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ON THE CARTAN AND BERWALD EXPRESSIONS OF FINSLER CONNECTIONS

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Abstract

The purpose of the present paper is to give two kinds of general expressions of Cartan and Berwald types for any Finsler connection in a Finsler space or in a generalized Finsler space, and to consider what kinds of Finsler tensor fields are essential in order to determine a Finsler connection.

Introduction

Let (M, L) be a Finsler space, that is, a differentiable manifold M endowed with a fundamental function $L(x, y)$ $(y^i = \dot{x}^i)$. The fundamental tensor field g_{ij} is given by $g_{ij} = (\dot{\partial}_i \dot{\partial}_j L^2)/2$, where $\dot{\partial}_i = \partial/\partial y^i$. We shall express a Finsler connection $F\Gamma$ in terms of its coefficients as $F\Gamma = (N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$.

In a Finsler space there are known two canonical Finsler connections, that is, the Cartan one $C\Gamma$ and the Berwald one $B\Gamma$. They are uniquely determined by the following systems of axioms respectively:

$C\Gamma$ (Matsumoto [7])	$B\Gamma$ (Okada [12])
(C1) $g_{ij\|k} = 0$,	(B1) $L_{\|k} = 0$,
(C2) $D^i{}_k\ (= y^j F_j{}^i{}_k - N^i{}_k) = 0$,	(B2) $D^i{}_k = 0$,
(C3) $T_j{}^i{}_k\ (= F_j{}^i{}_k - F_k{}^i{}_j) = 0$,	(B3) $T_j{}^i{}_k = 0$,
(C4) $S_j{}^i{}_k\ (= C_j{}^i{}_k - C_k{}^i{}_j) = 0$,	(B4) $P^i{}_{jk}\ (= \dot{\partial}_k N^i{}_j - F_k{}^i{}_j) = 0$,
(C5) $g_{ij}\|_k = 0$,	(B5) $C_j{}^i{}_k = 0$,

where the short and long bars denote the respective h- and v-covariant differentiations, $D^i{}_k$ the deflection tensor field, and $T_j{}^i{}_k$, $S_j{}^i{}_k$ and $P^i{}_{jk}$ the $(h)\,h$-, $(v)\,v$- and $(v)\,hv$-

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torsion tensor fields respectively. It is noted that the coefficients $C_j{}^i{}_k$ of $F\Gamma$ are also the components of the $(h)hv$-torsion tensor field of $F\Gamma$.

If we omit some axioms from one of the above systems, we get various Finsler connections of *Cartan* or *Berwald type*. For example, a Finsler connection satisfying (C 1), (C 2), (C 4), (C 5) is called a *generalized Cartan connection* (cf. Hashiguchi-Ichijyō [6]), and one satisfying (B 1), (B 2), (B 5) is called a *generalized Berwald connection* (cf. Aikou-Hashiguchi [2]), and these have contributed to generalize the notion of Berwald space. More generally, a Finsler connection satisfying (C 1), (C 5) is called *metrical* (cf. Miron-Hashiguchi [10]), and one satisfying (B 1), (B 5) is called *L-metrical* (cf. Aikou-Hashiguchi [3]). On the other hand, from various standpoints, non-metrical Finsler connections have been also studied.

For a better understanding of the above systems of axioms, in the present paper we shall show that any Finsler connection $F\Gamma$ is uniquely determined by the tensor fields appeared in each of the above systems. After the preliminary Section 1, in Section 2 the coefficients of $F\Gamma$ are expressed in terms of its $g_{ij|k}$, $D^i{}_k$, $T_j{}^i{}_k$, $S_j{}^i{}_k$ and $g_{ij}|_k$ (Theorem 2.1), and in Section 3 the coefficients of $F\Gamma$ are expressed in terms of its $L_{|k}$, $D^i{}_k$, $T_j{}^i{}_k$, $P^i{}_{jk}$ and $C_j{}^i{}_k$ (Theorem 3.1 and Theorem 3.3). These expressions are called the *Cartan* and *Berwald expressions* of $F\Gamma$ respectively. The problems of arbitrariness of the above tensor fields are discussed (Theorem 2.2, Theorem 3.2 and Theorem 3.4). In the last Section 4 we shall treat the case of a generalized Finsler space (M, g_{ij}) (cf. Miron [9], Hashiguchi [5] and Watanabe-Ikeda [14]). Similar problems are found in Schouten [13, pp.131-137], Atanasiu-Ghinea [4] and Miron-Hashiguchi [11], etc., too.

As to the terminology and notations we use those in [10] and [3], which are essentially based on Matsumoto [8]. In reference to $T_j{}^i{}_k$, the components of the $(v)v$-torsion tensor field are denoted not by $S^i{}_{jk}$ but by $S_j{}^i{}_k$.

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1. The $*$-operation

Throughout the present paper we shall use the following abridged notations without comment: $K_0{}^i{}_k = y^j K_j{}^i{}_k$, $\mathbf{S}_{jk}\{K_j{}^i{}_k\} = K_j{}^i{}_k + K_k{}^i{}_j$, $\mathbf{A}_{jk}\{K_j{}^i{}_k\} = K_j{}^i{}_k - K_k{}^i{}_j$. The metric tensor field g_{ij} and its conjugate g^{ij} will serve for lowering and raising indices, e.g., $K_{jk}{}^i = g_{ks} g^{ir} K_j{}^s{}_r$, where the position and order of indices are important.

For any Finsler tensor fields K_{jhk} and $K_j{}^i{}_k$ we define the $*$-*operation* as follows:

$$K^*{}_{jhk} = (K_{jkh} + K_{kjh} - K_{khj})/2, \tag{1.1}$$

(1.2) $$K^*{}_j{}^i{}_k=(K_{jk}{}^i+K_{kj}{}^i-K_k{}^i{}_j)/2.$$

Then we have

Proposition 1.1. (1) *If K_{jhk} is symmetric (resp. alternate) with respect to j, k, then*

(1.3) $$\mathbf{S}_{jk}\{K^*{}_{jhk}\}=K_{jkh} \qquad (\text{resp. } =0).$$

(2) *If $K_j{}^i{}_k$ is symmetric (resp. alternate) with respect to j, k, then*

(1.4) $$\mathbf{A}_{jk}\{K^*{}_j{}^i{}_k\}=0 \qquad (\text{resp. } =K_j{}^i{}_k).$$

Let us consider so-called Obata's operators:

(1.5) $$\underset{1}{\Omega}{}^{ir}_{sj}=(\delta^i_s\delta^r_j-g_{sj}g^{ir})/2, \quad \underset{2}{\Omega}{}^{ir}_{sj}=(\delta^i_s\delta^r_j+g_{sj}g^{ir})/2.$$

These operators act on a Finsler tensor field $K_j{}^i{}_k$ as

(1.6) $$\underset{1}{\Omega}{}^{ir}_{sj}K_r{}^s{}_k=(K_j{}^i{}_k-K^i{}_{jk})/2, \quad \underset{2}{\Omega}{}^{ir}_{sj}K_r{}^s{}_k=(K_j{}^i{}_k+K^i{}_{jk})/2,$$

and if K_{jkh} is symmetric with respect to j, h, then $K^*{}_j{}^i{}_k$ is expressed as

(1.7) $$K^*{}_j{}^i{}_k=K_{jk}{}^i/2-\underset{1}{\Omega}{}^{ir}_{sj}K_r{}^s{}_k.$$

The $*$-operation is linear: if $H_j{}^i{}_k=K_j{}^i{}_k+L_j{}^i{}_k$, then $H^*{}_j{}^i{}_k=K^*{}_j{}^i{}_k+L^*{}_j{}^i{}_k$, and it acts on a Finsler tensor field expressed as $H_j{}^i{}_k=K_{jk}{}^i$ (resp. $H_j{}^i{}_k=K_{kj}{}^i$) as $H^*{}_j{}^i{}_k=(K_j{}^i{}_k+K_k{}^i{}_j-K_{kj}{}^i)/2$ (resp. $H^*{}_j{}^i{}_k=(K^i{}_{jk}+K^i{}_{kj}-K_{jk}{}^i)/2$). Hence we have

Proposition 1.2. (1) *Let K_{jkh} be symmetric with respect to j, h. If $H_j{}^i{}_k=\mathbf{A}_{jk}\{K_{jk}{}^i\}/2$, then $H^*{}_j{}^i{}_k=\underset{1}{\Omega}{}^{ir}_{sj}K_r{}^s{}_k$.*

(2) *Let K_{jkh} be alternate with respect to j, h. If $H_j{}^i{}_k=\mathbf{A}_{jk}\{K_{jk}{}^i\}$, then $H^*{}_j{}^i{}_k=K_{jk}{}^i$.*

(3) *If $H_j{}^i{}_k=\mathbf{A}_{jk}\{\underset{1}{\Omega}{}^{ir}_{sj}X_r{}^s{}_k\}$, then $H^*{}_j{}^i{}_k=\underset{1}{\Omega}{}^{ir}_{sj}X_r{}^s{}_k$.*

The last statement (3) follows directly from (2) by putting $K_{jk}{}^i=\underset{1}{\Omega}{}^{ir}_{sj}X_r{}^s{}_k$.

2. The Cartan expression of a Finsler connection

In a Finsler space, a Finsler connection $F\Gamma=(N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$ is expressed by the difference from the Cartan connection $C\Gamma=(G^i{}_k,\ \overset{c}{F}{}_j{}^i{}_k,\ \overset{c}{C}{}_j{}^i{}_k)$ as follows:

(2.1) $$N^i{}_k=G^i{}_k-X^i{}_k,$$

(2.2) $$F_j{}^i{}_k=\overset{c}{F}{}_j{}^i{}_k+\overset{c}{C}{}_j{}^i{}_mX^m{}_k-B_j{}^i{}_k,$$

(2.3) $$C_j{}^i{}_k=\overset{c}{C}{}_j{}^i{}_k-D_j{}^i{}_k.$$

Then we have

Proposition 2.1. *Let U_{ikj}, V_{ikj} and $D^i{}_k$ be any Finsler tensor fields.*

(1) $g_{ij|k}=U_{ikj}$, $g_{ij}|_k=V_{ikj}$ *are equivalent to*

(2.4) $\underset{2}{\Omega}{}^{ir}_{sj}B_r{}^s{}_k=U_{jk}{}^i/2,$ (2.4′) $\underset{2}{\Omega}{}^{ir}_{sj}D_r{}^s{}_k=V_{jk}{}^i/2$

respectively.

(2) $F_0{}^i{}_k-N^i{}_k=D^i{}_k$ *is equivalent to*

$$X^i{}_k=D^i{}_k+B_0{}^i{}_k. \tag{2.5}$$

Now, let $F\Gamma=(N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$ be any given Finsler connection, and we put

$$W_{ikj}=(\dot{\partial}_m g_{ij})X^m{}_k, \tag{2.6}$$

(2.7) $U_{ikj}=g_{ij|k}.$ (2.7′) $V_{ikj}=g_{ij}|_k.$

Since $B_j{}^i{}_k$ satisfies (2,4), from the result about Obata's operators (cf. [10]) it is expressed in the form $B_j{}^i{}_k=U_{jk}{}^i/2-\underset{1}{\Omega}{}^{ir}_{sj}X_r{}^s{}_k$ by some Finsler tensor field $X_j{}^i{}_k$, and so we have

$$F_j{}^i{}_k=\overset{c}{F}{}_j{}^i{}_k+(W_{jk}{}^i-U_{jk}{}^i)/2+\underset{1}{\Omega}{}^{ir}_{sj}X_r{}^s{}_k, \tag{2.8}$$

from which we have

$$T_j{}^i{}_k=\mathbf{A}_{jk}\{(W_{jk}{}^i-U_{jk}{}^i)/2+\underset{1}{\Omega}{}^{ir}_{sj}X_r{}^s{}_k\}. \tag{2.9}$$

Paying attention to Proposition 1.2 and applying the $*$-operation to (2.9), we have $T^*{}_j{}^i{}_k=\underset{1}{\Omega}{}^{ir}_{sj}(W_r{}^s{}_k-U_r{}^s{}_k+X_r{}^s{}_k)$, by which we can eliminate $\underset{1}{\Omega}{}^{ir}_{sj}X_r{}^s{}_k$ from (2.8). $C_j{}^i{}_k$ are similarly treated, and we have from (1.7)

Proposition 2.2. *In a Finsler space, let $F\Gamma=(N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$ be any Finsler connection. Then if we define $X^i{}_k$, W_{ikj}, U_{ikj} and V_{ikj} by* (2.1), (2.6), (2.7) *and* (2,7′) *respectively, we have*

$$F_j{}^i{}_k=\overset{c}{F}{}_j{}^i{}_k+W^*{}_j{}^i{}_k-U^*{}_j{}^i{}_k+T^*{}_j{}^i{}_k, \tag{2.10}$$

$$C_j{}^i{}_k=\overset{c}{C}{}_j{}^i{}_k-V^*{}_j{}^i{}_k+S^*{}_j{}^i{}_k. \tag{2.11}$$

On the other hand, since $X^i{}_k$ and $B_j{}^i{}_k$ of $F\Gamma$ satisfy (2.5), we have from (2.2) and (2.10)

$$X^i{}_k+W^*{}_0{}^i{}_k=K^i{}_k, \tag{2.12}$$

where

$$K^i{}_k=D^i{}_k+U^*{}_0{}^i{}_k-T^*{}_0{}^i{}_k. \tag{2.13}$$

Since $W^*{}_0{}^i{}_k = \overset{c}{C}_k{}^i{}_m X^m{}_0$, we have from (2.12)

$$X^i{}_k = K^i{}_k - g^{ir}(\dot\partial_m g_{kr}/2)K^m{}_0. \tag{2.14}$$

Thus we have proved

Theorem 2.1. *In a Finsler space, let* $F\Gamma = (N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$ *be any Finsler connection. If we construct* U_{ikj}, V_{ikj}, $K^i{}_k$, $X^i{}_k$ *and* W_{ikj} *from* (2.7), (2.7'), (2.13), (2.14) *and* (2.6) *successively, then the coefficients of* $F\Gamma$ *are expressed as* (2.1), (2.10) *and* (2.11).

The expression of $F\Gamma$ stated in Theorem 2.1 is called the *Cartan expression* of $F\Gamma$. It is noted a Finsler connection is uniquely determined by its $g_{ij|k}$, $g_{ij}|_k$, $D^i{}_k$, $T_j{}^i{}_k$ and $S_j{}^i{}_k$. This shows an excellence of Matsumoto's system of axioms for $C\Gamma$.

Conversely, the above tensor fields are arbitrarily given. In fact, we have

Theorem 2.2. *In a Finsler space, let* U_{ikj} $(=U_{jki})$, V_{ikj} $(=V_{jki})$, $D^i{}_k$, $T_j{}^i{}_k$ $(=-T_k{}^i{}_j)$ *and* $S_j{}^i{}_k$ $(=-S_k{}^i{}_j)$ *be any Finsler tensor fields. Then there exists a unique Finsler connection* $F\Gamma = (N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$, *in which the h- and v-covariant derivatives of* g_{ij}, *the deflection tensor field, the (h)h- and (v)v-torsion tensor fields are the given* U_{ikj}, V_{ikj}, $D^i{}_k$, $T_j{}^i{}_k$ *and* $S_j{}^i{}_k$ *respectively. If we construct* $K^i{}_k$, $X^i{}_k$ *and* W_{ikj} *by* (2.13), (2.14) *and* (2.6) *from the given tensor fields, then* $F\Gamma$ *is given by* (2.1), (2.10) *and* (2.11).

Proof. From (2.2) and (2.10) we have

$$B_{jhk} = W_{jkh}/2 - W^*{}_{jhk} + U^*{}_{jhk} - T^*{}_{jhk}. \tag{2.15}$$

Proposition 1.1 (1) yields $\mathbf{S}_{jh}\{B_{jhk}\} = U_{jkh}$, i.e., (2.4), which is equivalent to $g_{ij|k} = U_{ikj}$. Similarly $g_{ij}|_k = V_{ikj}$ is obtained. Also using (2.15), and $X^i{}_0 = K^i{}_0$, we have (2.5), which is equivalent to $F_0{}^i{}_k - N^i{}_k = D^i{}_k$. Finally, $F_j{}^i{}_k - F_k{}^i{}_j = T_j{}^i{}_k$ (resp. $C_j{}^i{}_k - C_k{}^i{}_j = S_j{}^i{}_k$) follows directly by applying Proposition 1.1 (2) to (2.10) (resp. (2.11)).

3. The Berwald expression of a Finsler connection

In a Finsler space (M, L), a Finsler connection $F\Gamma = (N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$ is expressed by the difference from the Berwald connection $B\Gamma = (G^i{}_k,\ G_j{}^i{}_k,\ 0)$ as follows:

$$N^i{}_k = G^i{}_k - X^i{}_k, \tag{3.1}$$

$$F_j{}^i{}_k = G_j{}^i{}_k - \dot\partial_j X^i{}_k - P^i{}_{kj}. \tag{3.2}$$

The coefficients $C_j{}^i{}_k$ of $F\Gamma$ coincide with the components of the (h) hv-torsion tensor field of $F\Gamma$. Then we have

Proposition 3.1. *Let* L_k, $D^i{}_k$, $T_j{}^i{}_k$ *and* $P^i{}_{jk}$ *be any Finsler tensor fields, and we put*

(3.3) $$F_k = LL_k,$$

(3.4) $$Q_j{}^i{}_k = T_j{}^i{}_k - (P^i{}_{jk} - P^i{}_{kj}),$$

(3.5) $$H^i{}_k = D^i{}_k + T_k{}^i{}_0 + P^i{}_{0k}.$$

(1) $L_{|k} = L_k$ *is equivalent to*

(3.6) $$y_m X^m{}_k = F_k,$$

which is also equivalent to

(3.6′) $$X_0{}^i = F^i.$$

(2) $F_0{}^i{}_k - N^i{}_k = D^i{}_k$ *is equivalent to*

(3.7) $$X^i{}_k - y^r \dot{\partial}_r X^i{}_k = D^i{}_k + P^i{}_{k0}.$$

(3) $F_j{}^i{}_k - F_k{}^i{}_j = T_j{}^i{}_k$ *is equivalent to*

(3.8) $$\dot{\partial}_k X^i{}_j - \dot{\partial}_j X^i{}_k = Q_j{}^i{}_k.$$

Under the assumption (3.8), *the condition* (3.7) *is equivalent to*

(3.7′) $$X^i{}_k - y^r \dot{\partial}_k X^i{}_r = H^i{}_k,$$

which is also expressed as

(3.7″) $$2X^i{}_k - \dot{\partial}_k X^i{}_0 = H^i{}_k.$$

Now, let $F\Gamma = (N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$ be any given Finsler connection, and we put

(3.9) $$L_k = L_{|k},\quad F_k = (L^2/2)_{|k}\ (= LL_k),$$

(3.10) $$E_k = y^r \dot{\partial}_k F_r - F_k\ (= \dot{\partial}_k F_0 - 2F_k).$$

Since $X^i{}_k$ satisfies (3.6), operating $g^{ir} y^k \dot{\partial}_r$ to the both-sides of (3.6), we have from (3.6′)

(3.11) $$X^i{}_0 - X_0{}^i + g^{ir} y^k y_m \dot{\partial}_r X^m{}_k = E^i.$$

On the other hand, since $X^i{}_k$ satisfies (3.7′), we have

(3.12) $$X_0{}^i - g^{ik} y^r y_m \dot{\partial}_k X^m{}_r = H_0{}^i.$$

Hence we have from (3.11)

(3.13) $$X^i{}_0 = H_0{}^i + E^i,$$

and from (3.7″) and (3.13)

(3.14) $$X^i{}_k=(H^i{}_k+\dot\partial_k(H_0{}^i+E^i))/2.$$

Thus we have proved

Theorem 3.1. *In a Finsler space, let* $F\Gamma=(N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$ *be any Finsler connection. If we construct* F_k, E_k, $H^i{}_k$ *and* $X^i{}_k$ *from* (3.9), (3.10), (3.5) *and* (3.14) *successively, then the coefficients* $N^i{}_k$ *and* $F_j{}^i{}_k$ *of* $F\Gamma$ *are expressed as* (3.1) *and* (3.2).

The expression of $F\Gamma$ stated in Theorem 3.1 is called the *Berwald expression* of $F\Gamma$. It is noted a Finsler connection is uniquely determined by its $L_{|k}$, $D^i{}_k$, $T_j{}^i{}_k$, $P^i{}_{jk}$ and $C_j{}^i{}_k$. This shows an excellence of Okada's system of axioms for $B\Gamma$.

Contrary to the Cartan expression, the above tensor fields are not arbitrarily given. In fact, from Proposition 3.1 we have

Theorem 3.2. *In a Finsler space* (M, L), *let* L_k, $D^i{}_k$, $T_j{}^i{}_k$ $(=-T_k{}^i{}_j)$ *and* $P^i{}_{jk}$ *be any Finsler tensor fields, and we construct* F_k, E_k, $Q_j{}^i{}_k$, $H^i{}_k$ *and* $X^i{}_k$ *by* (3.3), (3.10), (3.4), (3.5) *and* (3.14) *from the given tensor fields. Then there exists a unique Finsler connection* $F\Gamma=(N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$, *in which the h-covariant derivative of* L, *the deflection tensor field, the (h)h- and (v)hv-torsion tensor fields are the given* L_k, $D^i{}_k$, $T_j{}^i{}_k$ *and* $P^i{}_{jk}$ *respectively, if and only if* $X^i{}_k$ *satisfies the conditions* (3.6), (3.7) *and* (3.8). *The coefficients* $C_j{}^i{}_k$ *are arbitrarily given.*

It is noted that $X^i{}_k$ given by (3.14) satisfies (3.7) and (3.8) if and only if $X^i{}_k$ satisfies (3.7) and

(3.15) $$y^r\dot\partial_r(\dot\partial_k X^i{}_j-\dot\partial_j X^i{}_k)=y^r(\dot\partial_k Q_j{}^i{}_r-\dot\partial_j Q_k{}^i{}_r).$$

If we assume a Finsler connection is positively homogeneous, the conditions (3.7) and (3.15) become

(3.16) $$D^i{}_k+P^i{}_{k0}=0,$$

(3.17) $$y^r(\dot\partial_k Q_j{}^i{}_r-\dot\partial_j Q_k{}^i{}_r)=0$$

respectively, and $H^i{}_k$ and $X^i{}_k$ are given by

(3.18) $$H^i{}_k=Q_k{}^i{}_0,$$

(3.19) $$X^i{}_k=(Q_k{}^i{}_0+\dot\partial_k(Q^i{}_{00}+E^i))/2$$

respectively. Hence Theorem 3.1 and Theorem 3.2 are stated as follows.

Theorem 3.3. *In a Finsler space, let* $F\Gamma=(N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$ *be any positively homogeneous Finsler connection. If we construct* F_k, E_k, $Q_j{}^i{}_k$ *and* $X^i{}_k$ *from* (3.9), (3.10), *(3.4) and (3.19) successively, then the coefficients $N^i{}_k$ and $F_j{}^i{}_k$ of $F\Gamma$ are expressed as (3.1) and (3.2).*

Theorem 3.4. *In a Finsler space (M, L), let L_k, $D^i{}_k$, $T_j{}^i{}_k$ $(=-T_k{}^i{}_j)$, $P^i{}_{jk}$ and $C_j{}^i{}_k$ be any positively homogeneous Finsler tensor fields of respective degrees 1, 1, 0, 0, -1, and satisfy the conditions (3.16) and (3.17), where $Q_j{}^i{}_k$ is constructed by (3.4) from the given $T_j{}^i{}_k$, $P^i{}_{jk}$. Then there exists a unique positively homogeneous Finsler connection $F\Gamma=(N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$, in which the h-covariant derivative of L, the deflection tensor field, the (h)h-, (v)hv- and (h)hv-torsion tensor fields are the given L_k, $D^i{}_k$, $T_j{}^i{}_k$, $P^i{}_{jk}$ and $C_j{}^i{}_k$ respectively. If we construct F_k, E_k and $X^i{}_k$ by (3.3), (3.10) and (3.19) from the given tensor fields, then the coefficients $N^i{}_k$ and $F_j{}^i{}_k$ are expressed as (3.1) and (3.2).*

For the proof it suffices to show (3.6), which follows from (3.16) and

$$y_i \dot{\partial}_k E^i = 2F_k. \tag{3.20}$$

4. The case of a generalized Finsler space

Let (M, g_{ij}) be a generalized Finsler space, where g_{ij} is a generalized Finsler metric tensor field defined as a symmetric and non-degenerate Finsler tensor field of type (0,2). We assume that g_{ij} is regular in the sense of Miron [9]:

$$(\dot{\partial}_k g_{ij}) y^i y^j = 0, \tag{4.1}$$

$$\det(A^i{}_k) \neq 0, \tag{4.2}$$

where $A^i{}_k = \delta^i_k + g^{is}(\dot{\partial}_k g_{rs}) y^r$, $(g^{ij}) = (g_{ij})^{-1}$. Since the condition (4.1) yields $\dot{\partial}_j \dot{\partial}_k (g_{rs} y^r y^s)/2 = g_{jk} + (\dot{\partial}_k g_{rj}) y^r$, we have $A_{jk} = A_{kj}$. $(\dot{\partial}_k g_{rj}) y^r$ is symmetric with respect to j, k.

Since $(A^i{}_k)$ has the inverse, $(B^i{}_k)$, we can put $G^i = B^i{}_r \gamma_0{}^r{}_0/2$, where $\gamma_j{}^i{}_k$ denote the Christoffel symbols with respect to g_{ij}. Then $G^i{}_k = \dot{\partial}_k G^i$ define a non-linear connection. Two canonical Finsler connections $MC\Gamma$ and $MB\Gamma$ have been known by Miron [9].

The one $MC\Gamma = (G^i{}_k, \overset{m}{F}_j{}^i{}_k, \overset{m}{C}_j{}^i{}_k)$ called the *Miron-Cartan connection* is given by

$$\overset{m}{F}_j{}^i{}_k = g^{ir}(\delta_k g_{jr} + \delta_j g_{kr} - \delta_r g_{jk})/2, \tag{4.3}$$

$$\overset{m}{C}_j{}^i{}_k = g^{ir}(\dot{\partial}_k g_{jr} + \dot{\partial}_j g_{kr} - \dot{\partial}_r g_{jk})/2, \tag{4.4}$$

where $\partial_k = \partial/\partial x^k$, $\delta_k = \partial_k - G^r{}_k \dot{\partial}_r$. This is uniquely determined by Matsumoto's system of axioms for $C\Gamma$ in which (C2) is replaced by (C2*) $N^i{}_k = G^i{}_k$.

For any Finsler tensor field $X^i{}_k$ we have the expression $\overset{m}{C}_j{}^i{}_r X^r{}_k = (W_{jk}{}^i + K_{jk}{}^i)/2$, where $W_{jkh} = (\dot{\partial}_r g_{jh}) X^r{}_k$, $K_{jkh} = \mathbf{A}_{jh}\{(\dot{\partial}_j g_{hr}) X^r{}_k\}$, and W_{jkh} (resp. K_{jkh}) is symmetric (resp. alternate) with respect to j, h.

If we express any Finsler connection $F\Gamma$ by the difference from the Miron-Cartan connection $MC\Gamma$ as (2.1), (2.2) and (2.3), where $\overset{c}{F}{}_j{}^i{}_k$ and $\overset{c}{C}{}_j{}^i{}_k$ are replaced by $\overset{m}{F}{}_j{}^i{}_k$ and $\overset{m}{C}{}_j{}^i{}_k$ respectively, then by the similar way as the proof of Proposition 2.2 we have

Theorem 4.1. *In a generalized Finsler space* $(M,\ g_{ij})$, *let* $F\Gamma=(N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$ *be any Finsler connection. Then if we define* $X^i{}_k$, W_{ikj}, U_{ikj} *and* V_{ikj} *by* (2.1), (2.6), (2.7) *and* (2.7′) *respectively, the coefficients* $F_j{}^i{}_k$, $C_j{}^i{}_k$ *of* $F\Gamma$ *are expressed as*

$$F_j{}^i{}_k=\overset{m}{F}{}_j{}^i{}_k+W^*{}_j{}^i{}_k-U^*{}_j{}^i{}_k+T^*{}_j{}^i{}_k, \tag{4.5}$$

$$C_j{}^i{}_k=\overset{m}{C}{}_j{}^i{}_k-V^*{}_j{}^i{}_k+S^*{}_j{}^i{}_k. \tag{4.6}$$

Conversely, the tensor fields U_{ikj}, V_{ikj}, $X^i{}_k$, $T_j{}^i{}_k$ and $S_j{}^i{}_k$ appeared in Theorem 4.1 are arbitrarily given. In fact, we have

Theorem 4.2. *In a generalized Finsler space* $(M,\ g_{ij})$, *let* $U_{ikj}\ (=U_{jki})$, $V_{ikj}\ (=V_{jki})$, $X^i{}_k$, $T_j{}^i{}_k\ (=-T_k{}^i{}_j)$ and $S_j{}^i{}_k\ (=-S_k{}^i{}_j)$ *be any Finsler tensor fields. Then there exists a unique Finsler connection* $F\Gamma=(N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$, *in which the h- and v-covariant derivatives of* g_{ij}, *the difference of the non-linear connections of* $F\Gamma$ *from* $MC\Gamma$, *the* (h)h- *and* (v)v-*torsion tensor fields are the given* U_{ikj}, V_{ikj}, $X^i{}_k$, $T_j{}^i{}_k$ *and* $S_j{}^i{}_k$ *respectively.* $F\Gamma$ *is given by* (2.1), (4.5) *and* (4.6) *constructed from the given tensor fields.*

We shall here consider whether the role of $X^i{}_k$ in the above theorems is replaced by the deflection tensor field $D^i{}_k$. Let $F\Gamma=(N^i{}_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$ be any Finsler connection. Instead we define $K^i{}_k$ for a Finsler space by (2.13), we define $K^i{}_k$ by

$$K^i{}_k=D^i{}_k-\overset{m}{D}{}^i{}_k+U^*{}_0{}^i{}_k-T^*{}_0{}^i{}_k, \tag{4.7}$$

where $\overset{m}{D}{}^i{}_k$ is the deflection tensor field of $MC\Gamma$. Then it is shown in the same way as for a Finsler space that the difference $X^i{}_k$ given by (2.1) satisfies (2.12).

Contracting (2.12) by y^k, we have from (4.1)

$$X^i{}_0=B^i{}_rK^r{}_0. \tag{4.8}$$

Hence if we put

$$\Omega^{ir}_{sk}=\underset{2}{\Omega}{}^{ir}_{sk}+\underset{1}{\Omega}{}^{ir}_{mk}A^m{}_s, \tag{4.9}$$

we can rewrite (2.12) as

$$\Omega^{ir}_{sk}X^s{}_r=K^i{}_k-g^{ir}(\dot\partial_m g_{kr}/2)B^m{}_sK^s{}_0. \tag{4.10}$$

Conversely, we can show $F_0{}^i{}_k-N^i{}_k=D^i{}_k$ from (2.1), (4.5), (4.7), (4.8) and (4.10). Thus we have proved

Theorem 4.3. *In a generalized Finsler space* (M, g_{ij}), *let* $F\Gamma=(N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$ *be any Finsler connection. The difference tensor field* $X^i{}_k$ $(=G^i{}_k-N^i{}_k)$ *in Theorem* 4.1 *satisfies* (4.8) *and* (4.10), *where* $K^i{}_k$ *is given by* (4.7).

Conversely, let $F\Gamma=(N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$ *be the Finsler connection given by Theorem* 4.2 *from any Finsler tensor fields* U_{ikj} $(=U_{jki})$, V_{ikj} $(=V_{jki})$, $X^i{}_k$, $T_j{}^i{}_k$ $(=-T_k{}^i{}_j)$ *and* $S_j{}^i{}_k$ $(=-S_k{}^i{}_j)$. *Let* $D^i{}_k$ *be any Finsler tensor field, and we define* $K^i{}_k$ *by* (4.7) *from the given* $D^i{}_k$, U_{ikj} *and* $T_j{}^i{}_k$. *If* $X^i{}_k$ *satisfies* (4.8) *and* (4.10), *then the deflection tensor field of* $F\Gamma$ *is the given* $D^i{}_k$.

We shall consider a case the equation (4.10) with the unknown $X^i{}_k$ has solutions. If the $n^2\times n^2$ -matrix (Ω^{ir}_{sk}), with Ω^{ir}_{sk} as the (ik), (sr) component, is regular, that is, if there exists a Finsler tensor field Φ^{iq}_{pk} satisfying

$$\Phi^{iq}_{pk}\Omega^{pr}_{sq}=\Omega^{iq}_{pk}\Phi^{pr}_{sq}=\delta^i_s\delta^r_k, \tag{4.11}$$

then Φ^{iq}_{pk} is uniquely determined by Ω^{ir}_{sk}, and (4.10) has the unique solution

$$X^i{}_k=\Phi^{iq}_{pk}(K^p{}_q-g^{pr}(\dot{\partial}_m g_{qr}/2)B^m{}_sK^s{}_0). \tag{4.12}$$

Since Ω^{ir}_{sk} is given by g_{ij} only, the regularity for (Ω^{ir}_{sk}) imposes on g_{ij} a regularity condition. The tensor field

$$A^{pg}_{kj}=2\Omega^{pr}_{ks}g_{rj}g^{sq} \tag{4.13}$$

is just the one given by (3.1) of Watanabe-Ikeda [14]. Hence the above regularity condition is the one in their sense. They introduced this condition in order to assure the existence and uniqueness of the *Cartan-like connection*, i.e., the Finsler connection satisfying the same system of axioms as Matsumoto's one for $C\Gamma$. Thus we have

Theorem 4.4. *In a generalized Finsler space* (M, g_{ij}) *satisfying the regularity condition in the sense of Watanabe-Ikeda, let* $F\Gamma=(N^i{}_k, F_j{}^i{}_k, C_j{}^i{}_k)$ *be any Finsler connection. Defining* $X^i{}_k$ *by* (4.12) *the coefficients of* $F\Gamma$ *are expressed as* (2.1), (4.5) *and* (4.6) *in terms of its* $g_{ij|k}$, $g_{ij}|_k$, $D^i{}_k$, $T_j{}^i{}_k$ *and* $S_j{}^i{}_k$.

The above tensor fields $U_{ikj}=g_{ij|k}$, $V_{ikj}=g_{ij}|_k$, $D^i{}_k$, $T_j{}^i{}_k$ *and* $S_j{}^i{}_k$ *are arbitrarily given, under the assumption* (4.8).

In the following we shall assume that g_{ij} is positively homogeneous of degree 0 : $g_{ij}(x,\lambda y)=g_{ij}(x,y)$ for $\lambda>0$. Then calculating from (4.3) we can show $\overset{m}{F}{}_0{}^i{}_0=G^i{}_0$, i.e., $\overset{m}{D}{}^i{}_0=0$.

In Theorem 4.4, if we put $U_{ikj}=V_{ikj}=D^i{}_k=T_j{}^i{}_k=S_j{}^i{}_k=0$, we have $K^i{}_k=-\overset{m}{D}{}^i{}_k$. Since $K^i{}_0=0$, the condition (4.8) becomes $\underset{m}{X}{}^i{}_0=0$, and $X^i{}_k$ given by (4.12) becomes $X^i{}_k=-\Phi^{iq}_{pk}\overset{m}{D}{}^p{}_q$. $X^i{}_k$ satisfies $\Omega^{ir}_{sk}X^s{}_r=-\overset{m}{D}{}^i{}_k$, whose contraction by y^k yields $X^i{}_0+A^i{}_sX^s{}_0=0$. Hence, if the matrix $(A^i{}_k)$ does not have the eigenvalue -1, we have

$X^i{}_0=0$. Since $A^i{}_k y^k=y^i$, the matrix $(A^i{}_k)$ has the eigenvalue 1. By Theorem 3 of [14] the sum of two eigenvalues of $(A^i{}_k)$ is not equal to zero if and only if the matrix (Ω^{ir}_{sk}) is regular. Thus $(A^i{}_k)$ does not have the eigenvalue -1. This shows the existence of the Cartan-like connection in a generalized Finsler space.

The other canonical Finsler connection $MB\Gamma=(G^i{}_k,\ \ G_j{}^i{}_k,\ \ 0)$ called the *Miron-Berwald connection* is given by $G_j{}^i{}_k=\dot{\partial}_j G^i{}_k$. Putting $L=(g_{ij}y^iy^j)^{1/2}$, we have $\bar{g}_{ij}=\dot{\partial}_i\dot{\partial}_j(L^2/2)=g_{ir}A^r{}_j$. Since L is positively homogeneous of degree 1, and satisfies det $(\bar{g}_{ij})$ $\neq 0$, we have a Finsler space $(M,\ L)$. Hence Theorem 3.3 and Theorem 3.4 hold also in the case of a generalized Finsler space $(M,\ g_{ij})$, if the rowering and raising of indices are made by $\bar{g}_{ij}$ and $\bar{g}^{ij}$

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