Rep. Fac. Sci., Kagoshima Univ., (Math., Phys. & Chem.),
No. 20. p. 57–61, 1987.

# SOME REMARKS ON CONFORMAL CHANGES OF GENERALIZED FINSLER METRICS

*Dedicated to Professor Dr. Radu Miron*
*on the occasion of his sixtieth birthday*

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(Received Septemer 10, 1987)

**Abstrat**

The purpose of the present paper is to investigate conformal changes of Finsler metrics and generalized Finsler metrics, which are also projective or collinear.

## Introduction

Let $(M, g_{ij})$, $(M, \bar{g}^i{}_j)$ be two Riemannian spaces with a same underlying manifold $M$. As is well-known, if the change $g_{ij} \rightarrow \bar{g}_{ij}$ is conformal and projective, then it is homothetic.

In the present paper, we shall first extend this result to the case of Finsler metics (Theorem 1.1), and also to the case of generalized Finsler metrics (Theorem 2.4). On the other hand, in the previous paper [8], we introduced the notion of collinear change of Finsler connections and obtained some results. In the last section, we shall consider its application to conformal changes (Theorem 3.1).

Throughout the present paper the terminology and notations are referred to Matsumoto [6].

The author wishes to express his sincere gratitude to Professor Dr. M. Matsumoto and Professor Dr. Y. Ichijyō for the invaluable suggestions and encouragement.

## 1. Conformal and projective changes of Finsler mertics

Let $F^n=(M,L)$ be an $n$-dimensional Finsler space, where $L(x,y)$ is the fundamental function, and $x=(x^i)$ and $(x,y)=(x^i, y^i)$ denote a point and a supporting element of the underlying manifold $M$ respectively. The fundamental tensor field $g_{ij}(x,y)$ is given by $g_{ij}=\dot{\partial}_i\dot{\partial}_j L^2/2$, where $\dot{\partial}_i=\partial/\partial y^i$. A geodesic of $F^n$ is given by the system of equations in the form

(1.1) $$d^2x^i/ds^2+2G^i(x, dx/ds)=0,$$

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if we take the arc-length $s$ as a parameter.

According to Matsumoto [5], Finsler metics $g_{ij}$, $\bar{g}_{ij}$ are projectively related if and only if there exists a (1)$p$-homogenenous Finsler scalar field $P(x,y)$ on $F^n$ satisfying

$$\bar{G}^i(x,y)=G^i(x,y)+P(x,y)y^i, \tag{1.2}$$

where $\bar{G}^i$ is the quantity constructed from $\bar{g}_{ij}$.

On the other hand, according to Hashiguchi [1], if two Finsler metrics $g_{ij}$, $\bar{g}_{ij}$ are conformally related:

$$\bar{g}_{ij}=e^{2\alpha(x)}g_{ij}, \tag{1.3}$$

then we have

$$\alpha_i=B_{ij}(G^j-\bar{G}^j), \tag{1.4}$$

where $\alpha_i=\partial\alpha/\partial x^i$ and the tensor field $B_{ij}$ is a conformal invariant given by

$$B_{ij}=2(g_{ij}-2l_il_j)/L^2. \tag{1.5}$$

Now we assume that the conformal change (1.3) of Finsler metrics $g_{ij}$, $\bar{g}_{ij}$ is also projective. Since the quantities $G^i$, $\bar{G}^i$ satisfy the equations (1.2), (1.4), we have from (1.5)

$$L^2\alpha_i=2Py_i. \tag{1.6}$$

Differentiating (1.6) by $y^j$ and substituting into the resulting equation from (1.6), we have

$$Pg_{ij}=l_i(2Pl_j-LP_j), \tag{1.7}$$

where $P_j=\dot{\partial}_jP$. If $P\neq 0$, it follows from (1.7) that rank $(g_{ij})\leqq 1$. So we get $p=0$, that is, $\alpha_i=0$. Thus we have

**Theorem 1.1.** *Let $g_{ij}$ and $\bar{g}_{ij}$ be Finsler metrics. If the change $g_{ij}\rightarrow\bar{g}_{ij}$ is conformal and projective, then the change is homothetic.*

## 2. Conformal and projective changes of generalized Finsler metrics

Due to Miron [7], a generalized Finsler metric is defined as follows.

**Definition 2.1.** A Finsler tensor field $g_{ij}$ of type (0,2) on a differentiable manifold $M$ is called a *generalized Finsler metric*, if it satisfies the following conditions:

(1) $g_{ij}=g_{ji}$, (2) $\det(g_{ij})\neq 0$,

and $(M, g_{ij})$ is called a *generalized Finsler space*. A generalized Finsler metric $g_{ij}$ is called *regular* if it satisfies the following conditions:

(3) $(\dot{\partial}_k g_{ij})y^iy^j=0$, (4) $\det(A^i_j)\neq 0$,

where $A^i_j$ is a tensor field defined by

$$A^i_j=\delta^i_j+g^{im}(\dot{\partial}_jg_{km})y^k. \tag{2.1}$$

In the following, we shall assume that a generalized Finsler space $(M, g_{ij})$ is regular, that is, $g_{ij}$ is regular. The arc-length $s$ of a curve $x=(x^i)$ in $(M, g_{ij})$ is measured by $s=\int L(x, dx/dt)dt$, where

$$L=(g_{ij}y^i y^j)^{1/2}, \tag{2.2}$$

and a geodesic is defined as an extremal of $\delta\int L(x, dx/dt)dt=0$. The homogeneity is not assumed for $g_{ij}$. It is noted, however, $L$ is (1)$p$-homogeneous owing to the regularity condition (3). This was recently pointed out by Kikuchi [4]. Thus as a parameter of a geodesic in $(M, g_{ij})$ we can take the arc-length $s$, and the system of equations of a geodesic is expressed in the form (1.1).

On the other hand, since $L$ is (1)$p$-homogeneous, if we put

$$\tilde{g}_{ij}=\dot{\partial}_i\dot{\partial}_j L^2/2, \tag{2.3}$$

$\tilde{g}_{ij}$ is (0)$p$-homegeneous. Since we have

$$\tilde{g}_{ij}=A_i^m g_{mj}, \tag{2.4}$$

the matrix $(\tilde{g}_{ij})$ is regular owing to the regularity condition (4). Thus $(M, \tilde{g}_{ij})$ is a Finsler space and called the *Finsler space associated with* $(M, g_{ij})$. In these spaces $(M, g_{ij})$, $(M, \tilde{g}_{ij})$, the arc-length of a curve is measured by the same formula. So we have the following theorem (cf. [2]).

**Theorem 2.1.** *Let $(M, g_{ij})$ be a regular generalized Finsler space and $(M, \tilde{g}_{ij})$ be the Finsler space associated with $(M, g_{ij})$. Then any geodesic in $(M, \tilde{g}_{ij})$ is also a geodesic in $(M, g_{ij})$ and the converse is also true, and it is expressed by* (1.1).

Now we shall consider the conformal change $\bar{g}_{ij}=e^{2\alpha(x,y)}g_{ij}$ of generalized Finsler metrics $g_{ij}$, $\bar{g}_{ij}$. Then from $(\dot{\partial}\bar{g}_{ij})y^i y^j=2(\dot{\partial}_k\alpha)\bar{g}_{ij}y^i y^j+e^{2\alpha}(\dot{\partial}_k g_{ij})y^i y^j$, we have easily the following theorem(cf. [3]).

**Theorem 2.2.** *Let $g_{ij}$ be a regular generalized Finsler metric. The metric $\bar{g}_{ij}$ given by $\bar{g}_{ij}=e^{2\alpha(x,y)}g_{ij}$ is regular if and only if $\alpha$ depends on position $(x^i)$ alone.*

Thus we shall consider the conformal change of regular generalized Finsler metrics $g_{ij}$, $\bar{g}_{ij}$:

$$\bar{g}_{ij}=e^{2\alpha(x)}g_{ij}. \tag{2.5}$$

Since $\bar{A}_j^i=\delta_j^i+\bar{g}^{im}(\dot{\partial}_j\bar{g}_{km})y^k=\delta_j^i+g^{im}(\dot{\partial}_j g_{km})y^k=A_j^i$, the Finsler metric associated with $\bar{g}_{ij}$ is given by

$$\tilde{\bar{g}}_{ij}=e^{2\alpha(x)}\tilde{g}_{ij}. \tag{2.6}$$

Consequently we have

**Theorem 2.3.** *Let $g_{ij}$ and $\bar{g}_{ij}$ be regular generalized Finsler metrics, and suppose that they are conformally related: $\bar{g}_{ij}=e^{2\alpha(x)}g_{ij}$. Then the tensor field $A_j^i$ is a conformal invariant by the conformal change $g_{ij}\to\bar{g}_{ij}$, and the respective Finsler metrics $\tilde{g}_{ij}$, $\tilde{\bar{g}}_{ij}$ associated with*

*$g_{ij}$, $\bar{g}_{ij}$ are also conformally related: $\tilde{\bar{g}}_{ij}=e^{2\alpha(x)}\tilde{g}_{ij}$.*

Hence, by Theorem 2.1 and Theorem 2.3, if the conformal change (2.5) of regular generalized Finsler metrics $g_{ij}$, $\bar{g}_{ij}$ is projective, the associated conformal change (2.6) of the associated Finsler metrics $\tilde{g}_{ij}$, $\tilde{\bar{g}}_{ij}$ is also projective. So, by Theorem 1.1 we get

**Theorem 2.4.** *Let $g_{ij}$ and $\bar{g}_{ij}$ be regular generalized Finsler metrics. If the change $g_{ij}\to\bar{g}_{ij}$ is conformal and projective, then the change is homothetic.*

## 3. Conformal and collinear changes of generalized Finsler metrics

In the previous paper [8], from the standpoint that a Finsler connection on $M$ is a linear connection in its tangent bundle $T(M)$ satisfying a certain condition, we introduced the notion of collinear change of Finsler connections. By the definition, the change $F\Gamma\to F\bar{\Gamma}$ of Finsler connections $F\Gamma=(N^i{}_j, F_j{}^i{}_k, C_j{}^i{}_k)$, $F\bar{\Gamma}=(\bar{N}^i{}_j, \bar{F}_j{}^i{}_k, \bar{C}_j{}^i{}_k)$ is called *collinear*, if they are expressed as the form

$$(3.1)\qquad \bar{N}^i{}_j=N^i{}_j-B^i{}_j,\quad \bar{F}_j{}^i{}_k=F_j{}^i{}_k+C_j{}^i{}_rB^r{}_k,\quad \bar{C}_j{}^i{}_k=C_j{}^i{}_k,$$

where $B^i{}_j$ satisfies the conditions

$$(3.2h)\qquad B^i{}_{j|k}=0,\qquad\qquad (3.2v)\qquad B^i{}_j|_k=0,$$

with respect to $F\Gamma$.

Now we shall consider the conformal change (2.5) of regular generalized Finsler metrics $g_{ij}$, $\bar{g}_{ij}$. Then by Theorem 2.3, the change (2.6) of the associated Finsler metrics $\tilde{g}_{ij}$, $\tilde{\bar{g}}_{ij}$ is also conformal. Thus we can use a result of Hashiguchi [1], and the respective non-linear connections $G^i{}_j$, $\bar{G}^i{}_j$ of Cartan connections $C\Gamma$, $C\bar{\Gamma}$ of the Finsler spaces $(M, \tilde{g}_{ij})$, $(M, \tilde{\bar{g}}_{ij})$ satisfy the relation

$$(3.3)\qquad \bar{G}^i{}_j=G^i{}_j-B^i{}_j,$$

if we put

$$(3.4)\qquad B^i{}_j=y_j\tilde{\alpha}^i-\delta^i_j\alpha_0-y^i\alpha_j-L^2\tilde{C}_j{}^i,$$

where $y_j=\tilde{g}_{jr}y^r(=g_{jr}y^r)$, $\tilde{\alpha}^i=\tilde{g}^{ir}\alpha_r$, $\alpha_0=\alpha_ry^r$ and $\tilde{C}_j{}^i=-(\dot{\partial}_j\tilde{g}^{ir}/2)\alpha_r$.

It seems to be interesting to consider the case that in the conformal change (2.5) the corresponding change $C\Gamma\to C\bar{\Gamma}$ is collinear. In general, however, if the change $C\Gamma\to C\bar{\Gamma}$ is collinear, then we have $C\Gamma=C\bar{\Gamma}$ from the corollay of Theorem 3.1 in [8]. Hence the conformal change is projective, and so homothetic by Theorem 1.1. Since the collinear change of the Cartan connection is nothing but the identity, we hope to consider some changes satisfying weaker conditions.

Let $F\Gamma$, $F\bar{\Gamma}$ be Finsler connections on a same underlying manifold $M$. The change $F\Gamma\to F\bar{\Gamma}$ is called *h-weakly* (resp. *v-weakly*) *collinear* if the difference tensor field $B^i{}_j=N^i{}_j-\bar{N}^i{}_j$ satisfies the equation (3.2*h*) (resp. (3.2*v*)).

In regular generalized Finsler metrics $g_{ij}$, $\bar{g}_{ij}$, if the change $C\Gamma\to C\bar{\Gamma}$of the Cartan connections $C\Gamma$, $C\bar{\Gamma}$ of the associated $(M, \tilde{g}_{ij})$, $(M, \tilde{\bar{g}}_{ij})$ is $h$-weakly (resp. $v$-weakly) collinear, the change $g_{ij} \to \bar{g}_{ij}$ is called *h-weakly* (resp. *v-weakly*) *collinear*.

Now suppose that the change (2.5) is $h$-weakly collinear. Then by direct calculations from (3.4), we have

$$\alpha_{i|j}=0, \quad \alpha_r \tilde{C}_j{}^{ir}{}_{|k}=0. \tag{3.5}$$

The converse is also true.

Similarly, if the change (2.5) is $v$-weakly collinear, we get $\alpha_i=0$, that is, the change is homothetic.

Thus we have

**Theorem 3.1.** *Let $g_{ij}$ and $\bar{g}_{ij}$ be regular generalized Finsler metrics and suppose that they are confmally related*: $\bar{g}_{ij}=e^{2\alpha(x)}g_{ij}$.

(1) *If the change $g_{ij} \to \bar{g}_{ij}$ is h-weakly collinear if and only if $\alpha_i$ satisfies* (3.5).

(2) *If the change $g_{ij} \to \bar{g}_{ij}$ is v-weakly collinear if and only if the change is homothetic.*

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