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Abstract

In a Finsler space the Finsler connections compatible with the given Finsler metric g_{ij} are the metrical Finsler connections, which are characterized as ones preserving the length of a vector under parallel displacements. If we consider the conformal Finsler structure $e^{2\rho}g_{ij}$, the compatible Finsler connections are the conformal Finsler connections, which are characterized as ones preserving the angle between vectors under parallel displacements. The authors [3, 4] have recently investigated these connections in detail.

On the other hand, continued from them, the authors [5] have treated an almost symplectic Finsler structure a_{ij}, which is defined as an alternate, non-degenerate Finsler tensor field of type (0, 2), and have especially considered the problem of its integrability, in terms of the compatible Finsler connections.

The purpose of the present paper is to discuss a conformal almost symplectic Finsler structure $e^{2\sigma}a_{ij}$. We first introduce such a structure (§ 1) and define the compatible Finsler connections (§ 2). And the structure of the set of all such connections is discussed (§ 3), and it is shown that the group of their transformations preserving a non-linear connection gives various important invariants (§ 4). Finally, by lifting a conformal almost symplectic Finsler structure to the tangent bundle (§ 5), we solve the problem of integrability of the structure (§ 6).

As to the terminology and notations we retain those in our previous joint papers [4, 5], which are based on Matsumoto [1, 2]. And all the theorems are proved applying the methods given in [4, 5]; so the proofs and the detailed references are almost omitted.

§1. The notion of c.a.s-Finsler structure.

Let M be a differentiable manifold of dimension $2n$. $x=(x^i)$ and $y=(y^i)$ denote a point of M and a supporting element respectively. Let $\mathfrak{A}_2(M)$ be the set of all alternate Finsler tensor fields of type (0, 2) on M. The relation for a_{ij}, $b_{ij}\in\mathfrak{A}_2(M)$ defined by

$$(1.1)\qquad a_{ij}\sim b_{ij}\Longleftrightarrow \exists\,\sigma(x, y)\,|\,a_{ij}=e^{2\sigma}b_{ij}$$

is an equivalent relation of $\mathfrak{A}_2(M)$. Since the property $\det(a_{ij})\neq 0$ is preserved by the relation, we can define as follows.

Definition 1.1. An equivalence class $\hat{a}$ of $\mathfrak{A}_2(M)$ is called a *conformal almost symplectic* (abbreviated to *c.a.s-*) *Finsler structure,* if $a_{ij}\in\hat{a}$ is non-degenerate: $\det(a_{ij})\neq 0$.

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Every $a'_{ij} \in \hat{a}_{ij}$ is expressed by

(1.2) $$a'_{ij} = e^{2\sigma} a_{ij},$$

and defines an almost symplectic (abbreviated to a.s-) Finsler structure. An example of a c.a.s-Finsler structure is given by $\hat{a}_{ij}$ from an example of an a.s-Finsler structure a_{ij} [5].

Given a c.a.s-Finsler structure $\hat{a}$, we can associate Obata's operators:

(1.3) $$\Theta^{ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j - a_{sj} a^{ir}), \quad \Theta^{*ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j + a_{sj} a^{ir}),$$

where $a_{ij} \in \hat{a}$, and (a^{ij}) is the inverse matrix of (a_{ij}):

(1.4) $$a_{ij} a^{jk} = \delta^k_i.$$

Since the relation (1.2) implies

(1.5) $$a'^{ij} = e^{-2\sigma} a^{ij},$$

Obata's operators do not depend on the representative of $\hat{a}$.

§2. C.a.s-Finsler connections.

On the analogy of the conformal Finsler connections for a conformal Finsler structure, we shall define the Finsler connections which seem to be compatible with a given c.a.s-Finsler structure as follows.

Definition 2.1. Let $\hat{a}$ be a c.a.s-Finsler structure. A Finsler connection is called *conformal almost symplectic* with respect to $\hat{a}$, if for $a_{ij} \in \hat{a}$ there exists a 1-form $\omega = \tilde{\omega}_k dx^k + \dot{\omega}_k \delta y^k \in \Lambda^1(T(M))$ such that

(2.1) $$a_{ij|k} = 2\tilde{\omega}_k a_{ij}, \quad a_{ij}|_k = 2\dot{\omega}_k a_{ij}.$$

This definition is well-defined, because we have

Theorem 2.1. *Let a Finsler connection satisfy* (2.1) *for* $a_{ij} \in \hat{a}$. *Then for* $a'_{ij} = e^{2\sigma} a_{ij}$ *it yields*

(2.2) $$a'_{ij|k} = 2\tilde{\omega}'_k a'_{ij}, \quad a'_{ij}|_k = 2\dot{\omega}'_k a'_{ij},$$

where $\omega' = \omega + d\sigma$.

A c.a.s-Finsler connection with respect to $\hat{a}$ determines an equivalence class $\hat{\omega}$ of $\Lambda^1(T(M))$, classified by the equivalent relation defined by

(2.3) $$\omega' \sim \omega \Longleftrightarrow \omega' - \omega \text{ is exact.}$$

When we want to express explicitly that a c.a.s-Finsler connection $F\Gamma$ satisfies (2.1) for a fixed $a_{ij} \in \hat{a}$, we shall say a c.a.s-Finsler connection with respect to $\hat{a}_{ij}$, corresponding to ω, and denote $F\Gamma(\omega)$.

Let X, Y be two Finsler vector fields. Then $a_{ij} \in \hat{a}$ defines a Finsler scalar field

(2.4) $$a(X, Y) = a_{ij} X^i Y^j.$$

Given a c.a.s-Finsler connection $F\Gamma(\omega)$ with respect to $\hat{a}_{ij}$, if X, Y are parallel along a curve $C(x^i(t))$ in M, with respect to a curve $\tilde{C}(x^i(t), y^i(t))$ in $T(M)$ mapped on C by the canonical projection of $T(M)$, the above $a(X, Y)$ satisfies the property

(2.5) $$da(X, Y)/dt = (\tilde{\omega}_k(dx^k/dt) + \dot{\omega}_k(\delta y^k/dt))a(X, Y)$$

along C with respect to $\tilde{C}$. Since (1.2) implies

(2.6) $$a'(X, Y) = e^{2\sigma} a(X, Y),$$

the property (2.5) has the meaning depending on $\tilde{a}$, and we can also define a c.a.s-Finsler connection as a Finsler connection satisfying the property (2.5) for arbitrary parallel X, Y along arbitrary C with respect to arbitrary $\tilde{C}$.

If we consider the case $\omega'=0$ in Theorem 2.1, we have the following definition and theorem.

Definition 2.2. A c.a.s-Finsler connection $F\Gamma$ with respect to $\tilde{a}$ is called *almost symplectic* if there exists a representative $a'_{ij}\in\tilde{a}$ such that $a'_{ij|k}=0$, $a'_{ij}|_k=0$.

Theorem 2.2. *A c.a.s-Finsler connection $F\Gamma(\omega)$ with respect to $\tilde{a}_{ij}$ is almost symplectic if and only if the 1-form ω is exact.*

If we apply the Ricci identities to a_{ij}, we have similarly to Proposition 3.1 in [4]

Theorem 2.3. *A c.a.s-Finsler connection is almost symplectic if and only if*

$$R^s_{skl}=0,\ P^s_{skl}=0,\ S^s_{skl}=0. \tag{2.7}$$

Furthermore, if we put generally

$$\begin{cases} R^{*i}_{jkl}=R^i_{jkl}-\dfrac{1}{2n}\delta^i_jR^s_{skl}, \\ P^{*i}_{jkl}=P^i_{jkl}-\dfrac{1}{2n}\delta^i_jP^s_{skl}, \\ S^{*i}_{jkl}=S^i_{jkl}-\dfrac{1}{2n}\delta^i_jS^s_{skl}, \end{cases} \tag{2.8}$$

paying attention that Obata's operators are h- and v-covariantly constant, we have similarly to Theorem 3.4 in [4]

Theorem 2.4. *The Finsler tensor fields $\Theta^{*ir}_{sj}R^{*s}_{rkl}$, $\Theta^{*ir}_{sj}P^{*s}_{rkl}$, $\Theta^{*ir}_{sj}S^{*s}_{rkl}$ and their h- and v-covariant derivatives of every order vanish, for every c.a.s-Finsler connection.*

§3. The set of all c.a.s-Finsler connections.

The set of all c.a.s-Finsler connections is determined in the same way as in our paper [4].

Theorem 3.1. *Let $\overset{\circ}{F\Gamma}$ be a fixed Finsler connection. The set of all c.a.s-Finsler connections $F\Gamma$ with respect to $\tilde{a}_{ij}$ is given by*

$$\begin{cases} N^i_k=\overset{\circ}{N}{}^i_k-X^i_k, \\ F^i_{jk}=\overset{\circ}{F}{}^i_{jk}+\overset{\circ}{C}{}^i_{jm}X^m_k+\dfrac{1}{2}a^{im}(a_{mj\overset{\circ}{|}k}-2\tilde{\omega}_ka_{mj}+a_{mj}\overset{\circ}{|}_pX^p_k)+\Theta^{ir}_{sj}X^s_{rk}, \\ C^i_{jk}=\overset{\circ}{C}{}^i_{jk}+\dfrac{1}{2}a^{im}(a_{mj}\overset{\circ}{|}_k-2\dot{\omega}_ka_{mj})+\Theta^{ir}_{sj}Y^s_{rk}, \end{cases} \tag{3.1}$$

where ω is an arbitrary 1-form in $T(M)$, and X^i_k, X^i_{jk}, Y^i_{jk} are arbitrary Finsler tensor fields.

Putting $X^i_k=X^i_{jk}=Y^i_{jk}=0$ in Theorem 3.1, we have an example of a c.a.s-Finsler connection $F\Gamma(\omega)$ with respect to $\tilde{a}_{ij}$, which corresponds to the Kawaguchi metrical Finsler connection derived from $\overset{\circ}{F\Gamma}$ in a Finsler space.

Theorem 3.2. *Let $\overset{\circ}{F\Gamma}$ be a fixed Finsler connection, and let ω be a given 1-form in*

$T(M)$. *Then the following Finsler connection is a c.a.s-Finsler connection with respect to* $\tilde{a}_{ij}$, *corresponding to* ω :

$$\text{(3.2)}\qquad \begin{cases} N^i_k = \overset{\circ}{N}{}^i_k, \\ F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \dfrac{1}{2}a^{im}(a_{mj\overset{\circ}{|}k} - 2\tilde{\omega}_k a_{mj}), \\ C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \dfrac{1}{2}a^{im}(a_{mj}\overset{\circ}{|}_k - 2\dot{\omega}_k a_{mj}). \end{cases}$$

If we take a c.a.s-Finsler connection $F\overset{\circ}{\Gamma}(\omega)$ as $F\overset{\circ}{\Gamma}$ in Theorem 3.1, we have

Theorem 3.3. *Let* $F\overset{\circ}{\Gamma}(\omega)$ *be a fixed c.a.s-Finsler connections with respect to* $\tilde{a}_{ij}$. *Then the set of all c.a.s-Finsler connection* $F\Gamma(\omega)$ *with respect to* $\tilde{a}_{ij}$ *is given by*

$$\text{(3.3)}\qquad \begin{cases} N^i_k = \overset{\circ}{N}{}^i_k - X^i_k, \\ F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + (\overset{\circ}{C}{}^i_{jm} + \delta^i_j \dot{\omega}_m) X^m_k + \Theta^{ir}_{sj} X^s_{rk}, \\ C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \Theta^{ir}_{sj} Y^s_{rk}, \end{cases}$$

where X^i_k, X^i_{jk}, Y^i_{jk} *are arbitrary Finsler tensor fields.*

Let N be a fixed non-linear connection. We denote by $F\Gamma(N, \omega)$ a c.a.s-Finsler connection, corresponding to ω and having N as the non-linear connection. The set in Theorem 3.1 has the following subset.

Theorem 3.4. *Let* $F\overset{\circ}{\Gamma}$ *be a fixed Finsler connection. Then the set of all c.a.s-Finsler connection* $F\Gamma(\overset{\circ}{N}, \omega)$ *with respect to* $\tilde{a}_{ij}$ *is given by*

$$\text{(3.4)}\qquad \begin{cases} N^i_k = \overset{\circ}{N}{}^i_k, \\ F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \dfrac{1}{2}a^{im}(a_{mj\overset{\circ}{|}k} - 2\tilde{\omega}_k a_{mj}) + \Theta^{ir}_{sj} X^s_{rk}, \\ C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \dfrac{1}{2}a^{im}(a_{mj}\overset{\circ}{|}_k - 2\dot{\omega}_k a_{mj}) + \Theta^{ir}_{sj} Y^s_{rk}, \end{cases}$$

where X^i_{jk}, Y^i_{jk} *are arbitrary Finsler tensor fields.*

§4. The group of transformations of c.a.s-Finsler connections.

Let us consider the transformation $F\Gamma(N, \omega) \rightarrow F\bar{\Gamma}(N, \omega')$ of c.a.s-Finsler connections, which preserves the non-linear connection. Owing to Theorem 3.4 we have

Theorem 4.1. *Two c.a.s-Finsler connections* $F\Gamma(N, \omega)$, $F\bar{\Gamma}(N, \omega')$ *with respect to* $\tilde{a}_{ij}$ *are related as follows :*

$$\text{(4.1)}\qquad \begin{cases} \bar{N}^i_k = N^i_k, \\ \bar{F}^i_{jk} = F^i_{jk} - \delta^i_j \tilde{p}_k + \Theta^{ir}_{sj} X^s_{rk}, \\ \bar{C}^i_{jk} = C^i_{jk} - \delta^i_j \dot{p}_k + \Theta^{ir}_{sj} Y^s_{rk}, \end{cases}$$

where $p = \omega' - \omega$, *and* X^i_{jk}, Y^i_{jk} *are some Finsler tensor fields.*

Conversely, if Finsler tensor fields X^i_{jk}, Y^i_{jk} *and a* 1*-form* p *on* $T(M)$ *are given, then* (4.1) *is thought to be a transformation of a c.a.s-Finsler connection* $F\Gamma(N, \omega)$ *with respect to* $\tilde{a}_{ij}$ *to a c.a.s-Finsler connection* $F\bar{\Gamma}(N, \omega + p)$ *with respect to* $\tilde{a}_{ij}$.

Theorem 4.2. *The set* $\mathfrak{CS}$ *of all transformations of c.a.s-Finsler connections with respect to* $\hat{a}_{ij}$ *given by* (4.1) *and mapping product is an abelian group, which acts transitively on the set of all c.a.s-Finsler connections* $F\Gamma(N, \omega)$ *with respect to* $\hat{a}_{ij}$.

A transformation given by (4.1) is expressed by the product of the following two transformations:

$$\bar{N}^i_k = N^i_k,\ \bar{F}^i_{jk} = F^i_{jk} - \delta^i_j \tilde{p}_k,\ \bar{C}^i_{jk} = C^i_{jk} - \delta^i_j \dot{p}_k, \tag{4.2}$$

$$\bar{N}^i_k = N^i_k,\ \bar{F}^i_{jk} = F^i_{jk} + \Theta^{ir}_{sj} X^s_{rk},\ \bar{C}^i_{jk} = C^i_{jk} + \Theta^{ir}_{sj} Y^s_{rk}. \tag{4.3}$$

The former is a so-called co-parallel transformation, and the latter is a transformation of a.s-Finsler connections. Thus we have

Theorem 4.3. *The group* $\mathfrak{CS}$ *is the direct product of the group of all co-parallel transformations and the group of all transformations of a.s-Finsler connections.*

It is noted that the invariants of $\mathfrak{CS}$ are the invariants of each of these subgroups, and reciprocally. Therefore, in order to obtain the invariants of $\mathfrak{CS}$ we shall pay attention to the invariants t^i_{jk}, R^i_{jk}, $t^*{}_{ijk}$, $R^*{}_{ijk}$, $T^*{}_{ijk}$, $S^*{}_{ijk}$ and $\overset{a}{\chi}_{ijk}(a=1, 2, 3, 4)$ of the latter subgroup, which have been defined in [5] as follows:

$$\begin{cases} t^i_{jk} = \mathfrak{A}_{jk}\{\partial N^i_j/\partial y^k\}, \\ R^i_{jk} = \mathfrak{A}_{jk}\{\delta N^i_j/\delta x^k\},\ P^i_{jk} = \partial N^i_j/\partial y^k - F^i_{kj}, \\ T^i_{jk} = \mathfrak{A}_{jk}\{F^i_{jk}\},\ S^i_{jk} = \mathfrak{A}_{jk}\{C^i_{jk}\}, \end{cases} \tag{4.4}$$

$$\begin{cases} t^*{}_{ijk} = \mathfrak{S}_{ijk}\{a_{im} t^m_{jk}\},\ R^*{}_{ijk} = \mathfrak{S}_{ijk}\{a_{im} R^m_{jk}\}, \\ T^*{}_{ijk} = \mathfrak{S}_{ijk}\{a_{im} T^m_{jk}\},\ S^*{}_{ijk} = \mathfrak{S}_{ijk}\{a_{im} S^m_{jk}\}, \end{cases} \tag{4.5}$$

and

$$\begin{cases} \overset{1}{\chi}_{ijk} = a_{km} T^m_{ij} + \mathfrak{A}_{ij}\{a_{im} P^m_{jk}\},\ \overset{2}{\chi}_{ijk} = a_{im} S^m_{jk} + \mathfrak{A}_{jk}\{a_{km} C^m_{ij}\}, \\ \overset{3}{\chi}_{ijk} = \mathfrak{A}_{jk}\{a_{km} P^m_{ij}\},\ \overset{4}{\chi}_{ijk} = \mathfrak{A}_{ij}\{a_{im} C^m_{jk}\}, \end{cases} \tag{4.6}$$

where $\mathfrak{A}_{ij}\{\cdots\}$ and $\mathfrak{S}_{ijk}\{\cdots\}$ denote, for example, $\mathfrak{A}_{jk}\{A_{jk}\} = A_{jk} - A_{kj}$ and $\mathfrak{S}_{ijk}\{A_{ijk}\} = A_{ijk} + A_{jki} + A_{kij}$.

Calculating the transformations of these tensor fields by a co-parallel transformation (4.2), we have

Proposition 4.1. *By a transformation* (4.1) *of c.a.s-Finsler connections the above tensor fields are transformed as follows*:

$$\bar{t}^i_{jk} = t^i_{jk},\ \bar{R}^i_{jk} = R^i_{jk},\ \bar{t}^*{}_{ijk} = t^*{}_{ijk},\ \bar{R}^*{}_{ijk} = R^*{}_{ijk}, \tag{4.7}$$

$$\bar{T}^*{}_{ijk} = T^*{}_{ijk} - 2\mathfrak{S}_{ijk}\{a_{ij}\tilde{p}_k\},\ \bar{S}^*{}_{ijk} = S^*{}_{ijk} - 2\mathfrak{S}_{ijk}\{a_{ij}\dot{p}_k\}, \tag{4.8}$$

$$\begin{cases} \overset{\bar{1}}{\chi}_{ijk} = \overset{1}{\chi}_{ijk} - 2\mathfrak{A}_{ij}\{a_{jk}\tilde{p}_i\},\ \overset{\bar{2}}{\chi}_{ijk} = \overset{2}{\chi}_{ijk} - 2\mathfrak{A}_{jk}\{a_{ij}\dot{p}_k\}, \\ \overset{\bar{3}}{\chi}_{ijk} = \overset{3}{\chi}_{ijk} - 2a_{jk}\tilde{p}_i,\ \overset{\bar{4}}{\chi}_{ijk} = \overset{4}{\chi}_{ijk} - 2a_{ij}\dot{p}_k. \end{cases} \tag{4.9}$$

We can easily eliminate $\tilde{p}_k$, $\dot{p}_k$ from (4.8), (4.9). Putting

$$\begin{cases} \overset{1}{\chi}_k = a^{pq}\overset{1}{\chi}_{pqk},\ \overset{2}{\chi}_i = a^{pq}\overset{2}{\chi}_{ipq}, \\ \overset{3}{\chi}_k = a^{pq}\overset{3}{\chi}_{pqk},\ \overset{4}{\chi}_i = a^{pq}\overset{4}{\chi}_{ipq}, \\ \overset{3}{\chi}{}^*{}_i = a^{pq}\overset{3}{\chi}_{ipq},\ \overset{4}{\chi}{}^*{}_k = a^{pq}\overset{4}{\chi}_{pqk}, \end{cases} \tag{4.10}$$

we have from (4.9)

$$(4.11)\qquad \begin{cases} \tilde{p}_k = \frac{1}{4}(\overset{1}{\chi}_k - \overset{\bar{1}}{\chi}_k) = \frac{1}{2}(\overset{3}{\chi}_k - \overset{\bar{3}}{\chi}_k) = \frac{1}{4n}(\overset{\bar{3}}{\chi^*}_k - \overset{3}{\chi^*}_k), \\ \dot{p}_i = \frac{1}{4}(\overset{2}{\chi}_i - \overset{\bar{2}}{\chi}_i) = \frac{1}{2}(\overset{4}{\chi}_i - \overset{\bar{4}}{\chi}_i) = \frac{1}{4n}(\overset{\bar{4}}{\chi^*}_i - \overset{4}{\chi^*}_i). \end{cases}$$

As is easily shown from (4.6) it holds

$$(4.12)\qquad \overset{1}{\chi}_k = 2\overset{3}{\chi}_k, \quad \overset{2}{\chi}_i = 2\overset{4}{\chi}_i.$$

We can recognize from (4.11) that the following tensor fields are invariants of $\mathfrak{C}\mathfrak{S}$:

$$(4.13)\qquad \overset{1}{\rho}_k = n\overset{1}{\chi}_k + \overset{3}{\chi^*}_k, \quad \overset{2}{\rho}_i = n\overset{2}{\chi}_i + \overset{4}{\chi^*}_i.$$

If we substitute from (4.11) into (4.8), (4.9), we have the following invariants of $\mathfrak{C}\mathfrak{S}$:

$$(4.14)\qquad \overset{1}{\tau}_{ijk} = T^*{}_{ijk} - \frac{1}{2}\mathfrak{S}_{ijk}\{a_{ij}\overset{1}{\chi}_k\}, \quad \overset{2}{\tau}_{ijk} = S^*{}_{ijk} - \frac{1}{2}\mathfrak{S}_{ijk}\{a_{ij}\overset{2}{\chi}_k\},$$

$$(4.15)\qquad \begin{cases} \overset{1}{\sigma}_{ijk} = \overset{1}{\chi}_{ijk} - \frac{1}{2}\mathfrak{A}_{ij}\{a_{jk}\overset{1}{\chi}_i\}, & \overset{2}{\sigma}_{ijk} = \overset{2}{\chi}_{ijk} - \frac{1}{2}\mathfrak{A}_{jk}\{a_{ij}\overset{2}{\chi}_k\}, \\ \overset{3}{\sigma}_{ijk} = \overset{3}{\chi}_{ijk} - a_{jk}\overset{3}{\chi}_i, & \overset{4}{\sigma}_{ijk} = \overset{4}{\chi}_{ijk} - a_{ij}\overset{4}{\chi}_k. \end{cases}$$

Thus we have proved

Theorem 4.4. *The Finsler tensor fields* t^i_{jk}, R^i_{jk}, $t^*{}_{ijk}$, $R^*{}_{ijk}$, $\overset{a}{\rho}_k$ $(a=1, 2)$, $\overset{a}{\tau}_{ijk}$ $(a=1, 2)$ *and* $\overset{a}{\sigma}_{ijk}$ $(a=1, 2, 3, 4)$, *constructed for a c.a.s-Finsler connection with respect to a c.a.s-Finsler structure* $\hat{a}_{ij}$, *are invariants of the transformation group* $\mathfrak{C}\mathfrak{S}$. *They are uniquely determined by a given* $\hat{a}_{ij}$ *and a given non-linear connection N.*

§5. C.a.s-structures in the tangent bundle.

If a non-linear connection is given on the tangent bundle, a 2-form $\Omega \in \Lambda^2(T(M))$ is expressed as

$$(5.1)\qquad \Omega = \frac{1}{2}\tilde{a}_{ij}dx^i \wedge dx^j + \tilde{b}_{ij}dx^i \wedge \delta y^j + \frac{1}{2}\tilde{c}_{ij}\delta y^i \wedge \delta y^j,$$

where $\tilde{a}_{ij} = -\tilde{a}_{ji}$, $\tilde{c}_{ij} = -\tilde{c}_{ji}$. We say that Ω is *non-degenerate,* if the matrix

$$(5.2)\qquad A = \begin{bmatrix} \tilde{a}_{ij} & \tilde{b}_{ij} \\ -\tilde{b}_{ji} & \tilde{c}_{ij} \end{bmatrix}$$

is non-singular. In this case Ω determines a c.a.s-structure $\hat{A}$ on $T(M)$.

Definition 5.1. A non-degenerate 2-form $\Omega \in \Lambda^2(T(M))$ is called *conformally integrable,* if there exists a 1-form $\lambda \in \Lambda^1(T(M))$ such that

$$(5.3)\qquad d\Omega = 2\lambda \wedge \Omega.$$

Especially, if $d\Omega = 0$, then Ω is called *integrable.*

If Ω, $\Omega' \in \Lambda^2(T(M))$ have the relation

$$(5.4)\qquad \Omega' = e^{2\Sigma}\Omega$$

for some scalar field Σ in $T(M)$, (5.3) implies

$$(5.5)\qquad d\Omega' = 2\lambda' \wedge \Omega',$$

where $\lambda' = \lambda + d\Sigma$. Therefore, the above definition has the meaning depending on the c.a.s-structure $\hat{A}$, and we have

Theorem 5.1. *Let a non-degenerate 2-form* $\Omega \in \Lambda^2(T(M))$ *be conformally integrable. There exists an integrable 2-form* Ω' *such that* $\Omega' = e^{2\Sigma}\Omega$, *if and only if the form* λ *appearing in* (5.3) *is exact.*

Let us express the exterior differential $d\Omega$ of $\Omega \in \Lambda^2(T(M))$ given by (5.1) in the form

$$(5.6) \qquad d\Omega = \frac{1}{6}\overset{1}{\omega}_{ijk} dx^i \wedge dx^j \wedge dx^k + \frac{1}{2}\overset{2}{\omega}_{ijk} dx^i \wedge dx^j \wedge \delta y^k + \frac{1}{2}\overset{3}{\omega}_{ijk}\, dx^i \wedge \delta y^j \wedge \delta y^k + \frac{1}{6}\overset{4}{\omega}_{ijk} \delta y^i \wedge \delta y^j \wedge \delta y^k.$$

Since $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{c}_{ij}$ in (5.1) are considered as Finsler tensor fields on the base manifold M, we have

Proposition 5.1. *If a Finsler connection is given on M,* $\overset{a}{\omega}_{ijk}$ $(a=1, 2, 3, 4)$ *in* (5.6) *have the expressions*

$$(5.7) \qquad \begin{cases} \overset{1}{\omega}_{ijk} = \mathfrak{S}_{ijk}\{\tilde{a}_{ij|k} + \tilde{a}_{im}T^m_{jk} + \tilde{b}_{im}R^m_{jk}\}, \\ \overset{2}{\omega}_{ijk} = \tilde{a}_{ij}|_k + \tilde{b}_{mk}T^m_{ji} + \tilde{c}_{km}R^m_{ij} + \mathfrak{A}_{ij}\{\tilde{b}_{jk|i} + \tilde{a}_{im}C^m_{jk} + \tilde{b}_{im}P^m_{jk}\}, \\ \overset{3}{\omega}_{ijk} = \tilde{b}_{im}S^m_{jk} + \tilde{c}_{jk|i} + \mathfrak{A}_{jk}\{\tilde{b}_{ij}|_k + \tilde{b}_{mj}C^m_{ik} + \tilde{c}_{mj}P^m_{ik}\}, \\ \overset{4}{\omega}_{ijk} = \mathfrak{S}_{ijk}\{\tilde{c}_{ij}|_k + \tilde{c}_{im}S^m_{jk}\}. \end{cases}$$

Theorem 5.2. *A non-degenerate 2-form* $\Omega \in \Lambda^2(T(M))$ *given by* (5.1) *is conformally integrable, if and only if there exists a 1-form* $\lambda = \tilde{\lambda}_i\, dx^i + \dot{\lambda}_i \delta y^i \in \Lambda^1(T(M))$ *such that the Finsler tensor fields* $\overset{a}{\omega}_{ijk}$ $(a=1, 2, 3, 4)$ *given by* (5.7) *are expressed as*

$$(5.8) \qquad \begin{cases} \overset{1}{\omega}_{ijk} = 2\mathfrak{S}_{ijk}\{\tilde{a}_{ij}\tilde{\lambda}_k\}, \\ \overset{2}{\omega}_{ijk} = 2\mathfrak{A}_{ij}\{\tilde{b}_{jk}\tilde{\lambda}_i\} + 2\tilde{a}_{ij}\dot{\lambda}_k, \\ \overset{3}{\omega}_{ijk} = 2\tilde{c}_{jk}\tilde{\lambda}_i + 2\mathfrak{A}_{jk}\{\tilde{b}_{ij}\dot{\lambda}_k\}, \\ \overset{4}{\omega}_{ijk} = 2\mathfrak{S}_{ijk}\{\tilde{c}_{ij}\dot{\lambda}_k\}. \end{cases}$$

§6. Conformal integrabilities of a c.a.s-Finsler structure.

Assume that a non-linear connection N be given in the tangent bundle $T(M)$. Then a c.a.s-Finsler structure $\tilde{a}_{ij}$ is lifted to a 2-form $\Omega \in \Lambda^2(T(M))$ in various ways. We consider the lifted forms Ω of the types II, I+II, I+III and II+III given by (5.1) with the following coefficients.

	$\tilde{a}_{ij}$	$\tilde{b}_{ij}$	$\tilde{c}_{ij}$
II	0	a_{ij}	0
I+II	a_{ij}	a_{ij}	0
I+III	a_{ij}	0	a_{ij}
II+III	0	a_{ij}	a_{ij}

Proposition 6.1. *Each 2-form* Ω *of the types* II, I+II, I+III *and* II+III *is non-degenerate, and defines a c.a.s-structure on* $T(M)$.

Proposition 6.2. *Let $F\Gamma(N, \omega)$ be a c.a.s-Finsler connection with respect to a c.a.s-Finsler structure $\hat{a}_{ij}$. The coefficients $\overset{a}{\omega}_{ijk}$ $(a=1, 2, 3, 4)$ of the exterior differential $d\Omega$ of the 2-form Ω given in Proposition* 6.1 *are expressed as follows*:

II :
$$\overset{1}{\omega}_{ijk}=R^*{}_{ijk}, \qquad \overset{2}{\omega}_{ijk}=\overset{1}{\chi}_{ijk}+2\mathfrak{A}_{ij}\{a_{jk}\tilde{\omega}_i\},$$
$$\overset{3}{\omega}_{ijk}=\overset{2}{\chi}_{ijk}+2\mathfrak{A}_{jk}\{a_{ij}\dot{\omega}_k\}, \qquad \overset{4}{\omega}_{ijk}=0\ ;$$

I+ II :
$$\overset{1}{\omega}_{ijk}=R^*{}_{ijk}+T^*{}_{ijk}+2\mathfrak{S}_{ijk}\{a_{ij}\tilde{\omega}_k\}, \qquad \overset{2}{\omega}_{ijk}=\overset{1}{\chi}_{ijk}+\overset{4}{\chi}_{ijk}+2\mathfrak{A}_{ij}\{a_{jk}\tilde{\omega}_i\}+2a_{ij}\dot{\omega}_k,$$
$$\overset{3}{\omega}_{ijk}=\overset{2}{\chi}_{ijk}+2\mathfrak{A}_{jk}\{a_{ij}\dot{\omega}_k\}, \qquad \overset{4}{\omega}_{ijk}=0\ ;$$

I+III :
$$\overset{1}{\omega}_{ijk}=T^*{}_{ijk}+2\mathfrak{S}_{ijk}\{a_{ij}\tilde{\omega}_k\}, \qquad \overset{2}{\omega}_{ijk}=\overset{4}{\chi}_{ijk}+a_{km}R^m_{ij}+2a_{ij}\dot{\omega}_k,$$
$$\overset{3}{\omega}_{ijk}=\overset{3}{\chi}_{ijk}+2a_{jk}\tilde{\omega}_i, \qquad \overset{4}{\omega}_{ijk}=S^*{}_{ijk}+2\mathfrak{S}_{ijk}\{a_{ij}\dot{\omega}_k\}\ ;$$

II+III :
$$\overset{1}{\omega}_{ijk}=R^*{}_{ijk}, \qquad \overset{2}{\omega}_{ijk}=\overset{1}{\chi}_{ijk}+a_{km}R^m_{ij}+2\mathfrak{A}_{ij}\{a_{jk}\tilde{\omega}_i\},$$
$$\overset{3}{\omega}_{ijk}=\overset{2}{\chi}_{ijk}+\overset{3}{\chi}_{ijk}+2a_{jk}\tilde{\omega}_i+2\mathfrak{A}_{jk}\{a_{ij}\dot{\omega}_k\}, \qquad \overset{4}{\omega}_{ijk}=S^*{}_{ijk}+2\mathfrak{S}_{ijk}\{a_{ij}\dot{\omega}_k\}.$$

Definition 6.1. A c.a.s-Finsler structure $\hat{a}_{ij}$ is called *conformally integrable* of the type II, I+II, I+III or II+III, if there exists a non-linear connection such that the corresponding lifted 2-form of $T(M)$ is conformally integrable.

If $a'_{ij}=e^{2\sigma}a_{ij}$, then the lifted 2-forms have the relation $\Omega'=e^{2\sigma}\Omega$. Hence, from the remark followed by Definition 5.1 the above definition has the meaning depending on the c.a.s-Finsler structure a_{ij}. Then, from Theorem 5.2 and Proposition 6.2 we have

Theorem 6.1. *A c.a.s-Finsler structure $\hat{a}_{ij}$ is conformally integrable of the type of* II, I+II, I+III *or* II+III, *if and only if there exists c.a.s-Finsler connection $F\Gamma(N, \omega)$ with respect to $\hat{a}_{ij}$ and a* 1-*form λ on $T(M)$ satisfying the following conditions* :

II :
$$R^*{}_{ijk}=0,$$
$$\overset{1}{\chi}_{ijk}+2\mathfrak{A}_{ij}\{a_{jk}(\tilde{\omega}_i-\tilde{\lambda}_i)\}=0,$$
$$\overset{2}{\chi}_{ijk}+2\mathfrak{A}_{jk}\{a_{ij}(\dot{\omega}_k-\dot{\lambda}_k)\}=0\ ;$$

I+ II :
$$R^*{}_{ijk}+T^*{}_{ijk}+2\mathfrak{S}_{ijk}\{a_{ij}(\tilde{\omega}_k-\tilde{\lambda}_k)\}=0,$$
$$\overset{1}{\chi}_{ijk}+\overset{4}{\chi}_{ijk}+2\mathfrak{A}_{ij}\{a_{jk}(\tilde{\omega}_i-\tilde{\lambda}_i)\}+2a_{ij}(\dot{\omega}_k-\dot{\lambda}_k)=0,$$
$$\overset{2}{\chi}_{ijk}+2\mathfrak{A}_{jk}\{a_{ij}(\dot{\omega}_k-\dot{\lambda}_k)\}=0\ ;$$

I+III :
$$T^*{}_{ijk}+2\mathfrak{S}_{ijk}\{a_{ij}(\tilde{\omega}_k-\tilde{\lambda}_k)\}=0,$$
$$\overset{4}{\chi}_{ijk}+a_{km}R^m_{ij}+2a_{ij}(\dot{\omega}_k-\dot{\lambda}_k)=0,$$
$$\overset{3}{\chi}_{ijk}+2a_{jk}(\tilde{\omega}_i-\tilde{\lambda}_i)=0,$$
$$S^*{}_{ijk}+2\mathfrak{S}_{ijk}\{a_{ij}(\dot{\omega}_k-\dot{\lambda}_k)\}=0\ ;$$

II+III :
$$R^*{}_{ijk}=0,$$
$$\overset{1}{\chi}_{ijk}+a_{km}R^m_{ij}+2\mathfrak{A}_{ij}\{a_{jk}(\tilde{\omega}_i-\tilde{\lambda}_i)\}=0,$$
$$\overset{2}{\chi}_{ijk}+\overset{3}{\chi}_{ijk}+2a_{jk}(\tilde{\omega}_i-\tilde{\lambda}_i)+2\mathfrak{A}_{jk}\{a_{ij}(\dot{\omega}_k-\dot{\lambda}_k)\}=0,$$

$$S^*{}_{ijk}+2\mathfrak{S}_{ijk}\{a_{ij}(\dot{\omega}_k-\dot{\lambda}_k)\}=0.$$

Now, let a c.a.s-Finsler structure $\hat{a}_{ij}$ be conformally integrable of the type II. From Theorem 6.1 we have

$$R^*{}_{ijk}=0, \tag{6.1}$$

$$\overset{1}{\chi}_{ijk}+2\mathfrak{A}_{ij}\{a_{jk}(\tilde{\omega}_i-\tilde{\lambda}_i)\}=0, \tag{6.2}$$

$$\overset{2}{\chi}_{ijk}+2\mathfrak{A}_{jk}\{a_{ij}(\dot{\omega}_k-\dot{\lambda}_k)\}=0. \tag{6.3}$$

We can eliminate $\tilde{\omega}_i-\tilde{\lambda}_i$, $\dot{\omega}_k-\dot{\lambda}_k$ from (6.2), (6.3) respectively, we have

$$\overset{1}{\sigma}_{ijk}=0, \quad \overset{2}{\sigma}_{ijk}=0. \tag{6.4}$$

Conversely, if (6.4) holds for some $F\Gamma(N, \omega)$, then (6.2), (6.3) are satisfied by $\tilde{\lambda}_i=\tilde{\omega}_i+\frac{1}{4}\overset{1}{\chi}_i$, $\dot{\lambda}_k=\dot{\omega}_k+\frac{1}{4}\overset{2}{\chi}_k$. The other types can be solved in the same manner, and we have

Theorem 6.2. *A c.a.s-Finsler structure* $\hat{a}_{ij}$ *is conformally integrable of the type* II, I+II, I+III *or* II+III, *if and only if there exists a non-linear connection satisfying the following invariant conditions of* $\mathfrak{C}\mathfrak{S}$:

II : $R^*{}_{ijk}=0$, $\overset{1}{\sigma}_{ijk}=0$, $\overset{2}{\sigma}_{ijk}=0$;

I+ II : $R^*{}_{ijk}+\overset{1}{\tau}_{ijk}-\frac{1}{2}\mathfrak{S}_{ijk}\{a_{ij}\overset{2}{\rho}_k\}=0$,

$\overset{1}{\sigma}_{ijk}+\overset{4}{\sigma}_{ijk}-\frac{1}{2}\mathfrak{A}_{ij}\{a_{jk}\overset{2}{\rho}_i\}=0$, $\overset{2}{\sigma}_{ijk}=0$;

I+III : $\overset{1}{\tau}_{ijk}=0$, $\overset{4}{\sigma}_{ijk}-a_{ij}R^m_{km}+a_{km}R^m_{ij}=0$,

$\overset{3}{\sigma}_{ijk}=0$, $\overset{2}{\tau}_{ijk}-\mathfrak{S}_{ijk}\{a_{ij}R^m_{km}\}=0$;

II+III : $R^*{}_{ijk}=0$, $\overset{1}{\sigma}_{ijk}+a_{km}R^m_{ij}-\frac{1}{2}a^{pq}R^m_{pq}\mathfrak{A}_{ij}\{a_{jk}a_{im}\}=0$,

$\overset{2}{\sigma}_{ijk}+\overset{3}{\sigma}_{ijk}-\frac{1}{2}\mathfrak{A}_{jk}\{a_{ij}(\overset{1}{\rho}_k+na^{pq}R^m_{pq}a_{km})\}-\frac{1}{2}a^{pq}R^m_{pq}a_{jk}a_{im}=0$,

$\overset{2}{\tau}_{ijk}-\frac{1}{2}\mathfrak{S}_{ijk}\{a_{ij}(\overset{1}{\rho}_k+na_{km}a^{pq}R^m_{pq})\}=0$.

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