Rep. Fac. Sci., Kagoshima Univ., (Math., Phys. & Chem.)
No. 23, p. 31-40, 1990.

Stability of Variable-Step, Variable-Formula pseudo Runge-Kutta Methods

Masaharu. NAKASHIMA*

(Recieved September 7, 1990)

Abstract

Stabillity of Variable Step-size, Variable-Formula pseudo Runge-Kutta Methods. The present paper deals with the stability of Variable-Stepsize Variable formula of pseudo Runge-Kutta methods in the numerical solution of initial value problems.

1. Introduction

This paper deals with the variable step variable formula of solving the initial value problem:

$$y'=y(x,y),\, y(x_0)=y_0. \tag{1.1}$$

The methods based on variable step-size and variable order are widely used for the numerical solution of ordinary initial value problem and it is proved that the variable step variable formula is superior to the fix step method. However, in [1962] A Nordieck has pointed out instability in his interpolation versions of Adams formula if the step-size was varied too frequently. C.W. Gear and K.W. Tu [2] have also shown that Nordsieck method is unstable unless some restrictions are imposed on the step-size sequence. In general case, stability of variable step-size variable formula can be ascertained if some restrictions are imposed on the step-size sequence.

We have proposed the following pseudo Runge-Kutta method [6, 9]:

$$g^{(i)}(x_n, y_n, y_{n-1}; h) = (1+a_i)\, y_n - a_i y_{n-1} + h \sum_{j=0}^{i-1} a_{ij} f(g^{(j)}(x_n, y_n, y_{n-1})),$$

$$(a_0=-1,\ a_1=0,\ a_{0i}=a_{1i}=0) \quad (i=0, 1, \ldots, r),$$

$$y_{n+1}=b_{-2}y_{n-1}+b_{-1}y_n+ \sum_{i=0}^{\gamma} b_i f(g^{(i)}(x_n, y_n, y_{n-1}; h)),$$

* Department of Mathematics, Faculty of Science, kagoshima University, Kagoshima, 890, Japan.

and studied the variable step variable formula [10, 11, 12]:

$$g^{(i)}(x_n, y_n; h_n) = (1+a_i)y(x_n) - a_i y(x_n - h_n) + h_n \sum_{j=0}^{i-1} a_{ij} f(g^{(j)}(x_n, y_n; h_n)),$$

$$(a_0=-1,\ a_1=0,\ a_{0i}=a_{1i}=0)\quad (i=0, 1, \ldots, r), \tag{1.2}$$

$$y_{n+1} = b_{-2}(h_{n+1})y(x_n - h_n) + b_{-1}(h_{n+1})y(x_n) + \sum_{i=0}^{\gamma} b_i(h_{i+1}) f(g^{(i)}(x_n, y_n; h_n)),$$

providing 3-stage fourth order which is stable only for bounded step-size selection of the grid, so the aim of this paper is to investigate the stability (zero stability) of (1.2) in more detail and to give the numerical results showing the characters of the formula (1.2).

The outline of this paper is as follows. In § 2, we derive the order conditions, the derivation of order can be clearly by using the tree notations studied by J. C. Butcher [1], E, Haire & G, Wanner [3] and many peoples, then we use the those notation for the expression. In §3, we analyze the stability of (1.2) and give some sufficient conditions for stable, in the last section, we shall give some numerical examples justifying the results.

2. Derivation of the formulae

We define the order as follows. The method (1.2) is of order p if

$$|y_{n+1} - u(x_n + h_{n+1})| = 0(h_{n+1}^{p+1}),$$

where $u(x)$ is the true solution to (1.1).
In the first place, we study the other condition of (1.2) with the help of a "tree model". Let us define $\alpha(t)$ as the number of the ways of labelling tree t with a set of ordered symbol such that along each outwardly directed arc labels increase and $\beta(t)$ as the number of the ways of labelling a tree with $r(t)$ distinct labels on the condition that the root is not labeled but every other vertex is labelled. We also define the the elementary differential $F(t)$, corresponding to t, by

$$F(\tau)(y) = f(y),$$

where τ is the tree with a single vertex and by

$$F(t)(y) = f(y)(F(t_1), F(t_2), \ldots, F(t_s)),$$

where $t = [t_1, t_2, \ldots, t_s]$.
and the elementary weight $\Phi_i(t)$ for stage t by

$$\Phi_i(\tau) = c_i,$$

$$\Phi_i(t) = \sum_{j=1}^{s} a_{ij}\Phi_j(t_1)\,\Phi_j(t_2)\cdots\Phi_j(t_s).$$

Then the expansions for y_{n+1} and $u(x_n+h_n)$ are given by

$$y_{n+1} = y(x_n) + \sum_{\gamma(t)\leq p} \frac{\beta(t)\,\Phi(t)\,F(t)}{(r(t)-1)!} h_n^{\gamma(t)} + 0(h_n^{p+1}),$$

and the true solution

$$u(x_n+h_{n+1}) = y(x_n) + \sum_{\gamma(t)\leq p} \frac{\alpha(t)\,F(t)\,y(x_n)}{\gamma(t)!} h_{n+1}^{r(t)} + 0(h_{n+1}^{p+1}).$$

We see that the method (1.2) is of order p if

$$\Phi(t) = \frac{\alpha(t)}{\gamma(t)\,\beta(t)} \left(\frac{h_{n+1}}{h_n}\right)^{r(t)},$$

for all tree t such that $r(t) \leq$ p.
From (1.2) we have

$$(g_i)^{(0)}|_{h_n=0} = y(x_n),$$

$$(g_i)^{(1)}|_{h_n=0} = c_i f(x_n),$$

$$(g_i)^{(2)}|_{h_n=0} = (a_i + 2\sum_i a_i c_j) f(x_n),$$

$$(g_i)^{(3)}|_{h_n=0} = \sum_i(-a_i + 3\sum_j a_{ij} c_j^2) y_{yy}(f,f)\,(y(x_n)) +$$

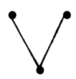
$$\sum_i(-a_i + 3\sum_j a_{ij} c_j (a_i + 2\sum_k a_{jk} c_k) c_j) f_y(f_y(f))\,(y(x_n)).$$

Therefor we have the following order conditions listing up to order 4.

t

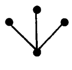
$$-b_{-2} + \sum_i b_i = \theta_n,$$

$$b_{-2} + 2\sum_i b_i c_i = \theta_n^{\ 2},$$

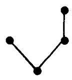


$$-b_{-2} + 3\sum_i b_i c_i^2 = \theta_n^{\ 3}, \qquad (2.1)$$

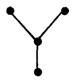
$$-b_{-2}+3\sum_i b_i(-a_i+2\sum_j a_{ij}c_j)=\theta_n{}^3,$$



$$b_{-2}+4\sum_i b_i c_i^3=\theta_n{}^4,$$



$$b_{-2}+4\sum_i b_i(-a_i+2\sum_j a_{ij}c_j)c_i=\theta_n{}^4,$$



$$b_{-2}+4\sum_i b_i(a_i+3\sum_j a_{ij}c_j^2=\theta_n{}^4,$$

$$b_{-2}+4\sum_i b_i\{a_i+3\sum_j a_{ij}(-a_j+2\sum_k a_{jk}c_k)\}=\theta_n{}^4,$$

where $a_{i\text{-}1}=-1$, $a_{i0}=0$ and $\theta_n=\dfrac{h_{n+1}}{h_n}$.

Solving (2.1) with $r=2$, we have the solutions, abbreviating $b_i=b_i(\theta_n)$, as follows:

order 2:

$$b_0=\frac{b_{-2}}{2}+c_2b_2-\frac{\theta^2}{2},\ b_1=\theta_n-(-b_{-1}+b_0+b_2), \tag{2.2}$$

$$a_{21}=c_2-(a_2+a_{20}),\ b_{-1}=1-b_{-2},$$

b_{-2}, b_2, c_2, θ_n, a_2, a_{20}; free parameters.

order 3:

$$b_2=\frac{\{\theta_n{}^2(3+2\theta_n)-b_{-2}\}}{6c_2(1+c_2)},\ b_0=\frac{b_{-2}}{2}+c_2b_2-\frac{\theta_n{}^2}{2},$$

$$b_1=\theta_n-\{-b_{-2}+b_0+b_2\},\ b_{-1}=1-b_{-2}, \tag{2.3}$$

$$a_{20}=-\frac{a_2}{2}-\frac{c_2^2}{2},\ a_{21}=c_2-(a_2+a_{20}),$$

b_{-2}, c_2, θ_n, a_2; free parameters.

order 4.

$$b_2=\frac{\theta_n{}^2(\theta_n+1)^2}{\{2c_2(2c_2+1)(c_2+1)\}},\ b_0=(3c_2^2+4c_2^3)-(\theta_n{}^3+\theta_n{}^4)\ , \tag{2.4}$$

$b_{-2}=\theta_n^{\ 2}-2c_2b_2+2b_0,\ b_{-1}=1-b_{-2},\ b_1=\theta_n-(-b_{-1}+b_0+b_2),$

$a_2=-(3c_2^2+2c_2^3),\ a_{20}=c_2^2+c_2^3,\ a_{21}=c_2-(a_2+a_{20}),$

$c_2,\ \theta_n$; free parameters.

3. Stability properties of pseudo Runge-Kutta Method

We now turn to the stability problem.

Definition.

Consider

$$\overline{y_{n+1}}=b_{-2}\overline{y_{n-1}}+b_{-1}\overline{y_n}+\Sigma b_i(h_{n+1})f(g^{(i)}(x_n,h_n,\overline{y_n}))+\gamma_n, \tag{3.1}$$

where the function f $(g^{(i)}(x_n,h_n,y_n))$ are same to that in (1.2).
The formula (3.1) is called zero-stability if there exist a constant u and for any $\varepsilon>0$, $\delta(\varepsilon)$ such that

$$|y_n-\bar{y}_n|\leq\varepsilon,$$

whenever

$$\sum_{i=0}^{n}|\gamma_i|\leq\delta(\varepsilon),$$

uniformly in $h_n\leq u$.
Introducing the notations:

$$Y_{n+1}=(y_n,y_{n+1})^t,\qquad \overline{Y_{n+1}}=(\overline{y_n},\overline{y_{n+1}})^t,$$

$$F(Y_n)=(0,\ \sum_{i=0}^{r}b_i(h_{n+1})f(g^{(i)}(x_n,h_n,y_n)))^t,$$

$$F(\overline{Y_n})=(0,\ \sum_{i=0}^{r}b_i(h_{n+1})f(g^{(i)}(x_n,h_n,\overline{y_n})))^t,$$

$$A_n=\begin{pmatrix}0 & 1\\ b_{-2} & b_{-1}\end{pmatrix},$$

the equation (1.2) and (3.1) can be written in the form

$$Y_{n+1}=A_nY_n+F(Y_n), \tag{3.2}$$

$$\overline{Y_{n+1}}=A_n\overline{Y}_n+F(\overline{Y}_n)+(0,\ \gamma_{n+1})^t, \tag{3.3}$$

Subtracting (3.2) from (3.3), we have

$$\varepsilon_{n+1}=A_n\,\varepsilon_n+B_n\,\varepsilon_n+C_{n+1}, \tag{3.4}$$

where

$$\varepsilon_n=Y_n-\overline{Y}_n,$$

$$B_n=\begin{pmatrix} 0 & 0 \\ p_n & q_n \end{pmatrix},\ C_{n+1}=\begin{pmatrix} 0 \\ \gamma_{n+1} \end{pmatrix},$$

with

$$p_n=-a_2b_2f_y(\delta)+b_2a_{20}h_nf_y(\delta)f_y(y_{n-1}+\eta_1(y_{n-1}-\overline{y_{n-1}})),$$

$$q_n=b_2(1+a_2)f_y(\delta)+b_2a_{21}h_nf_y(\delta)f_y(y_n+\eta_2(y_n-\overline{y_n})),$$

$$\delta=f_y(g^{(2)}(y_n)+\eta_3(g^{(2)}(y_n)-g^{(2)}(\overline{y_n}))\quad(0\leq\eta_1,\ \eta_2,\ \eta_3\leq1).$$

We call (3.4) as the stability equation and we want to know the behavior of δ_n. Using standard teandard techniques one obtain the following Lemmas.

Lemma 3.1. *The solution of*

$$\delta_{n+1}=A_n\,\delta_n+B_n\,\delta_{n-1}+C_n$$

is

$$\delta_n=\sum_{j=1}^{n}S_{nj}B_{j-1}\,\delta_{j-1}+\sum_{j=1}^{n}S_{nj}C_j, \tag{3.5}$$

where

$$S_{nj}=A_{n-1}A_{n-2}\cdots A_j,\ \textit{and}\ S_{nn}=I.$$

From the result stated above we know that the behavior of δ_n is determined by the norm of S_{nj}. The following conditions will be shown to be sufficient for δ_n to be bounded.

Lemma 3.2. *If there exists constants K_1 and K_2 such that*

(1) $|S_{nj}|\leq K_1,$

(2) $|B_j|\leq K_2$ for all $n, j,$

then

$$|\delta_n|\leq K\sum_{j=1}^{n}\gamma_j,$$

where K is some constant.

Proof

We can prove the result by using the Bellman-Gronwell inequality. By assumptions (1) and (2), we have

$$|\delta_n| \leq K_1K_2 \sum_{j=1}^{n} |\delta_{j-1}| + K_1 \sum_{j=1}^{n} C_j. \tag{3.6}$$

Solving this equation yields the result.

When the step size-ratio θ_n is constatnt, A_n is constant matrix with the eigenvalue of 1 and b_{-2}, so it is easy to derive the condition for the boundedness of S_{nj}.

Lemma 3.3. *When the step size-ratio θ_n is constant and*

$$|b_{-2}| = |1-b_{-1}| < 1,$$

Then

$$|S_{nj}| \leq K.$$

From the result, we have the following Theorem.

Theorem 1. *The method (1.2) with (2.4) is stable if the constant θ_n satisfies*

$$\frac{\theta_n^2 |-3\theta_n^2+4(a_2-1)\theta_n+6a_2|}{(2a_2+1)} < 1, \tag{3.7}$$

which lead to

$$0<a_2\leq 1 \quad (0<\theta_n\leq\bar{\theta}),$$

$$\frac{3\theta_n^4+4\theta_n^3-1}{4\theta_n^3+6\theta_n^2+2}<a_2\leq 1 \quad (\bar{\theta}<a_2<1.5\cdots),$$

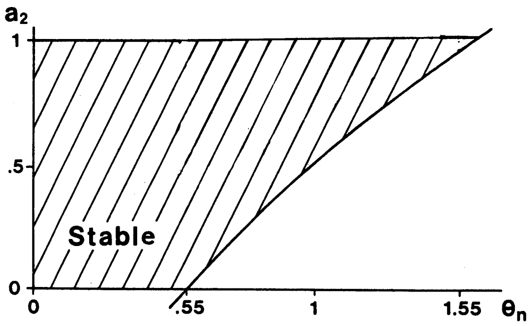
where $\bar{\theta}$ is the positive root to $3x^4+4x^3-1=0$.

We have plotted the region $A(\theta_n, a_2)$ satisfing (3.7) in Figure (I).

Noting that the coefficients b_{-2} given by (2.2) and (2.3) are independent upon θ_n, we may say the following corollary.

Corollary. *The method (2.2) given by (2.2) and (2.3) is A_0-stable under for any θ_n.*

when A_i is the constant matrix, as we have seen, it is easy to study the stability conditions. however for the variable matrix A_i, it is difficult to obtain the conditions for stability. We [13] shall study these problem by using the spectral decomposition notations.



Stability region in the plane (θ_n, a_2) for Theorem 1

4. Numerical Examples

In order to test the method (1.2), we wish to present some nimerical results to show how our scheme compares with R-K method. The descrided method is programmed in FORTRAN and run on the Personal Computer 9801RA (NEC). The computations are done in double precision.

The test problems, where (2) and (3) are considered from DETEST [14], are the following:

(1) $y' = -y + x^2, y(0) = 3,$

(2) $y' = -\frac{y^3}{2}, y(0) = 1,$

(3) $y' = (y(1 - y/20))/4, y(0) = 14.$

The true solutions to the problems (1), (2), and (3) are

$$y(x) = exp(-x) + 2 - 2x + x^2,$$

$$y(x) = 1/\sqrt{(x+1)},$$

$$y(x) = 20/(1 + 19.\exp(-x/4)),$$

respectively. The initial value y_1 necessary for the method (1.2), is computed by Runge-Kutta of order five.

Table Result usig the $\theta_n = 1.1$ and $h = 1/2^7$

Problem 1

x (h_n)	Absolute error 0.062.. (0.012..)	0.500.. (0.052..)	3.811.. (0.3535.)
R-K 3	0.666E-10	0.188E-7	0.177E-4
F N	18	63	123
R-K 4	0.173E-12	0.193E-9	0.142E-5
F N	24	84	164
R-K 5	0.642E-10	0.341E-9	0.909E-7
F N	36	126	246
(1.2)3	0.765E-8	0.182E-5	0.183E-3
F N	12	41	81
(1.2)4	0.275E-10	−0.269E-7	−0.353E-4
F N	12	41	81

Problem 2

x (h_n)	Absolute error 0.062.. (0.012..)	0.500.. (0.052..)	3.811.. (0.3535.)
R-K 3	0.610E-10	0.549E-8	0.103E-6
R-K 4	0.463E-14	0.217E-11	0.175E-9
R-K 5	0.104E-10	0.529E-10	0.516E-10
(1.2)3	0.259E-7	0.254E-5	0.509E-4
(1.2)4	−0.246E-9	−0.728E-7	−0.381E-5

Problem 3

x (h_n)	Absolute error 0.062.. (0.012..)	0.500.. (0.052..)	3.811.. (0.3535.)
R-K 3	0.693E-13	0.246E-10	0.668E-7
R-K 4	0.463E-16	0.677E-13	0.133E-8
R-K 5	0.538E-11	0.490E-10	0.698E-9
(1.2)3	0.171E-10	0.636E-8	0.119E-4
(1.2)4	0.874E-14	0.114E-10	−0.664E-7

FN : number of function evaluation.
P-K 3 : Heun third-order formulas.
P-K 4 : Heun fourth-order formulas.
P-K 5 : Heun fifth-order formulas.
(1.2)3 : Method (1.2) with (2.3) taking $c_2=0.5$, $a_2=0.01$ and $b_{-2}=0.01$,
(1.2)4 : Method (1.2) with (2.4) taking $c_2=(40.67)/52$.

References

[1] Butcher, J. C.: Coefficients for the study of Runge-Kutta integration processes. J. Austral. Math. Soc. **3**. 185-201 (1963).

[2] Gear, C. W. & Tu, K. W.: The effect of variable mesh size on the stability of multistep methods. SIAMJ. Numer. Anal. **II**, 1025-1043 (1974).

[3] Hairer, E & Wanner. G.: Multistep-multistage-multiderivative methods for ordinary differntial equations. Computing. **11** (1973). 287-303.

[4] Jackson, L. W. & Skeel, R. D.: The stability of variable-stepsize Nordsieck methods. SIAM J. Nnmer. Anal. **20**. 840-853 (1983).

[5] Piotrowski, P.: Stability, consistensy and convergence of variable K-step methods for numerical integration of large systems of ordinary differential equations. In: Conference on Numerical Solution of Differential Equations. 221-227. Berlin-Heidelberg New York: Springer 1969.

[6] Nakashima, M.: On Pseudo Runge-Kutta Methods with 2 and 3 stage. Publ. RIMS, Kyoto Univ. 18 (1982), 895-909.

[7] ———: Implicit Pseudo Runge-Kutta Processes. Publ. RIMS, Kyoto Univ. **20** (1984), 39-56.

[8] ———: Some Implicit Fourth and Fifth Order Methods with Optimun Processes for Numerical Initial Value Problems. Publ. RIMS, Kyoto Univ. **21** (1985), 255-277.

[9] ———: Pseudo Runge-Kutta Processes. Publ. RIMS, Kyoto Univ. 23 (1987), 583-611.

[10] ———: Some Methods of step size control for Explicit pseudo Runge-Kutta Methods, Intern. Confere. 1988 Initial Value Problems for ODE's at the Univ of Tront.

[11] ———: Stability of Variable-Step, Variable-Formula Pseudo Runge-Kutta Methods. Intrn. Confere. 1990 Scientific Computation at Technical Univ Vienna.

[12] ———: Embedding pseudo Runge-Kutta Methods. To appear in SIAM J. Numer. Anal.

[13] ———: Some another Variable-Step, Variable-Formula pseudo Runge-Kutta Methods. In preparation.

[14] Hull, T. E., Enright, W. H., Fellen, B. M. & Sedgwick, A. E: Comparing numerical methods for ordinary differential equations, SIAM J. Numer. Anal. **9** (1972), 603-637.

[15] Zlatev, Z.: Stability properties of variable stepsize variable formula methods. Numer. Math. **37**, 157-182 (1978).