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# METRICAL FINSLER CONNECTIONS

By

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In the present paper we determine all metrical Finsler connections to study the group of transformations of these connections and its invariants.

## § 0. Introduction.

In the ordinary treatments of the Finsler geometry the Cartan connection [3] is adopted as a typical metrical Finsler connection, but sometimes the others are useful. For example, V. Wagner [25] introduced the notion of generalized Cartan connection to study the so-called cubic metric. So, it is important to consider the relations between various metrical Finsler connections and to investigate what reasonable places they hold among the general Finsler connections. In fact, it has been studied by many authors from their respective standpoints (e.g. L. Berwald [2], M. Hashiguchi [5, 6, 8], A. Kawaguchi [12], M. Matsumoto [13], R. Miron [17, 18], T. Ohkubo [21], A. Sanini [23] etc.).

The purpose of the present paper is to determine all metrical Finsler connections to study the group of transformations of these connections and its invariants. As the results of these considerations we have three Finsler tensor fields $H^i_{jkl}$, $M^i_{jkl}$ and $N^i_{jkl}$, which are invariant for the semi-symmetric metrical Finsler connections having a common non-linear connection (Theorem 6. 2).

All metrical connections are determined in §3 using Obata's operators [20] explained in §2, and expressed in the various forms in §4, where the semi-symmetric metrical Finsler connections make an important class. In §5 we treat the transformations of the general metrical Finsler connections, and in the following § 6 we obtain the above invariants and consider their some properties.

Continued from this paper we should try to characterize the Finsler spaces satisfying $H^i_{jkl}=M^i_{jkl}=N^i_{jkl}=0$ for some semi-symmetric metrical Finsler connection. On the other hand, our method is applicable also to the cases of conformal Finsler, almost symplectic, almost complex etc. structures (cf. V. Cruceanu and R. Miron [4], R. Miron [16], R. Miron and V. Oproiu [19]). These shall be treated in the following papers.

The notations and terminologies are those of M. Matsumoto [14, 15] with few

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modifications. For convenience' sake, we shall devote § 1 to sketching the materials necessary for our discussions about the metrical Finsler connections.

This paper was mainly prepared by R. Miron, and was reconstituted by M. Hashiguchi adding some historical notes. The authors wish to express their sincere gratitude to Professor Dr. M. Matsumoto for his kind recommendation to this joint work.

## § 1. The notion of metrical Finsler connection.

**1.1.** Let $F_n$ be an $n$-dimensional Finsler space having $L(x, y)$ as the fundamental function, where $x=(x^i)$ and $y=(y^i)$ denote a point of the base manifold $M$ and a supporting element respectively. The components $g_{ij}$ of the fundamental tensor field are given by $g_{ij}=1/2\partial^2 L^2/\partial y^i \partial y^j$, and the components $y^i$ of the supporting element become also $y^i=g^{ij}y_j$, where $(g^{ij})=(g_{ij})^{-1}$, $y_j=L\partial L/\partial y^j$.

By Matsumoto's theory [14, 15] a Finsler connection $F\Gamma$ is defined in three manners as a pair $(\Gamma, N)$, as a pair $(\Gamma^h, \Gamma^v)$ or as a triad $(\Gamma_V, N, \Gamma^v)$, where $\Gamma$ and $\Gamma^h$ (resp. $\Gamma^v$) are a connection and a horizontal (resp. vertical) connection in the Finsler bundle $F(M)$, $N$ is a non-linear connection in the tangent bundle $T(M)$, and $\Gamma_V$ is a $V$-connection in the linear frame bundle $L(M)$. When $\Gamma^i_{jk}$, $C^i_{jk}$ are the coefficients of $\Gamma$, and $N^i_k$ and $F^i_{jk}$ are the respective ones of $N$ and $\Gamma_V$, they are related in

$$F^i_{jk}=\Gamma^i_{jk}-C^i_{jm}N^m_k\,. \tag{1.1}$$

The coefficients of $\Gamma^h$ (resp. $\Gamma^v$) are $F^i_{jk}$, $N^i_k$ (resp. $C^i_{jk}$).

Throughout the paper the Finsler connection $F\Gamma$ having $N^i_k$, $F^i_{jk}$, $C^i_{jk}$ as the coefficients is denoted by $F\Gamma=(N, F, C)$ for brevity, and also we use the following abridged notations:

$$A^i_{0k}=y^jA^i_{jk}\,, \quad \mathfrak{A}_{jk}\{A^i_{jk}\}=A^i_{jk}-A^i_{kj}\,, \quad \mathfrak{S}_{jkl}\{A^i_{jkl}\}=A^i_{jkl}+A^i_{klj}+A^i_{ljk}\,.$$

**1.2.** Given a Finsler connection $F\Gamma=(N, F, C)$, for a Finsler tensor field, for example, $K^i_j\,(x, y)$, the $h$- and $v$-covariant differentiations are defined by

$$K^i_{j|k}=\delta K^i_j/\delta x^k+K^m_jF^i_{mk}-K^i_mF^m_{jk}\,, \quad K^i_j|_k=\partial K^i_j/\partial y^k+K^m_jC^i_{mk}-K^i_mC^m_{jk}\,, \tag{1.2}$$

where $\delta/\delta x^k=\partial/\partial x^k-N^m_k\partial/\partial y^m$.

For the supporting element $y^i$ we have $y^i{}_{|k}=F^i_{0k}-N^i_k$, $y^i|_k=\delta^i_k+C^i_{0k}$. The Finsler tensor field $D$ given by

$$D^i_k=F^i_{0k}-N^i_k \tag{1.3}$$

is called the *deflection tensor field.*

The Ricci identities applying to $g_{ij}$ are

$$g_{ij|k|l}-g_{ij|l|k}=-g_{sj}R^s_{ikl}-g_{is}R^s_{jkl}-g_{ij|s}T^s_{kl}-g_{ij}|_sR^s_{kl}\,, \tag{1.4}$$

$$g_{ij|k}|_l-g_{ij}|_{l|k}=-g_{sj}P^s_{ikl}-g_{is}P^s_{jkl}-g_{ij|s}C^s_{kl}-g_{ij}|_sP^s_{kl}\,, \tag{1.5}$$

$$g_{ij}|_k|_l-g_{ij}|_l|_k=-g_{sj}S^s_{ikl}-g_{is}S^s_{jkl}-g_{ij}|_sS^s_{kl}\,, \tag{1.6}$$

where five torsion tensor fields $T$, $R^1$, $C$, $P^1$, $S^1$ and three curvature tensor fields $R^2$, $P^2$, $S^2$ appear, and their components are given by

$$T\colon T^i_{jk} = \mathfrak{A}_{jk}\{F^i_{jk}\}\,, \qquad S^1\colon S^i_{jk} = \mathfrak{A}_{jk}\{C^i_{jk}\}\,, \qquad C\colon C^i_{jk}\,, \tag{1.7}$$

$$R^1\colon R^i_{kl} = \mathfrak{A}_{kl}\{\delta N^i_k/\delta x^l\}\,, \qquad P^1\colon P^i_{kl} = \partial N^i_k/\partial y^l - F^i_{lk}\,, \tag{1.8}$$

$$R^2\colon R^i_{jkl} = \mathfrak{A}_{kl}\{\delta F^i_{jk}/\delta x^l + F^m_{jk}F^i_{ml}\} + C^i_{jm}R^m_{kl}\,, \tag{1.9}$$

$$P^2\colon P^i_{jkl} = \partial F^i_{jk}/\partial y^l - C^i_{jl|k} + C^i_{jm}P^m_{kl}\,, \tag{1.10}$$

$$S^2\colon S^i_{jkl} = \mathfrak{A}_{kl}\{\partial C^i_{jk}/\partial y^l + C^m_{jk}C^i_{ml}\}\,. \tag{1.11}$$

**1.3.** Let $C(x^i(t))$ be a differentiable curve in $M$ and $\tilde{C}(x^i(t), y^i(t))$ be a differentiable curve in $T(M)$ mapped on $C$ by the natural projection $\tau\colon T(M)\to M$. Given a Finsler connection $F\Gamma = (\Gamma, N)$, tangent vectors $X(t)$ along $C$ are called *parallel along $C$ with respect to $\tilde{C}$*, if the equations

$$\dot{X}^i + \Gamma^i_{jk}X^j\dot{x}^k + C^i_{jk}X^j\dot{y}^k = 0 \tag{1.12}$$

are satisfied, where a dot means $d/dt$.

A Finsler connection $F\Gamma$ is called *metrical* if the length $(g_{ij}(x, y)\,X^iX^j)^{1/2}$ of a vector $X(t)$ remains unchanged under the parallel displacement along any curve $C$ with respect to any $\tilde{C}$. Then the necessary and sufficient conditions that $F\Gamma$ be metrical are

$$g_{ij|k} = 0\,, \qquad g_{ij}|_k = 0\,. \tag{1.13}$$

A remarkable metrical Finsler connection is the Cartan connection $C\Gamma = (\overset{c}{N}, \overset{c}{F}, \overset{c}{C})$, whose coefficients are given by

$$\overset{c}{N}{}^i_k = \frac{1}{2}\partial\gamma^i_{00}/\partial y^k\,, \qquad \text{where} \quad \gamma^i_{jk} = \frac{1}{2}g^{im}(\partial g_{jm}/\partial x^k + \partial g_{km}/\partial x^j - \partial g_{jk}/\partial x^m)\,, \tag{1.14}$$

$$\overset{c}{F}{}^i_{jk} = \frac{1}{2}g^{im}(\delta g_{jm}/\delta x^k + \delta g_{km}/\delta x^j - \delta g_{jk}/\delta x^m)\,, \tag{1.15}$$

$$\overset{c}{C}{}^i_{jk} = \frac{1}{2}g^{im}(\partial g_{jm}/\partial y^k + \partial g_{km}/\partial y^j - \partial g_{jk}/\partial y^m) = \frac{1}{2}g^{im}\,\partial g_{jm}/\partial y^k\,. \tag{1.16}$$

Throughout the paper we specify the objects concerned with $C\Gamma$ by putting $c$. It holds $y^i{}_{\overset{c}{|}k} = 0$, $y^i\overset{c}{|}_k = \delta^i_k$, that is,

$$\overset{c}{D}{}^i_k = 0\,, \qquad \overset{c}{C}{}^i_{0k} = 0\,. \tag{1.17}$$

In the following we determine all metrical Finsler connections.

## § 2. Obata's operators of the Finsler space.

**2.1.** Let us consider the Finsler tensor fields

$$\Omega^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j - g_{sj}g^{ir})\,, \qquad \Omega^{*}{}^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j + g_{sj}g^{ir})\,, \tag{2.1}$$

which have the following properties and play the same role as the operators $A_1$, $A_2$ given by M. Obata [20] in the Riemannian case. We shall call them *Obata's operators.*

**Proposition 2.1.** *Obata's operators have the following properties*:

$$\Omega^{ir}_{sj}+\Omega^{*ir}_{sj}=\delta^i_s\delta^r_j\,, \tag{2.2}$$

$$\Omega^{ir}_{sj}\Omega^{sn}_{mr}=\Omega^{in}_{mj}\,,\qquad \Omega^{*ir}_{sj}\Omega^{*sn}_{mr}=\Omega^{*in}_{mj}\,,\qquad \Omega^{ir}_{sj}\Omega^{*sn}_{mr}=\Omega^{*ir}_{sj}\Omega^{sn}_{mr}=0\,, \tag{2.3}$$

$$\Omega^{ir}_{rj}=\Omega^{ir}_{si}=0\,,\qquad \Omega^{ir}_{ij}=\frac{1}{2}(n-1)\,\delta^r_j\,,\qquad \Omega^{*ir}_{ij}=\frac{1}{2}(n+1)\,\delta^r_j\,. \tag{2.4}$$

**Proposition 2.2.** *For any Finsler tensor field* $A^i_{jk}$ *the following three conditions* (1), (2), (3) *are equivalent*:

(1) $\Omega^{ir}_{sj}A^s_{rk}=0$ (*resp.* $\Omega^{*ir}_{sj}A^s_{rk}=0$),

(2) $\Omega^{*ir}_{sj}A^s_{rk}=A^i_{jk}$ (*resp.* $\Omega^{ir}_{sj}A^s_{rk}=A^i_{jk}$),

(3) $g_{sj}A^s_{ik}$ *is symmetric* (*resp. alternate*) *with respect to the indeces* $i$ *and* $j$.

Using the above propositions we have easily

**Theorem 2.1.** *A system of tensor equations*

$$\Omega^{*ir}_{sj}X^s_{rk}=A^i_{jk}\qquad (\textit{resp.}\ \Omega^{ir}_{sj}X^s_{rk}=A^i_{jk}) \tag{2.5}$$

*with* $X^i_{jk}$ *as unknowns, has solutions if and only if*

$$\Omega^{ir}_{sj}A^s_{rk}=0\qquad (\textit{resp.}\ \Omega^{*ir}_{sj}A^s_{rk}=0)\,. \tag{2.6}$$

*If* (2.6) *holds, then the general solutions of* (2.5) *are*

$$X^i_{jk}=A^i_{jk}+\Omega^{ir}_{sj}Y^s_{rk}\qquad (\textit{resp.}\ X^i_{jk}=A^i_{jk}+\Omega^{*ir}_{sj}Y^s_{rk})\,, \tag{2.7}$$

*where* $Y^i_{jk}$ *is an arbitrary Finsler tensor field.*

**2.2.** Now, assume a Finsler connection to be metrical. Then Obata's operators are $h$- and $v$-covariant constants:

$$\Omega^{ir}_{sj|l}=0\,,\qquad \Omega^{ir}_{sj}|_l=0\,,\qquad \Omega^{*ir}_{sj|l}=0\,,\qquad \Omega^{*ir}_{sj}|_l=0\,, \tag{2.8}$$

and the Ricci identity (1.4) becomes $g_{sj}R^s_{ikl}+g_{is}R^s_{jkl}=0$. As to $P^i_{jkl}$, $S^i_{jkl}$ the same relations hold, and we have from Proposition 2.2

**Theorem 2.2.** *The curvature tensor fields* $R^i_{jkl}$, $P^i_{jkl}$, $S^i_{jkl}$ *of a metrical Finsler connection have the following properties*:

$$\Omega^{*ir}_{sj}R^s_{rkl}=0\,,\quad \Omega^{*ir}_{sj}R^s_{rkl|l_1\cdots l_p}=0\,,\quad \Omega^{*ir}_{sj}R^s_{rkl}|_{l_1\cdots l_p}=0\quad (p=1,2,\cdots), \tag{2.9}$$

$$\Omega^{*ir}_{sj}P^s_{rkl}=0\,,\quad \Omega^{*ir}_{sj}P^s_{rkl|l_1\cdots l_p}=0\,,\quad \Omega^{*ir}_{sj}P^s_{rkl}|_{l_1\cdots l_p}=0\quad (p=1,2,\cdots), \tag{2.10}$$

$$\Omega^{*ir}_{sj}S^s_{rkl}=0\,,\quad \Omega^{*ir}_{sj}S^s_{rkl|l_1\cdots l_p}=0\,,\quad \Omega^{*ir}_{sj}S^s_{rkl}|_{l_1\cdots l_p}=0\quad (p=1,2,\cdots). \tag{2.11}$$

Also, we observe that for the Cartan connection

$$\Omega^{*ir}_{sj}\overset{c}{C}{}^s_{rk}=\overset{c}{C}{}^i_{jk}\,,\qquad \Omega^{*ir}_{sj}\overset{c}{P}{}^s_{rk}=\overset{c}{P}{}^i_{jk}\quad (=\overset{c}{C}{}^i_{jk\overset{c}{|}0})\,. \tag{2.12}$$

## § 3. The set of metrical Finsler connections.

**3.1.** We shall determine all metrical Finsler connections by a well-known method based on Theorem 2.1. Let $F\overset{0}{\Gamma}=(\overset{0}{N}, \overset{0}{F}, \overset{0}{C})$ be a fixed Finsler connection in $F_n$. Then any Finsler connection $F\Gamma=(N, F, C)$ in $F_n$ can be expressed in the form

$$N^i_k=\overset{0}{N}{}^i_k-A^i_k\,,\qquad F^i_{jk}=\overset{0}{F}{}^i_{jk}+\overset{0}{C}{}^i_{jm}A^m_k-B^i_{jk}\,,\qquad C^i_{jk}=\overset{0}{C}{}^i_{jk}-D^i_{jk}\,,\tag{3.1}$$

where $A^i_k$, $B^i_{jk}$, $D^i_{jk}$ are components of the difference tensor fields of $F\Gamma$ from $F\overset{0}{\Gamma}$ [14, 15]. There exists a one-to-one correspondence between the set of all Finsler connections $F\Gamma$ and the set of all triads of Finsler tensor fields $A^i_k$, $B^i_{jk}$, $D^i_{jk}$. So, to determine $F\Gamma$ in some geometrical conditions, for example, a metrical one, is the same as to determine $A^i_k$, $B^i_{jk}$, $D^i_{jk}$ in some tensorial conditions.

In order that $F\Gamma$ is metrical, that is, (1.13) holds for $F\Gamma$, it is necessary and sufficient that $A^i_k$, $B^i_{jk}$, $D^i_{jk}$ satisfy

$$g_{ij\overset{0}{|}k}+g_{ij}\overset{0}{|}_n A^n_k+g_{sj}B^s_{ik}+g_{is}B^s_{jk}=0\,,\qquad g_{ij}\overset{0}{|}_k+g_{sj}D^s_{ik}+g_{is}D^s_{jk}=0\,,$$

which is equivalent to

$$\Omega^{*ir}_{sj}B^s_{rk}=-\frac{1}{2}g^{im}(g_{mj\overset{0}{|}k}+g_{mj}\overset{0}{|}_n A^n_k)\,,\qquad \Omega^{*ir}_{sj}D^s_{rk}=-\frac{1}{2}g^{im}g_{mj}\overset{0}{|}_k\,,\tag{3.2}$$

where $\overset{0}{|}$ and $\overset{0}{|}$ denote the $h$- and $v$-covariant differentiations with respect to $F\overset{0}{\Gamma}$. Thus we have

**Proposition 3.1.** *Let $F\overset{0}{\Gamma}$ be a fixed Finsler connection. Then the set of all metrical Finsler connections $F\Gamma$ is given by* (3.1), *where $A^i_k$, $B^i_{jk}$, $D^i_{jk}$ are arbitrary Finsler tensor fields satisfying* (3.2). *Especially, if $F\overset{0}{\Gamma}$ is metrical, then* (3.2) *becomes $\Omega^{*ir}_{sj}B^s_{rk}=0$, $\Omega^{*ir}_{sj}D^s_{rk}=0$.*

From Theorem 2.1, however, the system (3.2) has solutions in $B^i_{jk}$, $D^i_{jk}$ for any Finsler tensor field $A^i_k=X^i_k$. Substituting in (3.1) from the general solution we have

**Theorem 3.1.** *Let $F\overset{0}{\Gamma}$ be a fixed Finsler connection. The set of all metrical Finsler connections $F\Gamma$ is given by*

$$\left.\begin{aligned} N^i_k&=\overset{0}{N}{}^i_k-X^i_k\,,\\ F^i_{jk}&=\overset{0}{F}{}^i_{jk}+\overset{0}{C}{}^i_{jm}X^m_k+\frac{1}{2}g^{im}(g_{mj\overset{0}{|}k}+g_{mj}\overset{0}{|}_n X^n_k)+\Omega^{ir}_{sj}X^s_{rk}\,,\\ C^i_{jk}&=\overset{0}{C}{}^i_{jk}+\frac{1}{2}g^{im}g_{mj}\overset{0}{|}_k+\Omega^{ir}_{sj}Y^s_{rk}\,,\end{aligned}\right\}\tag{3.3}$$

*where $X^i_k$, $X^i_{jk}$, $Y^i_{jk}$ are arbitrary Finsler tensor fields.*

**3.2.** As the particular case $X^i_k=X^i_{jk}=Y^i_{jk}=0$ in Theorem 3.1 we have

**Theorem 3.2.** *Let $F\overset{0}{\Gamma}$ be a given Finsler connection. Then the following Finsler connection $F\Gamma$ is metrical:*

$$N^i_k=\overset{0}{N}{}^i_k\,,\qquad F^i_{jk}=\overset{0}{F}{}^i_{jk}+\frac{1}{2}g^{im}g_{mj\overset{0}{|}k}\,,\qquad C^i_{jk}=\overset{0}{C}{}^i_{jk}+\frac{1}{2}g^{im}g_{mj}\overset{0}{|}_k\,.\tag{3.4}$$

In many cases of Finsler geometry, for example, in the theory of extremals, a non-metrical Finsler connection is determined easily and naturally. In order to derive a metrical Finsler connection from a given non-metrical one, A. Kawaguchi [12] obtained the metrization method essentially stated in the above theorems and showed that the Cartan connection is derived from the Berwald one. We shall call the connection $F\Gamma$ in Theorem 3.2 the *Kawaguchi metrical connection derived from* $F\overset{0}{\Gamma}$ and denoted by $K\overset{0}{\Gamma}$.

As examples of non-metrical Finsler connections $F\overset{0}{\Gamma}$ we know the ones $B\Gamma$, $R\Gamma$ and $H\Gamma$ given by L. Berwald [1], H.Rund [22] and M. Hashiguchi [9]:

$$B\Gamma\colon\ \overset{0}{N}{}^i_k = \overset{c}{N}{}^i_k\,, \qquad \overset{0}{F}{}^i_{jk} = \overset{c}{F}{}^i_{jk} + \overset{c}{P}{}^i_{jk}\,, \qquad \overset{0}{C}{}^i_{jk} = 0\,,$$

$$R\Gamma\colon\ \overset{0}{N}{}^i_k = \overset{c}{N}{}^i_k\,, \qquad \overset{0}{F}{}^i_{jk} = \overset{c}{F}{}^i_{jk}\,, \qquad \overset{0}{C}{}^i_{jk} = 0\,,$$

$$H\Gamma\colon\ \overset{0}{N}{}^i_k = \overset{c}{N}{}^i_k\,, \qquad \overset{0}{F}{}^i_{jk} = \overset{c}{F}{}^i_{jk} + \overset{c}{P}{}^i_{jk}\,, \qquad \overset{0}{C}{}^i_{jk} = \overset{c}{C}{}^i_{jk}\,.$$

Paying attention to (2.12) we have from Theorem 3.2

**Theorem 3.3.** *Let $F\overset{0}{\Gamma}$ be the Cartan, Berwald, Rund or Hashiguchi connection. Then the Kawaguchi metrical connection $K\overset{0}{\Gamma}$ derived from $F\overset{0}{\Gamma}$ is the Cartan one.*

**3.3.** We shall consider various expressions of Theorem 3.1. If we take a metrical Finsler connection (e.g. $C\Gamma$) as $F\overset{0}{\Gamma}$ in Theorem 3.1, we have

**Theorem 3.4.** *The set of all metrical Finsler connections is given by*

$$N^i_k = \overset{c}{N}{}^i_k - X^i_k\,, \qquad F^i_{jk} = \overset{c}{F}{}^i_{jk} + \overset{c}{C}{}^i_{jm}X^m_k + \Omega^{ir}_{sj}X^s_{rk}\,, \qquad C^i_{jk} = \overset{c}{C}{}^i_{jk} + \Omega^{ir}_{sj}Y^s_{rk}\,, \tag{3.5}$$

*where $X^i_k$, $X^i_{jk}$, $Y^i_{jk}$ are arbitrary Finsler tensor fields.*

In the above theorem we can replace $X^i_k$ by the deflection tensor field $D^i_k$. We shall denote by $F\Gamma(D)$ a Finsler connection having $D^i_k$ as the deflection tensor field. Especially, a Finsler connection is called a *F-connection if* $D^i_k=0$.

If we contract the second of (3.5) by $y^j$, and use (1.3), (1.17), then $X^i_k$ in (3.5) is expressed as

$$X^i_k = D^i_k - \Omega^{ir}_{s0}X^s_{rk}\,. \tag{3.6}$$

Conversely, $D^i_k$ may be arbitrarily given instead of $X^i_k$. Thus we have

**Theorem 3.5.** *Let $D^i_k$ be a given Finsler tensor field. Then the set of all metrical Finsler connections $F\Gamma(D)$ is given by* (3.5), *where $X^i_{jk}$, $Y^i_{jk}$ are arbitrary Finsler tensor fields and $X^i_k$ is given by* (3.6).

**Theorem 3.6.** *The set of all metrical F-connections is given by* (3.5), *where $X^i_{jk}$, $Y^i_{jk}$ are arbitrary Finsler fields and $X^i_k = -\Omega^{ir}_{s0}X^s_{rk}$.*

**3.4.** The arbitrariness of $X^i_k$ in Theorem 3.1 tells us any non-linear connection $N$ may become the non-linear connection of a metrical Finsler connection. We shall denote by $F\Gamma(N)$ a Finsler connection having $N$ as the non-linear connection. Theorem 3.1 is also restated as

**Theorem 3.7.** *Let* $\overset{0}{F\Gamma}$ *be a fixed Finsler connection. Given a non-linear connection* $N$, *the set of all metrical Finsler connections* $F\Gamma(N)$ *is given by*

$$\left.\begin{aligned} F^i_{jk} &= \overset{0}{F}{}^i_{jk}+\overset{c}{C}{}^i_{jm}X^m_k+\frac{1}{2}g^{im}(g_{mj\overset{0}{|}k}+g_{mj}\overset{0}{|}_nX^n_k)+\Omega^{ir}_{sj}X^s_{rk}\,,\\ C^i_{jk} &= \overset{0}{C}{}^i_{jk}+\frac{1}{2}g^{im}g_{mj}\overset{0}{|}_k+\Omega^{ir}_{sj}Y^s_{rk}\,, \end{aligned}\right\}\tag{3.7}$$

*where* $X^i_k=\overset{0}{N}{}^i_k-N^i_k$ *and* $X^i_{jk}$, $Y^i_{jk}$ *are arbitrary Finsler tensor fields.*

If we take $N=\overset{0}{N}$, we have Sanini's result [23]:

**Theorem 3.8.** *Let* $\overset{0}{N}$ *be a given non-linear connection. Then the set of all metrical Finsler connections* $F\Gamma(\overset{0}{N})$ *is given by*

$$F^i_{jk}=\overset{0}{F}{}^i_{jk}+\frac{1}{2}g^{im}g_{mj\overset{0}{|}k}+\Omega^{ir}_{sj}X^s_{rk}\,,\qquad C^i_{jk}=\overset{0}{C}{}^i_{jk}+\frac{1}{2}g^{im}g_{mj}\overset{0}{|}_k+\Omega^{ir}_{sj}Y^s_{rk}\,,\tag{3.8}$$

*where* $\overset{0}{F\Gamma}$ *is a fixed Finsler connection* $F\Gamma(\overset{0}{N})$ *and* $X^i_{jk}$, $Y^i_{jk}$ *are arbitrary Finsler tensor fields.*

If we take a metrical Finsler connection (e.g. $C\Gamma$) as $\overset{0}{F\Gamma}$ in the above theorem, we have

**Theorem 3.9.** *The set of all metrical Finsler connections* $F\Gamma\,(\overset{c}{N})$ *is given by*

$$F^i_{jk}=\overset{c}{F}{}^i_{jk}+\Omega^{ir}_{sj}X^s_{rk}\,,\qquad C^i_{jk}=\overset{c}{C}{}^i_{jk}+\Omega^{ir}_{sj}Y^s_{rk}\,,\tag{3.9}$$

*where* $X^i_{jk}$, $Y^i_{jk}$ *are arbitrary Finsler tensor fields.*

## § 4. Some special classes of metrical Finsler connections.

**4.1.** We shall here try to replace the arbitrary tensor fields $X^i_{jk}$, $Y^i_{jk}$ in Theorem 3.4 by the torsion tensor fields $T^i_{jk}$, $S^i_{jk}$. We put

$$\left.\begin{aligned} T^{*}{}^i_{jk} &= \frac{1}{2}g^{il}\,(g_{lh}T^h_{jk}-g_{jh}T^h_{lk}+g_{kh}T^h_{jl})\,,\\ S^{*}{}^i_{jk} &= \frac{1}{2}g^{il}(g_{lh}S^h_{jk}-g_{jh}S^h_{lk}+g_{kh}S^h_{jl})\,. \end{aligned}\right\}\tag{4.1}$$

Since $\overset{c}{T}{}^i_{jk}=0$, we have from the second of (3.5)

$$T^i_{jk}=\mathfrak{A}_{jk}\{F^i_{jk}\}=\mathfrak{A}_{jk}\{\overset{c}{C}{}^i_{jm}X^m_k+\Omega^{ir}_{sj}X^s_{rk}\}\,.\tag{4.2}$$

Substituting in (4.1) from (4.2), we can express $\Omega^{ir}_{sj}X^s_{rk}$ in terms of $X^i_k$ and $T^i_{jk}$:

$$\Omega^{ir}_{sj}X^s_{rk}=\overset{c}{C}{}^i_{km}X^m_j-\overset{c}{C}_{jkm}g^{il}X^m_l+T^{*}{}^i_{jk}\,.\tag{4.3}$$

Conversely, if for any alternate Finsler tensor field $T^i_{jk}$ we see (4.3) as a system of Obata's equations in $X^i_{jk}$, the compatibility condition (2.6) is easily verified. So, $T^i_{jk}$ may be arbitrarily given instead of $X^i_{jk}$. As to $Y^i_{jk}$ the same argument holds and we have

**Theorem 4.1.** *Let $T^i_{jk}$, $S^i_{jk}$ be given alternate Finsler tensor fields. Then the set of all metrical Finsler connections having $T^i_{jk}$, $S^i_{jk}$ as the torsion tensor fields $T$, $S^1$ is given by*

$$\left.\begin{aligned} N^i_k &= \overset{c}{N}{}^i_k - X^i_k\,, \\ F^i_{jk} &= \overset{c}{F}{}^i_{jk} + \overset{c}{C}{}^i_{jm} X^m_k + \overset{c}{C}{}^i_{km} X^m_j - \overset{c}{C}_{jkm} g^{il} X^m_l + T^{*i}_{jk}\,, \\ C^i_{jk} &= \overset{c}{C}{}^i_{jk} + S^{*i}_{jk}\,, \end{aligned}\right\} \tag{4.4}$$

*where $X^i_k$ is an arbitrary Finsler tensor field and $T^{*i}_{jk}$, $S^{*i}_{jk}$ are given by* (4.1).

**4.2.** In the same way as in §3.3 the arbitrary Finsler tensor field $X^i_k$ in Theorem 4.1 may be replaced by the deflection tensor field $D^i_k$.

**Theorem 4.2.** *Let $D^i_k$, $T^i_{jk}$, $S^i_{jk}$ be given Finsler tensor fields and assume that $T^i_{jk}$ and $S^i_{jk}$ are alternate. Then there exists a unique metrical Finsler connection $F\Gamma\,(D)$ having $T^i_{jk}$, $S^i_{jk}$ as the torsion tensor fields $T$, $S^1$. It is given by* (4.4) *with*

$$X^i_k = D^i_k + \overset{c}{C}{}^i_{km}(g^{ml} T^r_{0l} y_r - D^m_0) - T^{*i}_{0k}\,. \tag{4.5}$$

**Theorem 4.3.** *Let $T^i_{jk}$, $S^i_{jk}$ be given alternate Finsler tensor fields. Then there exists a unique metrical F-connection having $T^i_{jk}$, $S^i_{jk}$ as the torsion tensor fields $T$, $S^1$. It is given by* (4.4) *with*

$$X^i_k = \overset{c}{C}{}^i_{km} g^{ml} T^r_{0l} y_r - T^{*i}_{0k}\,. \tag{4.6}$$

**Theorem 4.4.** *There exists a unique metrical Finsler connection with the properties $D^i_k=0$, $T^i_{jk}=0$, $S^i_{jk}=0$, that is, $N^i_k=F^i_{0k}$, $F^i_{jk}=F^i_{kj}$, $C^i_{jk}=C^i_{kj}$. This is the Cartan connection $C\Gamma$.*

This characterization of the Cartan connection is due to M. Matsumoto [13], in which he noted that the Cartan connection has only one essential torsion $C^i_{jk}$, because of $T^i_{jk}=S^i_{jk}=0$, $R^i_{kl}=R^i_{0kl}$, $P^i_{kl}=P^i_{0kl}$, and gave an example of a metrical Finsler connection with many torsions, which is obtained from Theorem 4.2 by taking

$$D^i_k = L^{-1}(L^2\delta^i_k - y^i y_k)\,, \qquad T^i_{jk} = L^{-1}(\delta^i_j y_k - \delta^i_k y_j)\,, \qquad S^i_{jk} = L^{-2}(\delta^i_j y_k - \delta^i_k y_j)\,.$$

**4.3.** Paying attention to the non-linear connection, Theorem 4.1 is also restated as

**Theorem 4.5.** *Let $N$ be a given non-linear connection, and let $T^i_{jk}$, $S^i_{jk}$ be given alternate Finsler tensor fields. Then there exists a unique metrical Finsler connection $F\Gamma\,(N)$ having $T^i_{jk}$, $S^i_{jk}$ as the torsion tensor fields $T$, $S^1$. It is given by*

$$F^i_{jk} = \overset{c}{F}{}^i_{jk} + \overset{c}{C}{}^i_{jm} X^m_k + \overset{c}{C}{}^i_{km} X^m_j - \overset{c}{C}_{jkm} g^{il} X^m_l + T^{*i}_{jk}\,, \qquad C^i_{jk} = \overset{c}{C}{}^i_{jk} + S^{*i}_{jk}\,, \tag{4.7}$$

*where $X^i_k = \overset{c}{N}{}^i_k - N^i_k$ and $T^{*i}_{jk}$, $S^{*i}_{jk}$ are given by* (4.1).

*The former of* (4.7) *is also written as*

$$F^i_{jk} = \gamma^i_{jk} - \overset{c}{C}{}^i_{jm} N^m_k - \overset{c}{C}{}^i_{km} N^m_j + \overset{c}{C}_{jkm} g^{il} N^m_l + T^{*i}_{jk}\,. \tag{4.8}$$

Especially, if we take $N=\overset{c}{N}$, we have

**Theorem 4.6.** *Let $T^i_{jk}$, $S^i_{jk}$ be given alternate Finsler tensor fields. Then there exists a unique metrical Finsler connection $F\Gamma(\overset{c}{N})$ having $T^i_{jk}$, $S^i_{jk}$ as the torsion tensor fields $T$, $S^1$. It is given by*

$$F^i_{jk} = \overset{c}{F}{}^i_{jk} + T^{*i}_{jk}, \qquad C^i_{jk} = \overset{c}{C}{}^i_{jk} + S^{*i}_{jk}, \tag{4.9}$$

*where $T^{*i}_{jk}$, $S^{*i}_{jk}$ are given by* (4.1).

**Theorem 4.7.** *There exists a unique metrical Finsler connection $F\Gamma(\overset{c}{N})$ whose torsion tensor fields $T$, $S^1$ vanish. This is the Cartan connection.*

From (4.9) we have $D^i_k = T^{*i}_{0k}$. Conversely, the substitution in (4.5) from $D^i_k = T^{*i}_{0k}$ yields $X^i_k = 0$. Thus we have

**Theorem 4.8.** *A metrical Finsler connection having $T^i_{jk}$ as the torsion tensor field $T$ has the Cartan non-linear connection $\overset{c}{N}$ if and only if the deflection tensor field is given by*

$$D^i_k = T^{*i}_{0k}. \tag{4.10}$$

**4.4.** A Finsler connection is called *semi-symmetric* if the torsion tensor fields $T^i_{jk}$, $S^i_{jk}$ have the form

$$T^i_{jk} = (T_j\delta^i_k - T_k\delta^i_j)/(n-1), \qquad S^i_{jk} = (S_j\delta^i_k - S_k\delta^i_j)/(n-1), \tag{4.11}$$

where $T_j = T^i_{ji}$ and $S_j = S^i_{ji}$ will be called the *h- and v-torsion vector fields* respectively.

The Cartan connection is considered as a $F$-connection, or a Finsler connection $F\Gamma(\overset{c}{N})$, which is semi-symmetric and metrical and whose torsion vector fields $T_j$, $S_j$ vanish.

If we put $\sigma_j = T_j/(n-1)$, $\tau_j = S_j/(n-1)$, that is, $T^i_{jk} = \mathfrak{A}_{jk}\{\sigma_j\delta^i_k\}$, $S^i_{jk} = \mathfrak{A}_{jk}\{\tau_j\delta^i_k\}$, then $T^{*i}_{jk}$, $S^{*i}_{jk}$ given by (4.1) become

$$\left.\begin{aligned} T^{*i}_{jk} &= \sigma_j\delta^i_k - g_{jk}\sigma^i = 2\Omega^{ir}_{kj}\sigma_r \qquad (\sigma^i = g^{ir}\sigma_r), \\ S^{*i}_{jk} &= \tau_j\delta^i_k - g_{jk}\tau^i = 2\Omega^{ir}_{kj}\tau_r \qquad (\tau^i = g^{ir}\tau_r). \end{aligned}\right\} \tag{4.12}$$

From Theorems 4.3 and 4.6 we have the generalizations of Hashiguchi's results [8], in which the semi-symmetry was defined as $T^i_{jk} = \mathfrak{A}_{jk}\{\delta^i_j\sigma_k\}$, $S^i_{jk} = 0$:

**Theorem 4.9.** *The set of all semi-symmetric metrical F-connections is given by*

$$\left.\begin{aligned} N^i_k &= \overset{c}{N}{}^i_k + L^2\overset{c}{C}{}^i_{km}\sigma^m + \sigma_0\delta^i_k - y_k\sigma^i, \\ F^i_{jk} &= \overset{c}{F}{}^i_{jk} - L^2(\overset{c}{C}{}^i_{jm}\overset{c}{C}{}^m_{ks} + \overset{c}{C}{}^i_{km}\overset{c}{C}{}^m_{js} - \overset{c}{C}{}^i_{sm}\overset{c}{C}{}^m_{jk})\sigma^s \\ &\quad + (\overset{c}{C}{}^i_{js}y_k + \overset{c}{C}{}^i_{ks}y_j - \overset{c}{C}_{jks}y^i)\sigma^s - \overset{c}{C}{}^i_{jk}\sigma_0 + \sigma_j\delta^i_k - g_{jk}\sigma^i, \\ C^i_{jk} &= \overset{c}{C}{}^i_{jk} + \tau_j\delta^i_k - g_{jk}\tau^i, \end{aligned}\right\} \tag{4.13}$$

*where $\sigma_j$, $\tau_j$ are arbitrary Finsler vector fields.*

**Theorem 4.10.** *The set of all semi-symmetric metrical Finsler connections $F\Gamma(\overset{c}{N})$ is given by*

$$F^i_{jk} = \overset{c}{F}{}^i_{jk} + 2\Omega^{ir}_{kj}\sigma_r, \qquad C^i_{jk} = \overset{c}{C}{}^i_{jk} + 2\Omega^{ir}_{kj}\tau_r, \tag{4.14}$$

*where $\sigma_j$, $\tau_j$, are arbitrary Finsler vector fields.*

A Finsler connection obtained by putting $\tau_j=0$ from Theorem 4.9 is called a *Wagner connection* [8], and was used to study Wagner's generalized Berwald space [25]. It has moreover made the important contribution to the conformal theory of Finsler metrics [10, 11]. On the other hand, in a Finsler connection of Theorem 4.10 the deflection tensor field is alive: $D^i_k=\sigma_0\delta^i_k-y_k\sigma^i$. The simplicity will, however, lead us in §6 to the discussion of the transformations of such connections.

## § 5. The group of transformations of metrical Finsler connections.

**5.1.** Let $F\Gamma$, $F\bar{\Gamma}$ be any two general metrical Finsler connections. If we see $F\Gamma$, $F\bar{\Gamma}$ as $F\overset{0}{\Gamma}$, $F\Gamma$ in Proposition 3.1, the given $F\Gamma$, $F\bar{\Gamma}$ are expressed as

$$\bar{N}^i_k=N^i_k-A^i_k\,,\qquad \bar{F}^i_{jk}=F^i_{jk}+C^i_{jm}A^m_k-B^i_{jk}\,,\qquad \bar{C}^i_{jk}=C^i_{jk}-D^i_{jk} \tag{5.1}$$

for some uniquely determined Finsler tensor fields $A^i_k$, $B^i_{jk}$, $D^i_{jk}$ satisfying

$$\Omega^{*}{}^{ir}_{sj}B^s_{rk}=0\,,\qquad \Omega^{*}{}^{ir}_{sj}D^s_{rk}=0\,. \tag{5.2}$$

Conversely, given Finsler tensor fields $A^i_k$, $B^i_{jk}$, $D^i_{jk}$ satisfying (5.2), the above (5.1) is thought to be a transformation of a Finsler connection $F\Gamma$ to a Finsler connection $F\bar{\Gamma}$. Then it transforms a metrical one to a metrical one. We shall denote this transformation by $t(A^i_k, B^i_{jk}, D^i_{jk})$.

Let $\overset{m}{\mathfrak{T}}$ be the set of all such transformations $t(A^i_k, B^i_{jk}, D^i_{jk})$. For any $t(A^i_k, B^i_{jk}, D^i_{jk})$, $t(\bar{A}^i_k, \bar{B}^i_{jk}, \bar{D}^i_{jk})\in\overset{m}{\mathfrak{T}}$ their product becomes

$$t(\bar{A}^i_k, \bar{B}^i_{jk}, \bar{D}^i_{jk})\circ t(A^i_k, B^i_{jk}, D^i_{jk})=t(A^i_k+\bar{A}^i_k, B^i_{jk}+\bar{B}^i_{jk}+D^i_{jm}\bar{A}^m_k, D^i_{jk}+\bar{D}^i_{jk})\,, \tag{5.3}$$

which belongs to $\overset{m}{\mathfrak{T}}$. And, any $t(A^i_k, B^i_{jk}, D^i_{jk})\in\overset{m}{\mathfrak{T}}$ has the inverse $t(-A^i_k, -B^i_{jk}+D^i_{jm}A^m_k, -D^i_{jk})$ in $\overset{m}{\mathfrak{T}}$. Thus we have

**Theorem 5.1.** *The set* $\overset{m}{\mathfrak{T}}$ *of all transformations* $t(A^i_k, B^i_{jk}, D^i_{jk})$ *given by* (5.1) *with* (5.2), *together with the mapping product, is a group. This group acts on the set of all Finsler connections effectively, and acts on the set of all metrical Finsler connections transitively.*

**5.2.** The group $\overset{m}{\mathfrak{T}}$ is a subgroup of the general group $\mathfrak{T}$ of transformations of Finsler connections [18]. On the other hand, it has five remarkable subgroups:

$$\overset{m}{\mathfrak{T}}_N=\{t(0, B^i_{jk}, D^i_{jk})\in\overset{m}{\mathfrak{T}}\}\,,\qquad \overset{m}{\mathfrak{T}}_C=\{t(A^i_k, B^i_{jk}, 0)\in\overset{m}{\mathfrak{T}}\}\,,$$

$$\overset{m}{\mathfrak{T}}_\Gamma=\{t(A^i_k, 0, 0)\in\overset{m}{\mathfrak{T}}\}\,,\qquad \overset{m}{\mathfrak{T}}_{NC}=\{t(0, B^i_{jk}, 0)\in\overset{m}{\mathfrak{T}}\}\,,\qquad \overset{m}{\mathfrak{T}}_{NF}=\{t(0, 0, D^i_{jk})\in\overset{m}{\mathfrak{T}}\}\,.$$

These are all abelian, and we have from (5.1), (1.1)

**Proposition 5.1.** $\overset{m}{\mathfrak{T}}_\Gamma$ *and* $\overset{m}{\mathfrak{T}}_N$ *preserve* $\Gamma$ *and* $N$ *respectively.* $\overset{m}{\mathfrak{T}}_{NF}$ *preserves* $N$ *and* $\Gamma_V$ *and so* $\Gamma^h$, *and* $\overset{m}{\mathfrak{T}}_C$ *preserves* $C$ *and so* $\Gamma^v$. *And,* $\overset{m}{\mathfrak{T}}_{NC}$ *preserves* $N$ *and* $C$.

$\overset{m}{\mathfrak{T}}_N$, $\overset{m}{\mathfrak{T}}_C$ and so $\overset{m}{\mathfrak{T}}_{NC}$ are normal subgroups of $\overset{m}{\mathfrak{T}}$ [18]. $\overset{m}{\mathfrak{T}}_\Gamma$ and $\overset{m}{\mathfrak{T}}_{NF}$ are not normal in $\overset{m}{\mathfrak{T}}$, but normal in $\overset{m}{\mathfrak{T}}_C$ and $\overset{m}{\mathfrak{T}}_N$ respectively.

Any element of $\overset{m}{\mathfrak{T}}$ can be expressed as the product of elements of $\overset{m}{\mathfrak{T}}_\Gamma$ and $\overset{m}{\mathfrak{T}}_N$:

$$t(A^i_k, B^i_{jk}, D^i_{jk}) = t(0, B^i_{jk}, D^i_{jk})\circ t(A^i_k, 0, 0)\,, \tag{5.4}$$

which is not necessarily equal to $t\,(A^i_k,\ 0,\ 0)\circ t(0,\ B^i_{jk},\ D^i_{jk})$. Thus we have

**Theorem 5.2.** *The group* $\overset{m}{\mathfrak{T}}$ *is the semi-direct product of its subgroups* $\overset{m}{\mathfrak{T}}_\Gamma$ *and* $\overset{m}{\mathfrak{T}}_N$: $\overset{m}{\mathfrak{T}}=\overset{m}{\mathfrak{T}}_\Gamma\overset{\cdot}{\times}\overset{m}{\mathfrak{T}}_N$.

Also, any element of $\overset{m}{\mathfrak{T}}$ can be expressed as the product of elements of $\overset{m}{\mathfrak{T}}_C$ and $\overset{m}{\mathfrak{T}}_{NF}$:

$$t(A^i_k, B^i_{jk}, D^i_{jk}) = t(0, 0, D^i_{jk})\circ t(A^i_k, B^i_{jk}, 0)\,, \tag{5.5}$$

which is not necessarily equal to $t\,(A^i_k,\ B^i_{jk},\ 0)\circ t\,(0,\ 0,\ D^i_{jk})$. Thus we have

**Theorem 5.3.** *The group* $\overset{m}{\mathfrak{T}}$ *is the semi-direct product of its sbugroups* $\overset{m}{\mathfrak{T}}_C$ *and* $\overset{m}{\mathfrak{T}}_{NF}$: $\overset{m}{\mathfrak{T}}=\overset{m}{\mathfrak{T}}_C\overset{\cdot}{\times}\overset{m}{\mathfrak{T}}_{NF}$.

**5.3.** By Proposition 5.1 we can consider that Theorem 5.2 and Theorem 5.3 correspond to Matsumoto's first and second definitions $(\Gamma, N)$ and $(\Gamma^h, \Gamma^v)$ of a Finsler connection respectively.

As is easily seen, $\overset{m}{\mathfrak{T}}_N$ is the direct product of its subgroups $\overset{m}{\mathfrak{T}}_{NC}$ and $\overset{m}{\mathfrak{T}}_{NF}$: $\overset{m}{\mathfrak{T}}_N=\overset{m}{\mathfrak{T}}_{NC}\times\overset{m}{\mathfrak{T}}_{NF}$, and $\overset{m}{\mathfrak{T}}_C$ is the direct product of its subgroups $\overset{m}{\mathfrak{T}}_\Gamma$ and $\overset{m}{\mathfrak{T}}_{NC}$: $\overset{m}{\mathfrak{T}}_C=\overset{m}{\mathfrak{T}}_\Gamma\times\overset{m}{\mathfrak{T}}_{NC}$. Corresponding to his third definition $F\Gamma=(\Gamma_V,\ N,\ \Gamma^v)$ we have

**Theorem 5.4.** *The group* $\overset{m}{\mathfrak{T}}$ *is expressed as*

$$\overset{m}{\mathfrak{T}} = (\overset{m}{\mathfrak{T}}_\Gamma\times\overset{m}{\mathfrak{T}}_{NC})\overset{\cdot}{\times}\overset{m}{\mathfrak{T}}_{NF} = \overset{m}{\mathfrak{T}}_\Gamma\overset{\cdot}{\times}(\overset{m}{\mathfrak{T}}_{NC}\times\overset{m}{\mathfrak{T}}_{NF})\,.$$

*Corresponding to these decompositions the following commutative diagram holds for every transformation of metrical Finsler connections* $t(A^i_k,\ B^i_{jk},\ D^i_{jk})$: $(N,\ F,\ C)\to(\bar N,\ \bar F,\ \bar C)$.

$$\begin{tikzcd}
(N,F,C) \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, "\lambda"] & (\bar N,F^1,C) \arrow[r, "\gamma"] \arrow[d, "\beta"'] \arrow[dr, "\mu"] & (\bar N,F^1,\bar C) \arrow[d, "\beta"'] \\
(N,F^2,C) \arrow[r, "\alpha"] & (\bar N,\bar F,C) \arrow[r, "\gamma"] & (\bar N,\bar F,\bar C)
\end{tikzcd}$$

$$\lambda=t(A^i_k, B^i_{jk}, 0),\qquad \mu=t(0, B^i_{jk}, D^i_{jk}),$$
$$\alpha=t(A^i_k, 0, 0),\qquad \beta=t(0, B^i_{jk}, 0),\qquad \gamma=t(0, 0, D^i_{jk})$$

## § 6. The group of transformations of semi-symmetric metrical Finsler connections.

**6.1.** We apply the preceding considerations to the semi-symmetric case, and we will determine some invariants [18].

Let $N$ be a non-linear connection. Then any semi-symmetric metrical Finsler connection $F\Gamma\ (N)$ is given by (4.7) with (4.12). Hence two semi-symmetric metrical Finsler connections $F\Gamma\ (N)$, $F\bar\Gamma\ (N)$ are related in the form

$$\bar N^i_k = N^i_k\,,\qquad \bar F^i_{jk} = F^i_{jk}+2\Omega^{ir}_{kj}\sigma_r\,,\qquad \bar C^i_{jk} = C^i_{jk}+2\Omega^{ir}_{kj}\tau_r \tag{6.1}$$

for some uniquely determined Finsler vector fields $\sigma_j$, $\tau_j$. Conversely, given Finsler vector fields $\sigma_j$, $\tau_j$, the above (6.1) is thought to be a transformation of Finsler connections preserving the non-linear connection. Then it transforms a semi-symmetric metrical one to a semi-symmetric metrical one. We shall denote this transformation by $t(\sigma_j, \tau_j)$, Let $\overset{s}{\mathfrak{T}}_N$ be the set of all such transformations. Then it holds

$$t(\bar{\sigma}_j, \bar{\tau}_j)\circ t(\sigma_j, \tau_j) = t(\sigma_j+\bar{\sigma}_j, \tau_j+\bar{\tau}_j)$$

and we have

**Theorem 6.1.** *The set $\overset{s}{\mathfrak{T}}_N$ of all transformations $t(\sigma_j, \tau_j)$ given by* (6.1), *together with the mapping product, is an abelian group. This group acts on the set of all Finsler connections effectively, and for each non-linear connection $N$ it acts on the set of all semi-symmetric metrical Finsler connections $F\Gamma(N)$ transitively.*

*And, $\overset{s}{\mathfrak{T}}_N$ is the direct product of its subgroups $\overset{s}{\mathfrak{T}}_{NC}=\{t(\sigma_j,0)\in\overset{s}{\mathfrak{T}}_N\}$ and $\overset{s}{\mathfrak{T}}_{NF}=\{t(0,\tau_j)\in\overset{s}{\mathfrak{T}}_N\}$: $\overset{s}{\mathfrak{T}}_N=\overset{s}{\mathfrak{T}}_{NC}\times\overset{s}{\mathfrak{T}}_{NF}$.*

**6.2.** In order to find invariants of the group $\overset{s}{\mathfrak{T}}_N$, let us consider the transformation formulas of the curvature tensor fields by a transformation of Finsler connections

$$\bar{N}^i_k = N^i_k\,, \qquad \bar{F}^i_{jk} = F^i_{jk}-B^i_{jk}\,, \qquad \bar{C}^i_{jk} = C^i_{jk}-D^i_{jk}\,. \tag{6.2}$$

From Proposition 3.1 of [9] or directly from (1.9), (1.10), (1.11) we have

**Proposition 6.1.** *By a transformation* (6.2) *of Finlser connections $F\Gamma$, $F\bar{\Gamma}$ the curvature tensor fields are transformed as follows*:

$$\bar{R}^i_{jkl} = R^i_{jkl}-D^i_{jm}R^m_{kl}-B^i_{jm}T^m_{kl}+\mathfrak{A}_{kl}\{-B^i_{jk|l}+B^m_{jk}B^i_{ml}\}\,, \tag{6.3}$$

$$\bar{P}^i_{jkl} = P^i_{jkl}-D^i_{jm}P^m_{kl}-B^i_{jm}C^m_{kl}-B^i_{jk}|_l+D^i_{jl|k}+B^m_{jk}D^i_{ml}-D^m_{jl}B^i_{mk}\,, \tag{6.4}$$

$$\bar{S}^i_{jkl} = S^i_{jkl}-D^i_{jm}S^m_{kl}+\mathfrak{A}_{kl}\{-D^i_{jk}|_l+D^m_{jk}D^i_{ml}\}\,. \tag{6.5}$$

We can eliminate the torsion tensor fields $R^i_{kl}$ and $P^i_{kl}$ from (6.3), (6.4) and obtain the Finsler tensor fields which have the transformation formulas similar to (6.5).

**Proposition 6.2.** *The Finsler tensor fields defined by*

$$K^i_{jkl} = R^i_{jkl}-C^i_{jm}R^m_{kl}\,, \tag{6.6}$$

$$\mathfrak{P}^i_{jkl} = \mathfrak{A}_{kl}\{P^i_{jkl}-C^i_{jm}\partial N^m_k/\partial y^l\} \tag{6.7}$$

*are transformed by the transformation* (6.2) *as follows*:

$$\bar{K}^i_{jkl} = K^i_{jkl}-B^i_{jm}T^m_{kl}+\mathfrak{A}_{kl}\{-B^i_{jk|l}+B^m_{jk}B^i_{ml}\}\,, \tag{6.8}$$

$$\bar{\mathfrak{P}}^i_{jkl} = \mathfrak{P}^i_{jkl}-B^i_{jm}S^m_{kl}-D^i_{jm}T^m_{kl}+\mathfrak{A}_{kl}\{-B^i_{jk}|_l-D^i_{jk|l}+B^m_{jk}D^i_{ml}+D^m_{jk}B^i_{ml}\}\,. \tag{6.9}$$

**6.3.** Now, we shall treat the transformation (6.1) of semi-symmetric metrical Finsler connections. Substituting in (6.8), (6.5), (6.9) from

$$B^i_{jk} = -2\Omega^{ir}_{kj}\sigma_r\,, \qquad D^i_{jk} = -2\Omega^{ir}_{kj}\tau_r\,, \qquad T^i{}_l = \mathfrak{A}_{kl}\{T_k\delta^i_l\}/(n-1)\,,$$

$$S^i_{kl} = \mathfrak{A}_{kl}\{S_k\delta^i_l\}/(n-1)\,,$$

we have

**Proposition 6.3.** *The Finsler tensor fields $K^i_{jkl}$, $S^i_{jkl}$, $\mathfrak{P}^i_{jkl}$ of a semi-symmetric metrical Finsler connection $F\Gamma$ are transformed by the transformation* (6.1) *of $F\Gamma$ to $F\bar{\Gamma}$ as follows*:

$$\bar{K}^i_{jkl} = K^i_{jkl} + 2\mathfrak{A}_{kl}\{\Omega^{ir}_{kj}\sigma_{rl}\}\ , \tag{6.10}$$

$$\bar{S}^i_{jkl} = S^i_{jkl} + 2\mathfrak{A}_{kl}\{\Omega^{ir}_{kj}\tau_{rl}\}\ , \tag{6.11}$$

$$\bar{\mathfrak{P}}^i_{jkl} = \mathfrak{P}^i_{jkl} + 2\mathfrak{A}_{kl}\{\Omega^{ir}_{kj}\rho_{rl}\}\ , \tag{6.12}$$

*where we put for the h- and v-torsion vector fields $T_j$, $S_j$ of $F\Gamma$*

$$\sigma_{rl} = \sigma_{r|l} - \sigma_r\sigma_l + \frac{1}{2}g_{rl}\sigma - \sigma_r T_l/(n-1) \qquad (\sigma = g^{rs}\sigma_r\sigma_s)\ , \tag{6.13}$$

$$\tau_{rl} = \tau_r|_l - \tau_r\tau_l + \frac{1}{2}g_{rl}\tau - \tau_r S_l/(n-1) \qquad (\tau = g^{rs}\tau_r\tau_s)\ , \tag{6.14}$$

$$\rho_{rl} = \sigma_r|_l + \tau_{r|l} - (\sigma_r\tau_l + \tau_r\sigma_l) + g_{rl}\rho - (\tau_r T_l + \sigma_r S_l)/(n-1) \qquad (\rho = g^{rs}\sigma_r\tau_s)\ . \tag{6.15}$$

If we pay attention to the type of the transformation formulas shown in the above proposition, we have the following important invariants similar to the Weyl conformal curvature tensor field by a well-known elimination method.

**Theorem 6.2.** *For $n>2$ the following Finsler tensor fields $H^i_{jkl}$, $M^i_{jkl}$, $N^i_{jkl}$ of semi-symmetric metrical Finsler connections are invariants of the group $\overset{s}{\mathfrak{T}}_N$*:

$$H^i_{jkl} = K^i_{jkl} + 2\mathfrak{A}_{kl}\{\Omega^{ir}_{kj}(K_{rl} - Kg_{rl}/2(n-1))\}/(n-2)\ , \tag{6.16}$$

$$M^i_{jkl} = S^i_{jkl} + 2\mathfrak{A}_{kl}\{\Omega^{ir}_{kj}(S_{rl} - Sg_{rl}/2(n-1))\}/(n-2)\ , \tag{6.17}$$

$$N^i_{jkl} = \mathfrak{P}^i_{jkl} + 2\mathfrak{A}_{kl}\{\Omega^{ir}_{kj}(\mathfrak{P}_{rl} - \mathfrak{P}g_{rl}/2(n-1))\}/(n-2)\ , \tag{6.18}$$

*where* $\quad K_{jk} = K^i_{jki}\ ,\quad S_{jk} = S^i_{jki}\ ,\quad \mathfrak{P}_{jk} = \mathfrak{P}^i_{jki}\ ,\quad K = g^{jk}K_{jk}\ ,$
$S = g^{jk}S_{jk}\ ,\quad \mathfrak{P} = g^{jk}\mathfrak{P}_{jk}\ .$

**6.4.** In the following, let us assume the non-linear connection to be the Cartan one $\overset{c}{N}$ and consider some properties of the invariants $H^i_{jkl}$, $M^i_{jkl}$, $N^i_{jkl}$ of semi-symmetric metrical Finsler connections $F\Gamma(\overset{c}{N})$. Since $H^i_{jkl} = \overset{c}{H}{}^i_{jkl}$, $M^i_{jkl} = \overset{c}{M}{}^i_{jkl}$, $N^i_{jkl} = \overset{c}{N}{}^i_{jkl}$, it is sufficient to study the properties with respect to $C\Gamma$.

The tagent space $M_x$ at $x\epsilon M$ is considered to be a Riemannian space with $g_{ij}(x, y)$ as the metric tensor field. Since $\overset{c}{C}{}^i_{jk}$ is the Christoffel symbol from (1.16), $\overset{c}{S}{}^i_{jkl}$ is the curvature tensor field. Hence $\overset{c}{M}{}^i_{jkl}$ is nothing but the Weyl conformal curvature tensor field of the Riemannian space $M_x$. From the invariance of $M^i_{jkl}$ we have

**Theorem 6.3.** *Let the Finsler connection be a semi-symmetric metrical one $F\Gamma(\overset{c}{N})$. If $n=3$, then $M^i_{jkl}=0$. For $n>3$ $M^i_{jkl}=0$ if and only if at each point $x\epsilon M$ the tangent Riemannian space $M_x$ is conformally flat.*

Since $\partial\overset{c}{N}{}^i_k/\partial y^l = \partial\overset{c}{N}{}^i_l/\partial y^k$ from (1.14), we have $\mathfrak{P}^i_{jkl} = \mathfrak{A}_{kl}\{P^i_{jkl}\}$ for a Finsler connection $F\Gamma(\overset{c}{N})$. As is easily seen [7], we have $\overset{c}{\mathfrak{P}}{}^i_{jkl} = -\overset{c}{S}{}^i_{jkl\overset{c}{|}0}$, which implies $\overset{c}{N}{}^i_{jkl} = -\overset{c}{M}{}^i_{jkl\overset{c}{|}0}$. Hence we have

**Theorem 6.4.** *For a semi-symmetric metrical Finsler connection $F\Gamma(\overset{c}{N})$ it holds*

$$N^i_{jkl} = -M^i_{jkl\overset{c}{|}0}\,, \tag{6.19}$$

*and* $M^i_{jkl}=0$ *implies* $N^i_{jkl}=0$.

Next, with respect to $C\Gamma$ we have from (2.9), (2.12) $\Omega^{*ir}_{sj}\overset{c}{K}{}^s_{rkl}=-\overset{c}{C}{}^i_{jm}\overset{c}{R}{}^m_{kl}$, $\overset{c}{K}{}^i_{ikl}=-\overset{c}{C}_m\overset{c}{R}{}^m_{kl}$, where $\overset{c}{C}_m=\overset{c}{C}{}^i_{im}$. On the other hand, $\overset{c}{K}{}^i_{jkl}$ is the curvature tensor field of the Rund connection $R\Gamma$. As one of the Bianchi identities we have $S_{jkl}\{\overset{c}{K}{}^i_{jkl}\}=0$, from which it follows $\mathfrak{A}_{kl}\{\overset{c}{K}_{kl}\}=\overset{c}{C}_m\overset{c}{R}{}^m_{kl}$. As to $\overset{c}{S}{}^i_{jkl}$, $\overset{c}{\mathfrak{P}}{}^i_{jkl}$ we have the simliar relations. Since $C_m=C^i_{im}$ and $R^i_{kl}$ are invariants of $\overset{s}{\mathfrak{T}}_N$, we have from (2.3), (2.4)

**Theorem 6.5.** *For semi-symmetric metrical Finsler connection $F\Gamma(\overset{c}{N})$ the invariants $H^i_{jkl}$, $M^i_{jkl}$, $N^i_{jkl}$ have the following properties*:

$$\Omega^{*ir}_{sj}H^s_{rkl} = -\overset{c}{C}{}^i_{jm}R^m_{kl}\,, \qquad \Omega^{*ir}_{sj}M^s_{rkl} = 0\,, \qquad \Omega^{*ir}_{sj}N^s_{rkl} = 0\,, \tag{6.20}$$

$$H^i_{ikl} = -C_mR^m_{kl}\,, \qquad M^i_{ikl} = 0\,, \qquad N^i_{ikl} = 0\,, \qquad H^i_{jki} = 0\,, \qquad M^i_{jki} = 0\,, \quad N^i_{jki} = 0\,. \tag{6.21}$$

$$\mathfrak{S}_{jkl}\{H^i_{jkl}\} = -C_m\mathfrak{S}_{jkl}\{\delta^i_jR^m_{kl}\}/(n-2)\,, \qquad \mathfrak{S}_{jkl}\{M^i_{jkl}\} = 0\,, \qquad \mathfrak{S}_{jkl}\{N^i_{jkl}\} = 0\,. \tag{6.22}$$

It is noted from (6.21) that a Berwald space satisfying $H^i_{ikl}=0$ is just the space treated by G. Soós [24]. We shall finally give an example of the space satisfying $H^i_{jkl}=0$.

**Theorem 6.6.** *If the tensor field $K^i_{jkl}$ of a semi-symmetric metrical Finsler connection $F\Gamma(\overset{c}{N})$ has the isotropy property*

$$K^i_{jkl} = \lambda(x,y)(\delta^i_l g_{jk}-\delta^i_k g_{jl})\,, \tag{6.23}$$

*then* $H^i_{jkl}=0$.

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