1)\,\mathfrak{x}^{(6)}(1)]+O(h^4).$$

This completes the proof of Lemma 7.

LEMMA 8. *There exists the smooth function* $\bar{\tau}(t)$ *such that*

$$I(t_k; (P_3-P_4)\,g) = -(h^4/180)\,I(t_k; g^{(4)})+(h^5/360)\,\eta_4(t_k)+h^6\bar{\tau}(t_k)+O(h^7).$$

PROOF. Let $\phi(t)=g(t)-(P_3 g)(t)$, then we have

$$\int L_i(s)(P_3-P_4)\,g(s)\,ds = (h/15)(g_{i+1}-2g_i+g_{i-1})-(h^2/30)(g_{i+1}'-g_{i-1}')$$
$$+(h^2/30)(\phi_{i+1}'-\phi_{i-1}') = -(h^5/180)g_i^{(4)}-(h^7/2700)g_i^{(6)}+(h^2/30)(\phi_{i+1}'-\phi_{i-1}')$$
$$+O(h^8) \qquad (i=1, 2, \cdots, n-1);$$

$$\int L_0(s)(P_3-P_4)\,g(s)\,ds = (h^2/30)\,\phi_1', \quad \int L_n(s)(P_3-P_4)\,g(s)\,ds = (h^2/30)\,\phi_{n-1}'.$$

Thus we have

$$\sum H(t_k, t_i)\int L_i(s)(P_3-P_4)\,g(s)\,ds = -(h^4/180)\,I(t_k; g^{(4)})-(h^6/2700)\,I(t_k; g^{(6)})$$
$$+(h^5/360)\,\eta_4(t_k)-(h^6/2160)\,\mu_4(t_k)-(h^2/15)\sum H'(t_k, t_i)\,\phi_i'.$$

Next we shall consider the following quantity:

$$\sum H''(t_k, c_i)\int (s-t_i)(s-t_{i+1})(P_3-P_4)\,g(s)$$

$$= \sum H''(t_k, c_i) \int (s-t_i)(s-t_{i+1}) \sum \theta_j Q_4(s/h-j)\, ds$$

$$= -(h^3/180) \sum H''(t_k, c_i)(\theta_i+14\theta_{i-1}+14\theta_{i-2}+\theta_{i-3})$$

$$= -(h^3/6) \sum H''(t_k, t_i)\theta_{i-2}+O(h^7) \qquad (c_i=(t_i+t_{i+1})/2)$$

where the component θ_i ($i=-3, -2, \cdots, n-1$) satisfies:

$$\theta_{-3}-\theta_{-1}=0\,, \qquad (4\theta_{-3}+32\theta_{-2}+23\theta_{-1}+\theta_0)/60=O(h^5)$$

$$(\theta_i+26\theta_{i-1}+66\theta_{i-2}+26\theta_{i-3}+\theta_{i-4})/120=-(h^4/180)\, g_i^{(4)}+O(h^5)\,, \cdots .$$

Let $\gamma_i=H''(t_k,t_i)\theta_{i-2}$, then we have

$$(\gamma_{i+2}+26\gamma_{i+1}+66\gamma_i+26\gamma_{i-1}+\gamma_{i-2})/120=-(h^4/180)\, H''(t_k, t_i)\, g_i^{(4)}+O(h^5)\,.$$

Since $\gamma_i=O(h^4)$, we have

$$\sum \gamma_i=-(h^4/180) \sum H''(t_k,t_i)\, g_i^{(4)}+O(h^4)=-(h^3/180)\, I_2(t_k; g^{(4)})+O(h^4)\,.$$

Thus we have

$$\sum H''(t_k, c_i) \int (s-t_i)(s-t_{i+1})(P_3-P_4)\, g(s)\, ds=(h^6/1080)\, I_2(t_k, g^{(4)})+O(h^7)\,.$$

This completes the proof of Lemma 8.

Lemma 9. *There exists the smooth function* $\bar{p}_m(t)$ *such that*

$$I(t_k; (P_3-P_4)fe_4^{(m)})=h^6\bar{p}_m(t_k)+O(h^7) \qquad (m=0, 1).$$

Proof. Let $\psi_m(t)=(fe_4^{(m)})(t)-P_3(fe_4^{(m)})(t)$, then we have

(i) for $i=1, 2, \cdots, n-1$,

$$\int L_i(s)(P_3-P_4)(fe_4)(s)\, ds=-(h^5/180)f_ie_4^{(4)}(t_i)-(h^5/1200)f_i\{e_4^{(4)}(t_{i+1})$$
$$-2e_4^{(4)}(t_i)+e_4^{(4)}(t_{i-1})\}+(h^7/2160)f_i\mathfrak{X}^{(6)}+(h^2/30)\{\psi_0'(t_{i+1})-\psi_0'(t_{i-1})\}+O(h^8)$$

(ii) for $i=0, n$,

$$\int L_0(s)(P_3-P_4)(s)\, ds=(h^2/30)\, \psi_0'(t_0)=0\,, \cdots\cdots .$$

Since $\psi_0'(t_i)=O(h^5)$ ($i=1, 2, \cdots, n-1$), we have

$$h^2 \sum H(t_k, t_i)(\psi_0'(t_{i+1})-\psi_0'(t_{i-1}))=-2h^3 \sum H'(t_k,t_i)\, \psi'(t_i)=O(h^7)\,.$$

Therefore we have only to show:

$$\sum H(t_k, t_i) f_i \{e_4^{(4)}(t_{i+1})-2e_4^{(4)}(t_i)+e_4^{(4)}(t_{i-1})\}=h^2 \sum (H(t_k, s) f(s))''(t_i)\, e_4^{(4)}(t_i)+O(h^2)\,.$$

Similarly we have:

(i) for $i=1, 2, \cdots, n-1$,

$$\int L_i(s)(P_3-P_4)(fe_4')(s)\, ds=(h/15)\{(fe_4')(t_{i+1})-2(fe_4')(t_i)+(fe_4')(t_{i-1})\}$$
$$-(h^2/30)\{(fe_4')'(t_{i+1})-(fe_4')'(t_{i-1})\}+(h^2/30)\{\psi_1'(t_{i+1})-\psi_1'(t_{i-1})\}+O(h^8)$$

$$= -(h^4/360)f_i(e_4^{(4)}(t_{i+1})-e_4^{(4)}(t_{i-1}))-(h^5/45)f'_i e_4^{(4)}(t_i)+(h^7/1080)f_i\mathfrak{x}_i^{(7)}$$
$$+(h^7/240)f'_i\mathfrak{x}_i^{(6)}-(h^7/2700)\{f_i\mathfrak{x}_i^{(7)}+6f'_i\mathfrak{x}_i^{(6)}\}-(h^5/240)f'_i\{e_4^{(4)}(t_{i+1})-2e_4^{(4)}(t_i)$$
$$+e_4^{(4)}(t_{i-1})\}+(h^2/30)\{\psi'_1(t_{i+1})-\psi'_1(t_{i-1})\}+O(h^8)$$

(ii) for $i=0, n$,

$$\int L_0(s)(P_3-P_4)(s)(fe'_4)(s)\,ds = (h^2/30)\,\psi'_1(t_0) = 0\,, \cdots\cdots .$$

Thus we have

$$\sum H(t_k, t_i)\int L_i(s)(P_3-P_4)(fe'_4)(s)\,ds = -(h^4/360)\sum H(t_k, t_i)f_i(e_4^{(4)}(t_{i+1})-e_4^{(4)}(t_{i-1}))$$
$$+(h^6/1080)\,I(t_k;f\mathfrak{x}^{(7)})-(h^6/2700)\,I(t_k;f\mathfrak{x}^{(7)}+6f'\mathfrak{x}^{(6)})-(h^5/45)\sum H(t_k, t_i)f'_i e_4^{(4)}(t_i)$$
$$+(h^2/30)\sum H'(t_k, t_i)(\psi'_1(t_{i+1})-\psi'_1(t_{i-1}))+(h^6/240)\,I(t_k;f'\mathfrak{x}^{(6)})+O(h^7)$$
$$=(h^5/180)\sum H(t_k, s)f(s))'(t_i)\,e_4^{(4)}(t_i)-(h^5/45)\sum H(t_k, t_i)f_i e_4^{(4)}(t_i)$$
$$+(h^4/360)\,[H(t_k, 0)\,e_4^{(4)}(0)-H(t_k, 1)\,e_4^{(4)}(1)]-(h^6/2700)\,I(t_k;f\mathfrak{x}^{(7)}+6f'\mathfrak{x}^{(6)})$$
$$+(h^6/240)I(t_k;f'\mathfrak{x}^{(6)})-(h^3/15)\sum H'(t_k, t_i)\,\psi'_1(t_i)+(h^6/1080)\,I(t_k;f\mathfrak{x}^{(7)})+O(h^7)\,.$$

we have only to show:

$$\sum H'(t_k, t_i)\,\psi'_1(t_i) = (h^3/720)\,I_1(t_k;f\mathfrak{x}^{(6)})+O(h^4)\,.$$

Now $\psi'_1(t_i)$ $(i=0, 1, \cdots, n)$ satisfies:

$$\psi'_1(t_0)=0, \quad (\psi'_1(t_{i+1})+4\psi'_1(t_i)+\psi'(t_{i-1}))/6 = (h^2/144)f_i\{e_j^{(4)}(t_{i+1})-2e_j^{(4)}(t_i)+e_j^{(4)}(t_{i-1})\}$$
$$-(h^4/720)f_i\mathfrak{x}^{(6)}+O(h^5), \quad \psi'_1(t_n)=0\,.$$

Let $\beta_i = H'(t_k, t_i)\,\psi'_1(t_i)$, then we have

$$(\beta_{i+1}+4\beta_i+\beta_{i-1})/6 = (h^2/144)f_i H'(t_k, t_i)\{e_j^{(4)}(t_{i+1})-2e_j^{(4)}(t_i)+e_j^{(4)}(t_{i-1})\}$$
$$-(h^4/720)\,H'(t_k, t_i)f_i\mathfrak{x}_i^{(6)}+O(h^5) \qquad (i \neq k)$$

from which follows

$$\sum H'(t_k, t_i)\,\psi'_1(t_i) = (h^4/144)\sum (H'(t_k, s)f(s))''(t_i)\,e_j^{(4)}(t_i)$$
$$+(h^3/720)\,I_1(t_k;f\mathfrak{x}^{(6)})+O(h^4) = (h^3/720)\,I_1(t_k;f\mathfrak{x}^{(6)})+O(h^4)\,.$$

This completes the proof of Lemma 9.

LEMMA 10. *For $i=3, 4$, we have*

$$I(t;P_i(fe_i^2))\,, \qquad I(t;P_i(fe_ie'_i)) \qquad \text{and} \qquad I(t;P_i(fe'_ie'_i)) = O(h^7)\,.$$

PROOF. By a simple calculation, we have

$$\|(I-P_i)g\| \le ch\|g'\| \qquad (i=3, 4)$$

from which follows the desired result.

By the means of Lemmas 6–10, we obtain Theorem 2 as in a similar manner as in the proof of Theorem 1.

REMARK. *If* $b_0=b_1=0$, we *have* $\lambda_2(t)=\gamma_4(t)=0$.

PROOF. First we notice that $H(t, s)$ is the (1,2)-component of the matrix;

$$K(t, s) = \begin{cases} \Phi(t)\,[E-G^{-1}\begin{bmatrix} 0 & 0 \\ a_1 & b_1 \end{bmatrix}\Phi(1)]\,\Phi^{-1}(s) & (s \le t) \\ -\Phi(t)\,G^{-1}\begin{bmatrix} 0 & 0 \\ a_1 & b_1 \end{bmatrix}\Phi(1)\,\Phi^{-1}(s) & (t < s) \end{cases}$$

where

$$G = \begin{bmatrix} a_0 & -b_0 \\ a_1 y_1(1)+b_1 y_1'(1) & a_1 y_2(1)+b_1 y_2'(1) \end{bmatrix}, \qquad \Phi(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$$

and $y_k(t)$ ($k=1, 2$) is the solution of the first variation equation with respect to ($\hat{x}(t)$, $\hat{x}'(t)$), that is,

$$y_k'' = f_2(t, \hat{x}, \hat{x}')\, y_k + f_3(t, \hat{x}, \hat{x}')\, y_k'$$

subject to $y_k(0)=\delta_{1k}$, $y_k'(0)=\delta_{2k}$ ([3]).

By a simple calculation we have the desired result.

4. Numerical Examples

In this section, we discuss numerical results obtained from some concrete examples. These numerical results conform the theoretical accuracies established in previous sections. In the case of examples 1 and 2, the approximate problems ($k=4$) (8–10) are identical with the Numerov difference schemes. We now consider the numerical solutions of particular examples.

Eample 1 ([1]). As our first example, we consider the following simple linear problem:

$$x'' = 100x\,, \qquad x(0) = x(1) = 1\,.$$

Its exact solution $\hat{x}(t)=\cosh(10t-5)/\cosh 5$.
For this example, $\lambda_2(t)=\gamma_4(t)=0$.
(i) cubic spline approximations:

Table 1 ($h=1/10$)

	$z_1(t)$	$z_2(t)$	$w(t)$
0.1	1.39(−3)	−3.48(−4)	−1.80(−7)
0.2	7.80(−4)	−1.94(−4)	5.66(−6)
0.3	2.92(−4)	−7.24(−5)	4.99(−7)
0.4	7.43(−5)	−1.81(−5)	2.53(−7)
0.5	1.77(−5)	−4.05(−6)	2.93(−7)

Table 2 (h=1/20)

	$z_1(t)$	$z_2(t)$	$w(t)$
0.1	8.73(−5)	−2.18(−5)	1.84(−8)
0.2	4.93(−5)	−1.23(−5)	1.50(−8)
0.3	1.87(−5)	−4.67(−6)	8.88(−9)
0.4	4.97(−6)	−1.24(−6)	5.70(−9)
0.5	1.38(−6)	−3.40(−7)	4.38(−9)

Here $z_i(t)$ (i=1, 2) and $w(t)$ are defined as follows:

$$z_1(t) = \hat{x}(t)-[x_1(t;h)+x_2(t;h)]/2 = O(h^4),$$

$$z_2(t) = \hat{x}(t)-[4x_1(t;h/2)-x_1(t;h)+4x_2(t;h/2)-x_2(t;h)]/6$$

$$\fallingdotseq -z_1(t)/4,$$

$$w(t) = [z_1(t)+4z_2(t)]/5 = o(h^4).$$

(ii) quintic spline approximations:

Table 3 (h=1/20)

	$z_3(t)$	$z_4(t)$	$v(t)$
0.1	6.44(−9)	−1.65(−8)	9.78(−10)
0.2	5.42(−9)	−1.50(−8)	5.58(−10)
0.3	3.22(−9)	−9.32(−9)	2.32(−10)
0.4	1.86(−9)	−5.51(−9)	1.05(−10)
0.5	1.43(−9)	−4.36(−9)	5.14(−11)

Table 4 (h=1/40)

	$z_3(t)$	$z_4(t)$	$v(t)$
0.1	1.14(−10)	−3.08(−10)	1.35(−11)
0.2	8.95(−11)	−2.64(−10)	5.33(−12)
0.3	5.23(−11)	−1.59(−10)	1.99(−12)
0.4	3.00(−11)	−9.18(−11)	1.00(−12)
0.5	2.25(−11)	−7.12(−11)	1.98(−13)

Here $z_i(t)$ (i=3, 4) and $v(t)$ are defined as follows:

$$z_3(t) = \hat{x}(t)-[3x_1(t;h/2)+x_2(t;h/2)]/4 = O(h^6),$$

$$z_4(t) = \hat{x}(t)-[3\{16x_3(t;h/2)-x_3(t;h)\}$$

$$+16x_4(t;h/2)-x_4(t;h)\}]/60 \fallingdotseq -(16/5)\,z_3(t),$$

$$v(t) = [16z_3(t)+5z_4(t)]/21 = o(h^6).$$

Example 2. Next we consider the nonlinear equation:

$$x'' = 1.5x^2, \qquad x(0) = 4, \qquad x(1) = 1.$$

This problem has two isolated solutions such that $\hat{x}(t)=4/(t+1)^2$ and $\hat{x}(0.5)\fallingdotseq -10.53$.

A selection of numerical results for the solution $\hat{x}(t)=4/(t+l)^2$ is presented in Tables 5–7.
(i) cubic spline approximations:

Table 5 ($h=1/20$)

	$z_1(t)$	$z_2(t)$	$v(t)$
0.2	−5.55(−6)	1.38(−6)	6.0(− 9)
0.4	−4.40(−6)	1.15(−6)	0.0(− 9)
0.6	−2.91(−6)	7.30(−7)	2.0(− 9)
0.8	−1.39(−6)	3.43(−7)	3.6(−10)

(ii) quintic spline approximations:

Table 6 ($h=1/20$)

	$z_3(t)$	$z_4(z)$	$v(t)$
0.2	3.15(−10)	−1.32(−10)	7.43(−11)
0.4	2.64(−10)	−7.29(−10)	2.76(−11)
0.6	1.70(−10)	−4.91(−10)	1.50(−11)
0.8	0.83(−10)	−2.35(−10)	0.73(−11)

Table 7 ($h=1/40$)

	$z_3(t)$	$z_4(t)$	$v(t)$
0.2	5.18(−12)	−1.54(−11)	7.80(−13)
0.4	4.40(−12)	−1.29(−11)	2.81(−13)
0.6	2.78(−12)	−8.36(−12)	1.28(−13)
0.8	1.38(−12)	−4.07(−12)	8.95(−14)

Example 3 ([2]). As our final example, consider

$$x'' = x^3-(\cos t+1)^3-\cos t,$$

where $x(0)'=0$ *and* $x'(1)=-x^3(1)\sin 1/(\cos 1+1)^3$.
The unique solution is $\hat{x}(t)=\cos t+1$.
qunitic spline approximations:

Table 8 ($h=1/8$)

	$u_3(t)$	$u_4(t)$	$u(t)$
0	2.00(−10)	−8.33(−10)	3.62(−12)
1/4	9.23(−11)	−3.84(−10)	1.87(−12)
2/4	5.61(−11)	−2.33(−10)	1.30(−12)
3/4	5.71(−11)	−2.51(−10)	−1.42(−12)
1	9.15(−11)	−4.15(−10)	−4.59(−12)

Table 9 ($h=1/16$)

	$u_3(t)$	$u_4(t)$	$u(t)$
0	6.24(−12)	−2.65(−11)	3.20(−14)
1/4	2.88(−12)	−1.13(−11)	1.89(−13)
2/4	1.73(−12)	−7.49(−12)	−2.51(−14)
3/4	1.77(−12)	−7.70(−12)	−2.49(−14)
1	2.84(−12)	−9.66(−12)	4.69(−13)

$$u_3(t) = \hat{x}(t) - [3x_3(t; h/2) + x_4(t; h/2)]/4 ,$$

$$u_4(t) = \hat{x}(t) - [16x_4(t; h/2) - x_4(t; h)]/15 \fallingdotseq -(64/15)\, u_3(t) ,$$

$$u(t) = [64u_3(t) + 15u_4(t)]/79 = O(h^6) .$$

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