ON GENERALIZED BERWALD CONNECTIONS Dedicated to Professor Dr. Joyo Kanitani on the occasion of his ninetieth birthday

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By

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Abstract

In the present paper we discuss Finsler connections of Berwald type with surviving $(v)hv$-torsion $P^i{}_{jk}$ and consider what kind of Finsler connection should be reasonable as a generalization of the Berwald connection.

§ 0. Introduction.

In a Finsler space there are known two canonical Finsler connections, that is, the Cartan one $C\Gamma$ and the Berwald one $B\Gamma$. Various generalizations are possible for these connections. For example, suggested by Wagner [8], Hashiguchi, one of the authors, introduced the notion of generalized Cartan connection $GC\Gamma$ and defined a generalized Berwald space with respect to this $GC\Gamma$ (Hashiguchi [1], Hashiguchi-Ichijyō [2]). As a generalization of the Berwald connection, Matsumoto [4] defined the notion of Berwald connection with torsion $B\Gamma T$ and showed that a generalized Berwald space can be also defined with respect to this $B\Gamma T$.

Each of these generalizations has a surviving $(h)h$-torsion $T_j{}^i{}_k$. It is noted that $T_j{}^i{}_k$ of $B\Gamma T$ should satisfy the so-called $B\Gamma$-condition, whereas $T_j{}^i{}_k$ of $GC\Gamma$ is arbitrarily given. This situation should be cleared up.

On the other hand, in his recent paper [5], Matsumoto has treated a Finsler connection introduced on a hypersurface of a Finsler space, and obtained an interesting Finsler connection of Berwald type, which we could call a Berwald connection with surviving $(v)hv$-torsion $P^i{}_{jk}$.

The purpose of the present paper is to discuss such a Finsler connection generally. A meaning of Matsumoto's $B\Gamma$-condition is cleared up, and it is shown what kind of Finsler connection should be reasonable as a generalization of the Berwald connection.

Throughout the present paper the terminology and notations are referred to Matsumoto's monograph [3].

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§ 1. Finsler connections of Berwald type with torsion $P^i{}_{jk}$.

We are concerned with an n-dimensional Finsler space $F^n=(M, L)$, where $L(x, y)$ is the fundamental function, and x denotes a point of the underlying manifold M, and y denotes a supporting element. The fundamental tensor g_{ij} is given by $g_{ij}=(\dot{\partial}_i\dot{\partial}_j L^2)/2$, where $\dot{\partial}_j$ denotes the partial differentiation by y^j. We shall express a Finsler connection $F\Gamma$ by $F\Gamma=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ in terms of its coefficients. The Cartan connection $C\Gamma$ and the Berwald one $B\Gamma$ are uniquely determined by the following systems of axioms respectively. We line up them comparatively.

$C\Gamma$ (Matsumoto [3])	$B\Gamma$ (Okada [7])
(C1) $g_{ij\vert k}=0$,	(B1) $L_{\vert k}=0$,
(C2) $D^i{}_k\equiv F_0{}^i{}_k-N^i{}_k=0$,	(B2) $D^i{}_k=0$,
(C3) $T_j{}^i{}_k\equiv F_j{}^i{}_k-F_k{}^i{}_j=0$,	(B3) $T_j{}^i{}_k=0$,
(C4) $S^i{}_{jk}\equiv C_j{}^i{}_k-C_k{}^i{}_j=0$,	(B4) $P^i{}_{jk}\equiv\dot{\partial}_k N^i{}_j-F_k{}^i{}_j=0$,
(C5) $g_{ij}\vert_k=0$;	(B5) $C_j{}^i{}_k=0$.

If we omit some of axioms from each of the above systems, we get various Finsler connections of Cartan type or Berwald type. For example, a *generalized Cartan connection* $GC\Gamma$ is defined as a Finsler connection satisfying the axioms of $C\Gamma$ except (C3), and a *Berwald connection with torsion* $B\Gamma T$ is defined as a Finsler connection satisfying the axioms of $B\Gamma$ except (B3).

Both $GC\Gamma$ and $B\Gamma T$ are with surviving $(h)h$-torsion $T_j{}^i{}_k$. Furthermore we can consider a Finsler connection of Berwald type in which the $(v)hv$-torsion $P^i{}_{jk}$ is also surviving. In a similar way as shown in [4], we have

Theorem 1. *A Finsler connection* $(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ *satisfying* (B1), (B2), (B5) *can be expressed in terms of its torsions* $T_j{}^i{}_k$, $P^i{}_{jk}$ *as follows*:

$$(1.1)\qquad \begin{cases} N^i{}_k=G^i{}_k-((T_k{}^i{}_0+P^i{}_{0k})+\dot{\partial}_k(T^i{}_{00}+P_{00}{}^i))/2, \\ F_j{}^i{}_k=\dot{\partial}_j N^i{}_k-P^i{}_{kj}, \\ C_j{}^i{}_k=0, \end{cases}$$

where $G^i{}_k$ *is the non-linear connection of* $B\Gamma$.

Proof. Putting $F=L^2/2$, the condition (B1) is rewritten in an equivalent form

$$(1.2)\qquad \partial_i F=y_r N^r{}_i.$$

Differentiating (1.2) by y^j, we get $\dot{\partial}_j\partial_i F=g_{rj}N^r{}_i+y_r\dot{\partial}_j N^r{}_i$. By this equations and (1.2), the well-known quantities $2G_j=y^i\dot{\partial}_j\partial_i F-\partial_j F$ are rewritten in the form

$$2G_j=y^i g_{rj}N^r{}_i+y^i y_r\dot{\partial}_j N^r{}_i-y_r N^r{}_j.$$

Since $\dot{\partial}_j N^r{}_i=F_j{}^r{}_i+P^r{}_{ij}$, $N^i{}_j=F_0{}^i{}_j=T_0{}^i{}_j+F_j{}^i{}_0$, we have

$$2G^i=y^r N^i{}_r+T^i{}_{00}+P_{00}{}^i.$$

By differentiating by y^k, we get the expression for $N^i{}_k$. The expressions for $F_j{}^i{}_k$ and $C_j{}^i{}_k$ follow from the definition of $P^i{}_{jk}$ and (B5) respectively.

Putting $P^i{}_{jk}=0$ in Theorem 1, we get the coefficients of a Berwald connection with torsion $B\Gamma T$ of Matsumoto [4]. According to Miron-Hashiguchi [6], a generalized Cartan connection $GC\Gamma$ with torsion $T_j{}^i{}_k$ is uniquely determined for an arbitrarily given alternate tensor $T_j{}^i{}_k$. In the case of $B\Gamma T$, however, the torsion $T_j{}^i{}_k$ should satisfy the following $B\Gamma$*-condition* (Matsumoto [4]):

(1.3) $$y^r(\dot\partial_k T_j{}^i{}_r - \dot\partial_j T_k{}^i{}_r) = 0.$$

In Theorem 1, therefore, the torsions $T_j{}^i{}_k$, $P^i{}_{jk}$ should satisfy some conditions, too. We want to clear up such a different matter.

§ 2. Matsumoto's $B\Gamma$-condition.

For a given Finsler connection $F\Gamma = (F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ we get a Finsler connection $FN\Gamma = (\dot\partial_j N^i{}_K, N^i{}_k, 0)$ called the N*-connection* of $F\Gamma$. Since $\dot\partial_j N^i{}_k = F_j{}^i{}_K + P^i{}_{kj}$, the $(h)h$-torsion $Q_j{}^i{}_k$ of $FN\Gamma$ is exressed by the torsions $T_j{}^i{}_k$, $P^i{}_{jk}$ of $F\Gamma$ as

(2.1) $$Q_j{}^i{}_k = T_j{}^i{}_k - (P^i{}_{jk} - P^i{}_{kj}).$$

Calculating from $Q_j{}^i{}_k = \dot\partial_j N^i{}_k - \dot\partial_k N^i{}_j$, we have

Proposition 1. *Let $F\Gamma$ be a Finsler connection. If $N^i{}_k$ of $F\Gamma$ is* (1) *p-homogeneous, the tensor $Q_j{}^i{}_k$ given by* (2.1) *satisfies the condition*

(2.2) $$y^r(\dot\partial_k Q_j{}^i{}_r - \dot\partial_j Q_k{}^i{}_r) = 0.$$

In the case of $P^i{}_{jk} = 0$, the condition (2.2) *is the $B\Gamma$-condition* (1.3).

In the case of $GC\Gamma$ we can choose $T_j{}^i{}_k$ arbitrarily, but the condition (2.2) is implicitly imposed on $T_j{}^i{}_k$ together with $P^i{}_{jk}$. In the case of $B\Gamma T$ the condition (2.2) is explicitly imposed on $T_j{}^i{}_k$ as the $B\Gamma$-condition, since we assume $P^i{}_{jk} = 0$ for $B\Gamma T$. This is the reason for the difference between the arbitrariness of $T_j{}^i{}_k$ in $GC\Gamma$ and the one in $B\Gamma T$.

We shall here give remarks about the arbitrariness of $Q_j{}^i{}_k$ satisfying the condition (2.2). By Matsumoto [4] a (0) p-homogeneous alternate tensor $Q_j{}^i{}_k$ satisfying (2.2) is written in the form

(2.3) $$Q_j{}^i{}_k = (\dot\partial_k A^i{}_j - \dot\partial_j A^i{}_k)/2,$$

where $A^i{}_j$ is an arbitrary (1) p-homogeneous tensor. Such a $Q_j{}^i{}_k$ is also written in the form

(2.4) $$Q_j{}^i{}_k = A_j{}^i{}_k + y^r(\dot\partial_k A_j{}^i{}_r - \dot\partial_j A_k{}^i{}_r)/2,$$

where $A_j{}^i{}_k$ is an arbitrary (0) p-homogeneous alternate tensor. Then we get

Proposition 2. *For a given* (0) *p-homogeneous alternate tensor $T_j{}^i{}_k$, a tensor $P^i{}_{jk}$ satisfying* (2.2) *is expressed in the form*

(2.5) $$P^i{}_{jk} = (T_j{}^i{}_k - Q_j{}^i{}_k)/2 + B_j{}^i{}_k,$$

where $Q_j{}^i{}_k$ is a tensor given by (2.3) *or* (2.4), *and $B_j{}^i{}_k$ is an arbitrary* (0) *p-homogeneous symmetric tensor.*

For a given (0) *p-homogeneous tensor $P^i{}_{jk}$, an alternate tensor $T_j{}^i{}_k$ satisfying* (2.2) *is expressed in the form*

(2.6) $$T_j{}^i{}_k = P^i{}_{jk} - P^i{}_{kj} + Q_j{}^i{}_k,$$

where $Q_j{}^i{}_k$ is a tensor given by (2.3) *or* (2.4).

On the other hand, in order to consider the converse problem of Theorem 1, we need another relation satisfied by $P^i{}_{jk}$. If we assume the p-homogeneity of $N^i{}_k$ for a Finsler connection, we have $P^i{}_{j0} = -D^i{}_k$, since $P^i{}_{jk} = \dot\partial_k N^i{}_j - F_k{}^i{}_j$. Thus we have a well-known

Proposition 3. *Let $F\Gamma$ be a Finsler connection. If $N^i{}_k$ of $F\Gamma$ is* (1) *p-homogeneous, the de-*

flection tensor $D^i{}_k$ *vanishes if and only if*

(2.7) $$P^i{}_{j0}=0.$$

§ 3. Generalized Berwald connections.

In this section we assume Finsler connections to be p-homogeneous. The torsions $T_j{}^i{}_k$, $P^i{}_{jk}$ should satisfy the conditions (2.2) and (2.7). The converse is also true as follows.

Theorem 2. *Given* (0) *p-homogeneous tensors* $T_j{}^i{}_k(=-T_k{}^i{}_j)$, $P^i{}_{jk}$, *there exists a unique Finsler connection* $F\Gamma$ *satisfying* (B1), (B2), (B5) *whose* $(h)h$*-and* $(v)hv$*-torsion tensors are the given* $T_j{}^i{}_k$, $P^i{}_{jk}$ *respectively, if* $T_j{}^i{}_k$ *and* $P^i{}_{jk}$ *satisfy the conditions* (2.2), (2.7), *where* $Q_j{}^i{}_k$ *is a tensor given by* (2.1).

The coefficients of $F\Gamma$ *are given by* (1.1). $N^i{}_k$ *of* $F\Gamma$ *is also expressed in the form*

(3.1) $$N^i{}_k=G^i{}_k-(Q_k{}^i{}_0+\dot{\partial}_k Q^i{}_{00})/2.$$

Proof. It is directly shown that the Finsler connection given by (1.1) satisfies the conditions for $F\Gamma$ of Theorem 2. The uniqueness follows from Theorem 1. The p-homogeneity of $N^i{}_k$ yields (3.1).

We can show here that there exists a Finsler connection $F\Gamma$ of Berwald type whose $(h)h$-torsion is an arbitrarily given (0)p-homogeneous alternate tensor $T_j{}^i{}_k$, if we give up to impose the axiom (B4) on $F\Gamma$. Let $T_j{}^i{}_k$ be a (0)p-homogeneous alternate tensor. If we take the $GC\Gamma=(F_j{}^i{}_k,\ N^i{}_k,\ C_j{}^i{}_k)$ whose $(h)h$-torsion is the given $T_j{}^i{}_k$, its torsions $T_j{}^i{}_k$ and $P^i{}_{jk}$ satisfy the conditions (2.2) and (2.7). The Finsler connection given by Theorem 2 has the given $(h)h$-torsion. So we shall define

Definition 1. A Finsler connection $GB\Gamma=(F_j{}^i{}_k,\ N^i{}_k,\ G_j{}^i{}_k)$ satisfying (B1), (B2), (B5) is called a *generalized Berwald connection.*

We have shown a method to obtain a generalized Berwald connection $GB\Gamma$ whose $(h)h$-torsion is an arbitrarily given (0)p-homogeneous alternate tensor $T_j{}^i{}_k$, by taking $GC\Gamma=(F_j{}^i{}_k,\ N^i{}_k,\ C_j{}^i{}_k)$ whose $(h)h$-torsion is the given $T_j{}^i{}_k$. It is easily seen that this $GB\Gamma$ is nothing but the *C-zero connection* $(F_j{}^i{}_k,\ N^i{}_k,\ 0)$ of $GC\Gamma$. Especially, if we take $T_j{}^i{}_k=0$, the $GC\Gamma$ becomes $C\Gamma=(\Gamma^*{}_j{}^i{}_k,\ G^i{}_k,\ g_j{}^i{}_k)$ and the $GB\Gamma$ is the Rund connection $R\Gamma=(\Gamma^*{}_j{}^i{}_k,\ G^i{}_k,\ 0)$. Thus we have

Theorem 3. *The C-zero connection of a generalized Cartan connection is a generalized Berwald connection. Especially, the Rund connection is a generalized Berwald connection obtained from the Cartan connection as its C-zero connection.*

The above definition of a generalized Berwald connection $GB\Gamma$ has various advantages. First, Theorem 3 and Proposition 3 show a wide freedom of choosing $T_j{}^i{}_k$, $P^i{}_{jk}$ to construct a $GB\Gamma$. Next, we can give the following comparative definitions for $GC\Gamma$ and $GB\Gamma$.

Definition 2. A Finsler connection $GC\Gamma=(F_j{}^i{}_k,\ F_0{}^i{}_k,\ g_j{}^i{}_k)$ satisfying $g_{ij|k}=0$ is called a *generalized Cartan connection*, and a Finsler connection $GB\Gamma=(F_j{}^i{}_k,\ F_0{}^i{}_k,\ 0)$ satisfying $L_{|k}=0$ is called a *generalized Berwald connection.*

Last, we pay attention to a generalized Berwald space. A Finsler connection is called *linear*, if the coefficients $F_j{}^i{}_k$ depend on position alone. A Finsler space is a generalized Berwald space, if we can introduce a linear $B\Gamma T$ (Matsumoto [4]). Now, if a generalized Berwald connection $GB\Gamma$ is linear, we have $P^i{}_{jk}=\dot{\partial}_k F_0{}^i{}_j - F_k{}^i{}_j=0$. So the $GB\Gamma$ becomes a $B\Gamma T$. Consequently we can also define a generalized Berwald space in terms of a $GB\Gamma$, even if it is defined by weaker conditions than ones for a $B\Gamma T$.

Theorem 4. *A Finsler space is a generalized Berwald space, if we can introduce a linear* $GB\Gamma$.

By the above considerations a Finsler connection $GB\Gamma$ given by Definition 1 seems to be reasonable as a generalization of the Berwald connection.

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