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ADAPTIVE MESHES FOR THE GALERKIN'S METHOD

By

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Abstract

In this paper we consider adaptive meshes for the Galerkin's method to solve a two point boundary value problem. The interval is partitioned so that errors under the L^2-energy norm and the L^∞-norm are less than a wishful value. Some numerical results are given.

1. Introduction

Recently, many reseachers study the method of adaptive meshes for various approximate solutions of two point boundary value problems ([1], [2], [3], [4], [5], [6]). For example, Babuška and Rheinboldt published *a posteriori* error estimates which gave bases of adaptive meshes for the Galerkin's method and, using their estimates, they calculated examples of adaptive meshes ([1], [2], [3], [4]). Also the studies of adaptive meshes have been doing for the difference method and the collocation method ([5], [6]).

We shall consider adaptive meshes for the Galerkin's method to solve two point boundary value problem (3.1) in §3. Babuška and Rheinboldt considered adaptive meshes on condition that number of intervals used in the partition is fixed. In this paper, for a positive constant δ, we shall consider approach to optimal meshes which satisfy conditions

$$\|y-Y\|_{E(I)} \leq \delta$$

and

$$\|y-Y\|_{L^\infty(I)} \leq \delta\,,$$

where y is a genuine solution of (3.1) and Y is its Galerkin's approximation.

The examples are shown in §4. The actual errors are very small at the nodes against the wished errors. Also it isn't mentioned in this paper how we partition the interval at first. If the studies for these and the improvements of error estimates in §3 make progress in futuer, adaptive meshes mentioned in this paper will turn out to more significant meaning.

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2. Notation

Given real α and β with $\alpha<\beta$, let I be the open interval $\{x \in R^1 | \alpha<x<\beta\}$ and $\bar{I}$ its closure.

For an integer $k\geq 0$ let $C^k(\bar{I})$ be the space of k-times continuously differentiable functions on $\bar{I}$.

Also let $L^2(I)$ and $L^\infty(I)$ be the spaces of measurable functions f such that

$$\|f\|_{L^2(I)}=\left(\int_I f(x)^2dx\right)^{1/2} \qquad <+\infty$$

and

$$\|f\|_{L^\infty(I)}=\operatorname*{ess\,sup}_{x\in I}|f(x)| \qquad <+\infty ,$$

respectively. Further more let $(\cdot,\cdot)$ be the inner product in $L^2(I)$ and $H_0^1(I)$ be the space of locally integrable functions u such that $u,u' \in L^2(I)$ and $u(\alpha)=u(\beta)=0$. For each $u \in H_0^1(I)$, define

$$\|u\|_{H_0^1(I)}=(\|u\|^2_{L^2(I)}+\|u'\|^2_{L^2(I)})^{1/2} ,$$

which is the norm on $H_0^1(I)$.

Assume that functions a and b are in $C^0(\bar{I})$ and there are constants $\bar{a}$, $\underline{a}$ and $\bar{b}$ such that

$$\bar{a}\geq a(x)\geq \underline{a}>0 , \quad \bar{b}\geq b(x)\geq 0 , \qquad \forall x \in \bar{I} .$$

For these a and b, define for u and v in $H_0^1(I)$

$$B_I(u, v)=\int_I (au'v'+buv)dx ,$$

and

$$\|u\|_{E(I)}=(B_I(u, u))^{1/2}.$$

Then $B_I(u, v)$ is a bilinear form on $H_0^1(I)\times H_0^1(I)$. Also $\|\cdot\|_{E(I)}$ is a norm equivalent to the norm $\|\cdot\|_{H_0^1(I)}$ on $H_0^1(I)$.

On the interval $\bar{I}$ consider a partition

$$\Delta\ ;\ \alpha=x_0^\Delta<x_1^\Delta<\cdots<x_{m-1}^\Delta<x_m^\Delta=\beta , \qquad m=m(\Delta)\geq 1 ,$$

and introduce the notations

$$\begin{aligned} I_j(\Delta) &= (x_{j-1}^\Delta, x_j^\Delta) , \\ h_j(\Delta) &= x_j^\Delta-x_{j-1}^\Delta , \qquad j=1, \cdots, m \\ h(\Delta) &= \max_{1\leq j\leq m} h_j(\Delta) . \end{aligned}$$

Let $P_k(J)$ denote the space of polynomials of degree not greater than k restricted to a set $J(J\subset R^1)$. For the partition Δ and an integer $r\geq 1$, define

$$\mathscr{M}_\Delta^r=\{v \in C^0(\bar{I}) |\ v|_{I_j(\Delta)} \in P_r(I_j(\Delta)), j=1, \cdots, m; v(\alpha)=v(\beta)=0\} .$$

Obviously the relation $\mathscr{M}_\Delta^r\subset H_0^1(I)$ holds.

3. A Boundary value Problem and its Error Estimates

In this paper we consider the following two point boundary value problem:

$$
\begin{aligned}
&L[y] = -\frac{d}{dx}\left(a(x)\frac{dy}{dx}\right) + b(x)y = f(x)\,, \qquad x \in I = (0, 1)\,, \\
&y(0) = y(1) = 0\,,
\end{aligned}
\tag{3.1}
$$

where $a \in C^r(\bar{I})$, $b, f \in C^{r-1}(\bar{I})$ and, as before, assume that there are constants $\bar{a}$, $\underline{a}$ and $\bar{b}$ such that

$$\bar{a} \geq a(x) \geq \underline{a} > 0\,, \quad \bar{b} \geq b(x) \geq 0\,, \qquad \forall x \in \bar{I}\,. \tag{3.2}$$

From now on assume that (3.1) has a unique solution $y \in C^{r+1}(\bar{I})$ for each $f \in C^{r-1}(\bar{I})$.

Also for a partition Δ, let Y_Δ be the Galerkin's approximation to y determined by the relation

$$(aY'_\Delta, v') + (bY_\Delta, v) = (f, v)\,, \qquad \forall v \in \mathscr{M}^r_\Delta\,. \tag{3.3}$$

Then the following result is known by Babuška and Rheinboldt ([1]).

THEOREM 1. *Let y be the solution of* (3.1). *Also, for a partition Δ, let Y_Δ be the Galerkin's approximation to y determined by* (3.3). *Then the inequality*

$$\|y - Y_\Delta\|_{E(I)} \leq \left(\sum_{j=1}^{m} \frac{h_j(\Delta)^2 \mu_j^2}{\pi^2 a_{\min}^j}\right)^{1/2} (1 + O(h(\Delta))) \tag{3.4}$$

holds, where

$$
\left.
\begin{aligned}
\mu_j^2 &= \int_{x_{j-1}^\Delta}^{x_j^\Delta} r_j(x)^2\,dx\,, \\
r_j(x) &= L[Y_\Delta] - f\,, \qquad x \in I_j(\Delta) \\
a_{\min}^j &= \min_{x_{j-1}^\Delta \leq x \leq x_j^\Delta} |a(x)|
\end{aligned}
\right\}\; j = 1, \cdots, m\,.
$$

Using theorem 1, we shall show the following

THEOREM 2. *For the same assumptions in Theorem* 1, *there are positive constants C' and C'' such that*

$$\|y - Y_\Delta\|_{L\infty(I_i(\Delta))} \leq C'\left(\sum_{j=1}^{m} \frac{h_j(\Delta)^2 \mu_j^2}{\pi^2 a_{\min}^j}\right)^{1/2} + C''\|y^{(r+1)}\|_{L\infty(I_i(\Delta))} h_i(\Delta)^{r+1} \tag{3.5}$$

for each interval $I_i(\Delta)$.

Proof. Let $\zeta = y - Y_\Delta$, then

$$(a\zeta', v') + (b\zeta, v) = 0\,, \qquad v \in \mathscr{M}^r_\Delta\,.$$

Let $G(x, \xi)$ be the Green's function for (3.1); i.e.,

$$y(x) = (L[y], G(x, \cdot)) = \left(ay', \frac{\partial G}{\partial \xi}(x, \cdot)\right) + \left(by, G(x, \cdot)\right)$$

In particuler, this representation holds for $Y_\Delta \in H_0^1(I)$. Thus

$$\zeta(x_i^\Delta)=\left(a\zeta', \frac{\partial G}{\partial \xi}(x_i^\Delta, \cdot)\right)+\left(b\zeta, G(x_i^\Delta, \cdot)\right),$$

$$=\left(a\zeta', \frac{\partial G}{\partial \xi}(x_i^\Delta, \cdot)-v'\right)+\left(b\zeta, G(x_i^\Delta, \cdot)-v\right), \quad v\in \mathscr{M}_\Delta^r \tag{3.6}$$

and

$$|\zeta(x_i^\Delta)|\leq\|\zeta\|_{E(I)} \inf_{v\in\mathscr{M}_\Delta^r}\|G(x_i^\Delta, \cdot)-v\|_{E(I)}. \tag{3.7}$$

Also, since

$$G(x_i^\Delta, \cdot)\in H^{r+1}((0, x_i^\Delta))\cap H^{r+1}((x_i^\Delta, 1)),$$

then

$$\inf_{v\in\mathscr{M}_\Delta^r}\|G(x_i^\Delta, \cdot)-v\|_{E(I)}\leq\Big(\sum_{j=1}^m(\bar{a}C_{r,1}h_j(\Delta)^{2r}$$

$$+\bar{b}C_{r,2}h_j(\Delta)^{2(r+1)})\|G^{(r+1)}(x_i^\Delta,\cdot)\|^2_{L^2(I_j(\Delta))}\Big)^{1/2},$$

where, for example, it is found by [7] that

$$C_{r,1}=4/(r!)^2,$$

$$C_{r,2}=4/((r+1)!)^2.$$

Thus

$$|\zeta(x_i^\Delta)|\leq\Big(\sum_{j=1}^m\frac{h_j(\Delta)^2\mu_j^2}{\pi^2 a^j_{\min}}\Big)^{1/2}$$

$$\times\Big(\sum_{j=1}^m(\bar{a}C_{r,1}h_j(\Delta)^{2r}+\bar{b}C_{r,2}h_j(\Delta)^{2(r+1)})\|G^{(r+1)}(x_i^\Delta, \cdot)\|^2_{L^2(I_j(\Delta))}\Big)^{1/2}$$

$$\times(1+O(h(\Delta))). \tag{3.8}$$

From now on, for simplicity, denote the right-hand side of (3.8) by C_1.

Now, let function $\alpha(x)$ be the linear function on the interval $I_i(\Delta)$ such that

$$\alpha(x_{i-1}^\Delta)=\zeta(x_{i-1}^\Delta),$$

$$\alpha(x_i^\Delta)=\zeta(x_i^\Delta).$$

Also define

$$P_r^0(I_i(\Delta))=\{v\in P_r(I_i(\Delta))\,|\,v(x_{i-1}^\Delta)=v(x_i^\Delta)=0\},$$

and let $z\in P_r(I_i(\Delta))$ satisfy the relations $z(x_{i-1}^\Delta)=y(x_{i-1}^\Delta)$, $z(x_i^\Delta)=y(x_i^\Delta)$ and

$$B_{I_i(\Delta)}(z,w)=B_{I_i(\Delta)}(y, w), \qquad w\in P_r^0(I_i(\Delta)).$$

Then it follows from Peano's theorem ([8;P 108]) that there is a positive constant C_2 such that

$$\|z-y\|_{L^\infty(I_i(\Delta))}\leq C_2\|y^{(r+1)}\|_{L^\infty(I_i(\Delta))}h_i(\Delta)^{r+1}. \tag{3.9}$$

Set

$$\nu(x)=\begin{cases} Y_\Delta - z+\alpha\,, & x\in I_i\,,\\ 0\,, & x\in I\setminus I_i\,.\end{cases}$$

Then

$$\begin{aligned}\|\nu\|^2_{E(I_i(\Delta))} &= B_{I_i(\Delta)}(\nu,\nu)\,,\\ &= B_{I_i(\Delta)}(Y_\Delta,\nu)-B_{I_i(\Delta)}(y,\nu)+B_{I_i(\Delta)}(\alpha,\nu)\,,\\ &= B_{I_i(\Delta)}(\alpha,\nu)\,,\\ &=\|\alpha\|_{E(I_i(\Delta))}\|\nu\|_{E(I_i(\Delta))}\,.\end{aligned}$$

Thus

$$\|\nu\|_{E(I_i(\Delta))}\le\|\alpha\|_{E(I_i(\Delta))}\,.$$

Moreover by Markoff's inequality ([9; §I.3]) we have

$$\begin{aligned}\|\alpha\|_{E(I_i(\Delta))} &\le (\bar a\|\alpha'\|^2_{L^2(I_i(\Delta))}+\bar b\|\alpha\|^2_{L^2(I_i(\Delta))})^{1/2}\,,\\ &\le\left(\frac{4\bar a}{h_i(\Delta)}C_1^2+\bar b C_1^2 h_i(\Delta)\right)^{1/2},\\ &= C_1\left(\frac{4\bar a}{h_i(\Delta)}+\bar b h_i(\Delta)\right)^{1/2},\end{aligned}$$

i.e.

$$\|\nu\|_{E(I_i(\Delta))}\le C_1\left(\frac{4\bar a}{h_i(\Delta)}+\bar b\, h_i(\Delta)\right)^{1/2}.$$

It follows from [10; P845] and Poincare's inequality ([11; P57]) that there is a positive constant C_3 such that

$$\|\nu\|_{L\infty(I_i(\Delta))}\le C_1C_3(4\bar a+\bar b h_i(\Delta)^2)^{1/2}\,. \tag{3.10}$$

Since $\zeta=y-z-\nu+\alpha$, the estimates (3.9) and (3.10) show that the inequality

$$\|y-Y_\Delta\|_{L\infty(I_i(\Delta))}\le C_1(1+C_3(4\bar a+\bar b h_i(\Delta)^2)^{1/2})+C_2\|y^{(r+1)}\|_{L\infty(I_i(\Delta))}h_i(\Delta)^{r+1} \tag{3.11}$$

holds. Q.E.D.

Now the problem (3.1) is replaced by

$$L[y]=y''-a(x)y'-b(x)y=-f(x)\,, \qquad x\in I\,,$$
$$y(0)=y(1)=0\,,$$

where $a, b, f\in C^{r-1}(\bar I)$.

Similar results to Theorem 2 can be found without using Theorem 1 ([12]). But, compared with previous problem, it must be generally difficult to compute the constants of this problem.

4. Adaptive Meshes and their Examples

In this section, we consider the methods of partition in accordance with the error estimates in §3.

First define

$$C(\Delta)=\Big(\sum_{j=1}^{m}\frac{h_j(\Delta)^2\mu_j^2}{\pi^2 a_{\min}^j}\Big)^{1/2}.$$

Then it follows from Theorem 1 that

$$\|y-Y_\Delta\|_{E(I)}\le C(\Delta)\,(1+O(h(\Delta))).$$

Babuška and Rheinboldt considered adaptive meshes on condition that number of intervals used in the partition is fixed. But we consider the partition such that

$$C(\Delta)\le\delta \tag{4.1}$$

with a few number of intervals used in the partition and less computational time. There are various methods to get (4.1).

In this paper we use the method (Method I) which bisects all intervals such that

$$\frac{h_j(\Delta)\,\mu_j^2}{\pi^2 a_{\min}^j}>\delta^2$$

till the inequality (4.1) holds. We compare this method with the method (Method II) which bisects the only intervals $I_i(\Delta)$ such that

$$\frac{h_i(\Delta)\,\mu_i^2}{\pi^2 a_{\min}^i}=\max_{j=1,\cdots,m}\frac{h_j(\Delta)\,\mu_j^2}{\pi^2 a_{\min}^j}>\delta^2. \tag{4.2}$$

Generally speaking, Method I takes more number of intervals used in the partition than Method II, but on the other hand Method I takes less computational time than Method II. The computational results by each method are summarized in Table 1 and Table 4.

Next we consider the partition in accordance with Theorem 2. Then it is the important problem that the value $\|y^{(r+1)}\|_{L^\infty(I_i(\Delta))}$ isn't generally known. Using (3.1), we get the representation such that

$$y^{(r+1)}\equiv A(x)y'+B(x)y+C(x).$$

Also, using the same way as in [12],

$$\|y^{(j)}-Y_\Delta^{(j)}\|_{L^\infty(I)}\le C^j\|y^{(r+1)}\|_{L^\infty(I)}h(\Delta)^{r+1-j}\qquad (j=0,1).$$

Thus we substitute the norm

$$\|AY'_\Delta+BY_\Delta+C\|_{L^\infty(I_i(\Delta))} \tag{4.3}$$

for $\|y^{(r+1)}\|_{L^\infty(I_i(\Delta))}$.

Define

$$C(\Delta,I_i(\Delta))=C(\Delta)\Big(\sum_{j=1}^{m}(\bar a C_{r,1}h_j(\Delta)^{2r}+\bar b C_{r,2}h_j(\Delta)^{2(r+1)})\|G^{(r+1)}(x_i^\Delta,\cdot)\|^2_{L^2(I_j(\Delta))}\Big)^{1/2}$$
$$\times(C_3(4\bar a+\bar b\,h_i(\Delta)^2)^{1/2}+1)+C_2\|AY'_\Delta+BY_\Delta+C\|_{L^\infty(I_i(\Delta))}h_i(\Delta)^{r+1} \tag{4.4}$$

Obviously

$$\|y-Y_\Delta\|_{L\infty(I_i(\Delta))} \leq C(\Delta, I_i(\Delta))(1+O(h(\Delta))).$$

Using this inequality, we consider the partition such that

$$C(\Delta, I_i(\Delta)) \leq \delta \tag{4.5}$$

for each interval $I_i(\Delta)$.

In this paper the following methods are introduced.

(I) Designate the first partition. Next bisect all intervals such that

$$C(\Delta, I_i(\Delta)) > \delta$$

till the inequality (4.5) holds.

(II) Designate the first partition. Next seek the partition satisfied (4.1) and bisect all intervals such that

$$C(\Delta, I_i(\Delta)) > \delta$$

till the inequality (4.5) holds.

The following are examples in case of $r=2$.

Example 1.

$$\varepsilon y''-y=1, \qquad (\varepsilon>0),$$
$$y(0)=y(1)=0,$$

with the genuine solution $y=\frac{e^{x/\sqrt{\varepsilon}}}{e^{1/\sqrt{\varepsilon}}+1}+\frac{e^{-x/\sqrt{\varepsilon}}}{e^{-1/\sqrt{\varepsilon}}+1}-1$.

The results used the methods in accordance with Theorem 1 are summarized in Table 1 and the results used the methods in accordance with Theorem 2 are summarized in Table 2. Also the interval (0, 1) is divided into four equal parts at first in either table and the computations are done for the constants

$$C_2=1+\frac{15\varepsilon}{2(10\varepsilon+h_i(\Delta)^2)}+\frac{15h_i(\Delta)^2}{16(10\varepsilon+h_i(\Delta)^2)},$$
$$C_3=(15/8)^{1/2}$$

in Table 2.

Example 2.

$$y''=m(m-1)\,x^{m-2}, \qquad (m\geq 3),$$
$$y(0)=y(1)=0,$$

with the genuine solution $y=x^m-x$.

The Green's function for this problem is a linear function for each variables. Thus, according to the error estimate in §3, the errors at the nodes vanish. Consequently, the partition used Theorem 2 is done in accordance with the values

$$C_2\|AY'_\Delta+BY_\Delta+C\|_{L\infty(I_i(\Delta))}h_i(\Delta)^{r+1}.$$

The results in case of $C_2=\frac{7}{4}+\frac{3h_i(\Delta)^2}{32}$ are summarized in Table 3. Also the results used the methods in accordance with Theorem 1 are summarized in Table 4. In this case the first partition is the same as in Example 1.

The notations in each table are as follows;
k: number of equal intervals used in the first partition
k': number of intervals used in the last partition
E: maximum error at the nodes
T: computational time (s)

TABLE 1

ε	δ	Method	$C(\Delta)$	k'	E	T
0.1(+0)	0.1(+0)	I	0.23(−1)	4	0.16(−3)	2
		II	0.23(−1)	4	0.16(−3)	2
	0.1(−1)	I	0.85(−2)	6	0.60(−4)	2
		II	0.85(−2)	6	0.60(−4)	2
	0.1(−2)	I	0.57(−3)	24	0.36(−6)	5
		II	0.89(−3)	20	0.45(−5)	7
0.1(−1)	0.1(+0)	I	0.41(−1)	6	0.12(−2)	2
		II	0.41(−1)	6	0.12(−2)	2
	0.1(−1)	I	0.48(−2)	16	0.54(−4)	4
		II	0.66(−2)	14	0.55(−4)	4
	0.1(−2)	I	0.72(−3)	42	0.13(−5)	12
		II	0.96(−3)	40	0.23(−4)	36

TABLE 2

ε	δ	Method	k'	E	T
0.1(+0)	0.1(+0)	I	8	0.98(−5)	2
		II	8	0.98(−5)	2
	0.1(−1)	I	16	0.62(−6)	4
		II	14	0.42(−5)	4
	0.1(−2)	I	32	0.44(−7)	7
		II	24	0.36(−6)	6

TABLE 3

m	δ	k	k'	E	T
5	0.1(+0)	4	11	0.86(−5)	3
		5	10	0.10(−4)	2
	0.1(−1)	4	22	0.95(−6)	4
		7	20	0.19(−5)	4
	0.1(−2)	4	44	0.15(−6)	9
		11	41	0.19(−6)	8
10	0.1(+0)	4	13	0.21(−4)	3
		3	11	0.16(−4)	3
	0.1(−1)	4	25	0.90(−6)	5
	0.1(−2)	4	50	0.18(−6)	11
		5	51	0.97(−7)	12

TABLE 4

m	δ	Method	k'	E	T
5	0.1(+0)	I	16	0.92(−4)	4
		II	14	0.92(−4)	6

All the computations were carried out using the OKITAC system 50/40 disk oriented system.

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