

SOME REMARKS ON CONFORMAL CHANGES OF  
GENERALIZED FINSLER METRICS Dedicated to  
Professor Dr. Radu Miron on the occasion of  
his sixtieth birthday

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## SOME REMARKS ON CONFORMAL CHANGES OF GENERALIZED FINSLER METRICS

*Dedicated to Professor Dr. Radu Miron  
on the occasion of his sixtieth birthday*

Tadashi AIKOU\*

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### Abstract

The purpose of the present paper is to investigate conformal changes of Finsler metrics and generalized Finsler metrics, which are also projective or collinear.

### Introduction

Let  $(M, g_{ij})$ ,  $(M, \bar{g}_{ij})$  be two Riemannian spaces with a same underlying manifold  $M$ . As is well-known, if the change  $g_{ij} \rightarrow \bar{g}_{ij}$  is conformal and projective, then it is homothetic.

In the present paper, we shall first extend this result to the case of Finsler metrics (Theorem 1.1), and also to the case of generalized Finsler metrics (Theorem 2.4). On the other hand, in the previous paper [8], we introduced the notion of collinear change of Finsler connections and obtained some results. In the last section, we shall consider its application to conformal changes (Theorem 3.1).

Throughout the present paper the terminology and notations are referred to Matsumoto [6].

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### 1. Conformal and projective changes of Finsler metrics

Let  $F^n=(M,L)$  be an  $n$ -dimensional Finsler space, where  $L(x,y)$  is the fundamental function, and  $x=(x^i)$  and  $(x,y)=(x^i, y^i)$  denote a point and a supporting element of the underlying manifold  $M$  respectively. The fundamental tensor field  $g_{ij}(x,y)$  is given by  $g_{ij}=\partial_i\partial_j L^2/2$ , where  $\partial_i=\partial/\partial y^i$ . A geodesic of  $F^n$  is given by the system of equations in the form

$$(1.1) \quad d^2x^i/ds^2 + 2G^i(x, dx/ds) = 0,$$

\* Department of Mathematics, Faculty of Science, Kagoshima University, Kagoshima, Japan.

if we take the arc-length  $s$  as a parameter.

According to Matsumoto [5], Finsler metrics  $g_{ij}$ ,  $\bar{g}_{ij}$  are projectively related if and only if there exists a (1) $p$ -homogeneous Finsler scalar field  $P(x, y)$  on  $F^n$  satisfying

$$(1.2) \quad \bar{G}^i(x, y) = G^i(x, y) + P(x, y)y^i,$$

where  $\bar{G}^i$  is the quantity constructed from  $\bar{g}_{ij}$ .

On the other hand, according to Hashiguchi [1], if two Finsler metrics  $g_{ij}$ ,  $\bar{g}_{ij}$  are conformally related:

$$(1.3) \quad \bar{g}_{ij} = e^{2\alpha(x)} g_{ij},$$

then we have

$$(1.4) \quad \alpha_i = B_{ij}(G^j - \bar{G}^j),$$

where  $\alpha_i = \partial\alpha/\partial x^i$  and the tensor field  $B_{ij}$  is a conformal invariant given by

$$(1.5) \quad B_{ij} = 2(g_{ij} - 2l_i l_j)/L^2.$$

Now we assume that the conformal change (1.3) of Finsler metrics  $g_{ij}$ ,  $\bar{g}_{ij}$  is also projective. Since the quantities  $G^i$ ,  $\bar{G}^i$  satisfy the equations (1.2), (1.4), we have from (1.5)

$$(1.6) \quad L^2 \alpha_i = 2P y_i.$$

Differentiating (1.6) by  $y^j$  and substituting into the resulting equation from (1.6), we have

$$(1.7) \quad P g_{ij} = l_i (2P l_j - L P_j),$$

where  $P_j = \partial_j P$ . If  $P \neq 0$ , it follows from (1.7) that  $\text{rank}(g_{ij}) \leq 1$ . So we get  $p=0$ , that is,  $\alpha_i = 0$ . Thus we have

**Theorem 1.1.** *Let  $g_{ij}$  and  $\bar{g}_{ij}$  be Finsler metrics. If the change  $g_{ij} \rightarrow \bar{g}_{ij}$  is conformal and projective, then the change is homothetic.*

## 2. Conformal and projective changes of generalized Finsler metrics

Due to Miron [7], a generalized Finsler metric is defined as follows.

**Definition 2.1.** A Finsler tensor field  $g_{ij}$  of type (0,2) on a differentiable manifold  $M$  is called a *generalized Finsler metric*, if it satisfies the following conditions:

$$(1) \quad g_{ij} = g_{ji}, \quad (2) \quad \det(g_{ij}) \neq 0,$$

and  $(M, g_{ij})$  is called a *generalized Finsler space*. A generalized Finsler metric  $g_{ij}$  is called *regular* if it satisfies the following conditions:

$$(3) \quad (\partial_\kappa g_{ij}) y^i y^j = 0, \quad (4) \quad \det(A_j^i) \neq 0,$$

where  $A_j^i$  is a tensor field defined by

$$(2.1) \quad A_j^i = \delta_j^i + g^{im} (\partial_j g_{km}) y^k.$$

In the following, we shall assume that a generalized Finsler space  $(M, g_{ij})$  is regular, that is,  $g_{ij}$  is regular. The arc-length  $s$  of a curve  $x=(x^i)$  in  $(M, g_{ij})$  is measured by  $s = \int L(x, dx/dt) dt$ , where

$$(2.2) \quad L = (g_{ij} y^i y^j)^{1/2},$$

and a geodesic is defined as an extremal of  $\delta \int L(x, dx/dt) dt = 0$ . The homogeneity is not assumed for  $g_{ij}$ . It is noted, however,  $L$  is (1) $p$ -homogeneous owing to the regularity condition (3). This was recently pointed out by Kikuchi [4]. Thus as a parameter of a geodesic in  $(M, g_{ij})$  we can take the arc-length  $s$ , and the system of equations of a geodesic is expressed in the form (1.1).

On the other hand, since  $L$  is (1) $p$ -homogeneous, if we put

$$(2.3) \quad \bar{g}_{ij} = \partial_i \partial_j L^2 / 2,$$

$\bar{g}_{ij}$  is (0) $p$ -homogeneous. Since we have

$$(2.4) \quad \bar{g}_{ij} = A_i^m g_{mj},$$

the matrix  $(\bar{g}_{ij})$  is regular owing to the regularity condition (4). Thus  $(M, \bar{g}_{ij})$  is a Finsler space and called the *Finsler space associated with*  $(M, g_{ij})$ . In these spaces  $(M, g_{ij})$ ,  $(M, \bar{g}_{ij})$ , the arc-length of a curve is measured by the same formula. So we have the following theorem (cf. [2]).

**Theorem 2.1.** *Let  $(M, g_{ij})$  be a regular generalized Finsler space and  $(M, \bar{g}_{ij})$  be the Finsler space associated with  $(M, g_{ij})$ . Then any geodesic in  $(M, \bar{g}_{ij})$  is also a geodesic in  $(M, g_{ij})$  and the converse is also true, and it is expressed by (1.1).*

Now we shall consider the conformal change  $\bar{g}_{ij} = e^{2\alpha(x,y)} g_{ij}$  of generalized Finsler metrics  $g_{ij}$ ,  $\bar{g}_{ij}$ . Then from  $(\partial \bar{g}_{ij}) y^i y^j = 2(\partial_k \alpha) \bar{g}_{ij} y^i y^j + e^{2\alpha} (\partial_k g_{ij}) y^i y^j$ , we have easily the following theorem (cf. [3]).

**Theorem 2.2.** *Let  $g_{ij}$  be a regular generalized Finsler metric. The metric  $\bar{g}_{ij}$  given by  $\bar{g}_{ij} = e^{2\alpha(x,y)} g_{ij}$  is regular if and only if  $\alpha$  depends on position  $(x^i)$  alone.*

Thus we shall consider the conformal change of regular generalized Finsler metrics  $g_{ij}$ ,  $\bar{g}_{ij}$ :

$$(2.5) \quad \bar{g}_{ij} = e^{2\alpha(x)} g_{ij}.$$

Since  $\bar{A}^i = \delta^i_j + \bar{g}^{im} (\partial_j \bar{g}_{km}) y^k = \delta^i_j + g^{im} (\partial_j g_{km}) y^k = A^i_j$ , the Finsler metric associated with  $\bar{g}_{ij}$  is given by

$$(2.6) \quad \tilde{\bar{g}}_{ij} = e^{2\alpha(x)} \bar{g}_{ij}.$$

Consequently we have

**Theorem 2.3.** *Let  $g_{ij}$  and  $\bar{g}_{ij}$  be regular generalized Finsler metrics, and suppose that they are conformally related:  $\bar{g}_{ij} = e^{2\alpha(x)} g_{ij}$ . Then the tensor field  $A^i_j$  is a conformal invariant by the conformal change  $g_{ij} \rightarrow \bar{g}_{ij}$ , and the respective Finsler metrics  $\bar{g}_{ij}$ ,  $\tilde{\bar{g}}_{ij}$  associated with*

$g_{ij}, \bar{g}_{ij}$  are also conformally related:  $\tilde{g}_{ij} = e^{2\alpha(x)} \bar{g}_{ij}$ .

Hence, by Theorem 2.1 and Theorem 2.3, if the conformal change (2.5) of regular generalized Finsler metrics  $g_{ij}, \bar{g}_{ij}$  is projective, the associated conformal change (2.6) of the associated Finsler metrics  $\tilde{g}_{ij}, \tilde{\bar{g}}_{ij}$  is also projective. So, by Theorem 1.1 we get

**Theorem 2.4.** *Let  $g_{ij}$  and  $\bar{g}_{ij}$  be regular generalized Finsler metrics. If the change  $g_{ij} \rightarrow \bar{g}_{ij}$  is conformal and projective, then the change is homothetic.*

### 3. Conformal and collinear changes of generalized Finsler metrics

In the previous paper [8], from the standpoint that a Finsler connection on  $M$  is a linear connection in its tangent bundle  $T(M)$  satisfying a certain condition, we introduced the notion of collinear change of Finsler connections. By the definition, the change  $F\Gamma \rightarrow F\bar{\Gamma}$  of Finsler connections  $F\Gamma = (N^i_j, F^i_{jk}, C^i_{jk}), F\bar{\Gamma} = (\bar{N}^i_j, \bar{F}^i_{jk}, \bar{C}^i_{jk})$  is called *collinear*, if they are expressed as the form

$$(3.1) \quad \bar{N}^i_j = N^i_j - B^i_j, \quad \bar{F}^i_{jk} = F^i_{jk} + C^i_{j\tau} B^{\tau}_k, \quad \bar{C}^i_{jk} = C^i_{jk}$$

where  $B^i_j$  satisfies the conditions

$$(3.2h) \quad B^i_{jk} = 0, \quad (3.2v) \quad B^i_j|_k = 0,$$

with respect to  $F\Gamma$ .

Now we shall consider the conformal change (2.5) of regular generalized Finsler metrics  $g_{ij}, \bar{g}_{ij}$ . Then by Theorem 2.3, the change (2.6) of the associated Finsler metrics  $\tilde{g}_{ij}, \tilde{\bar{g}}_{ij}$  is also conformal. Thus we can use a result of Hashiguchi [1], and the respective non-linear connections  $G^i_j, \bar{G}^i_j$  of Cartan connections  $C\Gamma, C\bar{\Gamma}$  of the Finsler spaces  $(M, \tilde{g}_{ij}), (M, \tilde{\bar{g}}_{ij})$  satisfy the relation

$$(3.3) \quad \bar{G}^i_j = G^i_j - B^i_j,$$

if we put

$$(3.4) \quad B^i_j = y_j \tilde{\alpha}^i - \delta^i_j \alpha_0 - y^i \alpha_j - L^2 \tilde{C}^i_j,$$

where  $y_j = \tilde{g}_{jr} y^r (= g_{jr} y^r)$ ,  $\tilde{\alpha}^i = \tilde{g}^{ir} \alpha_r$ ,  $\alpha_0 = \alpha_r y^r$  and  $\tilde{C}^i_j = -(\partial_j \tilde{g}^{ir} / 2) \alpha_r$ .

It seems to be interesting to consider the case that in the conformal change (2.5) the corresponding change  $C\Gamma \rightarrow C\bar{\Gamma}$  is collinear. In general, however, if the change  $C\Gamma \rightarrow C\bar{\Gamma}$  is collinear, then we have  $C\Gamma = C\bar{\Gamma}$  from the corollary of Theorem 3.1 in [8]. Hence the conformal change is projective, and so homothetic by Theorem 1.1. Since the collinear change of the Cartan connection is nothing but the identity, we hope to consider some changes satisfying weaker conditions.

Let  $F\Gamma, F\bar{\Gamma}$  be Finsler connections on a same underlying manifold  $M$ . The change  $F\Gamma \rightarrow F\bar{\Gamma}$  is called *h-weakly* (resp. *v-weakly*) *collinear* if the difference tensor field  $B^i_j = N^i_j - \bar{N}^i_j$  satisfies the equation (3.2h) (resp. (3.2v)).

In regular generalized Finsler metrics  $g_{ij}, \bar{g}_{ij}$ , if the change  $C\Gamma \rightarrow C\bar{\Gamma}$  of the Cartan

connections  $C\Gamma$ ,  $C\bar{\Gamma}$  of the associated  $(M, \mathbf{g}_{ij})$ ,  $(M, \tilde{\mathbf{g}}_{ij})$  is  $h$ -weakly (resp.  $v$ -weakly) collinear, the change  $\mathbf{g}_{ij} \rightarrow \tilde{\mathbf{g}}_{ij}$  is called  $h$ -weakly (resp.  $v$ -weakly) collinear.

Now suppose that the change (2.5) is  $h$ -weakly collinear. Then by direct calculations from (3.4), we have

$$(3.5) \quad \alpha_{i|j} = 0, \quad \alpha_r \tilde{C}_j{}^{ir} = 0.$$

The converse is also true.

Similarly, if the change (2.5) is  $v$ -weakly collinear, we get  $\alpha_i = 0$ , that is, the change is homothetic.

Thus we have

**Theorem 3.1.** *Let  $\mathbf{g}_{ij}$  and  $\tilde{\mathbf{g}}_{ij}$  be regular generalized Finsler metrics and suppose that they are conformally related:  $\tilde{\mathbf{g}}_{ij} = e^{2\alpha(x)} \mathbf{g}_{ij}$ .*

(1) *If the change  $\mathbf{g}_{ij} \rightarrow \tilde{\mathbf{g}}_{ij}$  is  $h$ -weakly collinear if and only if  $\alpha_i$  satisfies (3.5).*

(2) *If the change  $\mathbf{g}_{ij} \rightarrow \tilde{\mathbf{g}}_{ij}$  is  $v$ -weakly collinear if and only if the change is homothetic.*

#### References

- [1] M. Hashiguchi, On conformal transformations of Finsler metrics, J. Math. Kyoto Univ. **16** (1976), 25–50.
- [2] M. Hashiguchi, On generalized Finsler spaces, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi, I a Mat. **30**–1 (1984), 69–73.
- [3] M. Hashiguchi, Some topics on Finsler geometry, Confer. Sem. Mat. Univ. Bari, **210** (1986), 1–26.
- [4] S. Kikuchi, On metrical Finsler connections of generalized Finsler spaces, Symp. Finsler Geom. at Asahikawa, Aug. 5–8, 1987.
- [5] M. Matsumoto, Projective changes of Finsler metrics and projectively flat Finsler spaces, Tensor N. S., **34** (1980), 303–315.
- [6] M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, Kaiseisha Press, Otsu, Japan, 1986, 365pp.
- [7] R. Miron, Metrical Finsler structures and metrical Finsler connections, J. Math. Kyoto Univ. **23** (1983), 219–224.
- [8] T. Nagano and T. Aikou, On collinear changes of Finsler connections, to appear in Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) **20** (1987).