

On iteration schemes $K_0(n)$ and $K_1(n)$

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$$\text{code}(x) = \sum_{j=1}^n x_j 2^{n-j} \quad \text{for each } x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in X$$

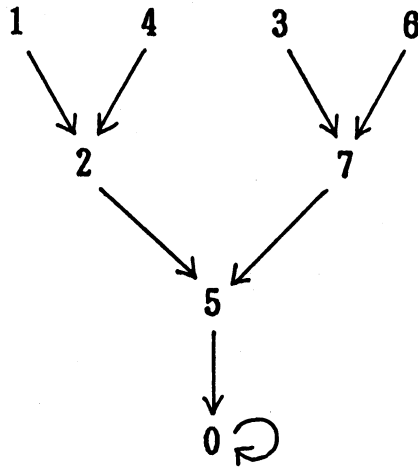
Let us denote an arrow

$$\text{code}(x) \longrightarrow \text{code}(y)$$

when $f(x) = y$ for x and y in X .

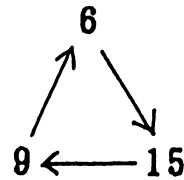
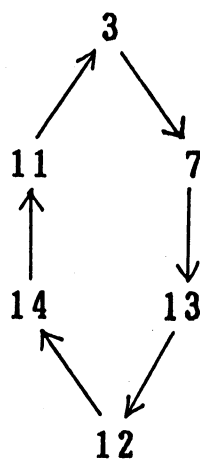
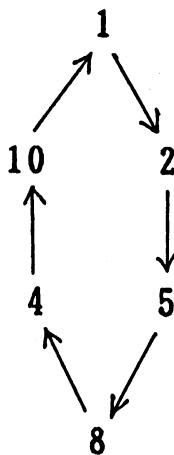
We get a diagram over $\text{code}(x)$ by writing all arrows $\text{code}(x) \longrightarrow \text{code}(y)$ for each x and y in X when $f(x) = y$ holds.

$K_0(3)$:

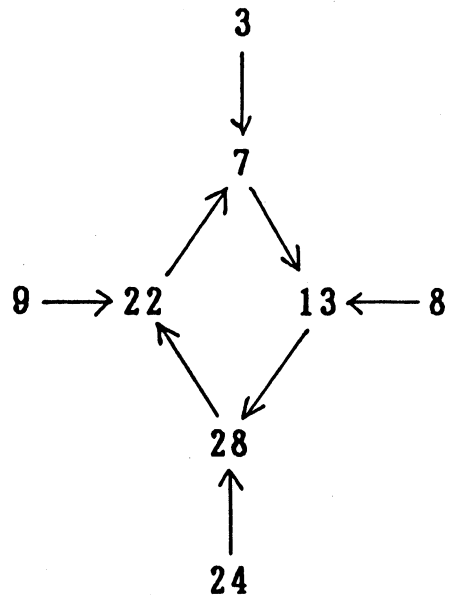
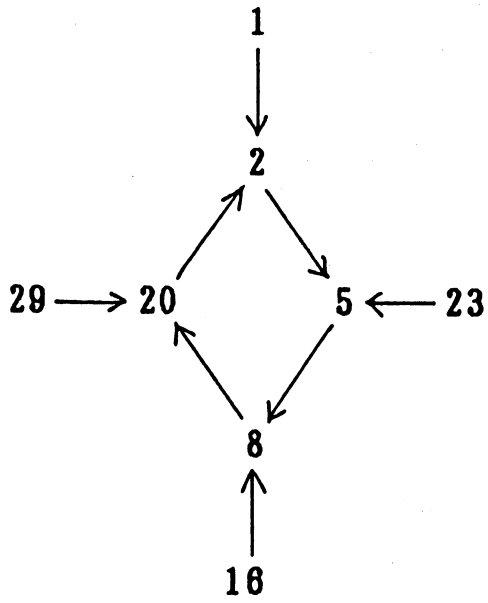


Type of scheme : $\{1_2[2[2]](1)\}$

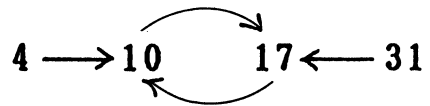
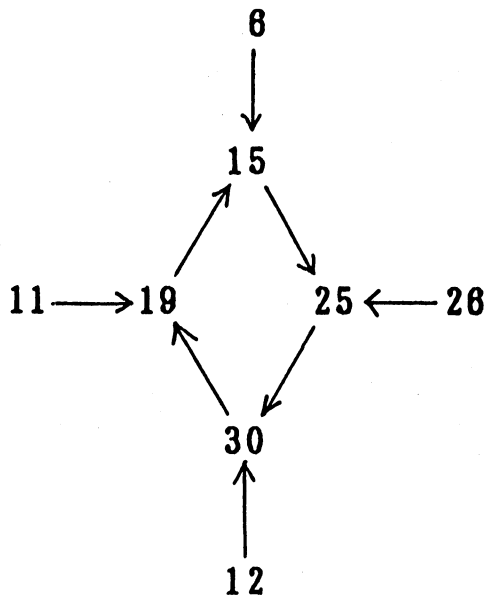
$K_0(4)$:



Type of scheme : $\{6(2), 3(1), 1(1)\}$

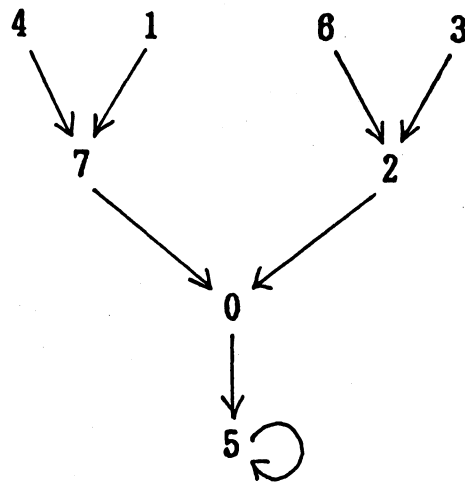


$K_0(5)$:



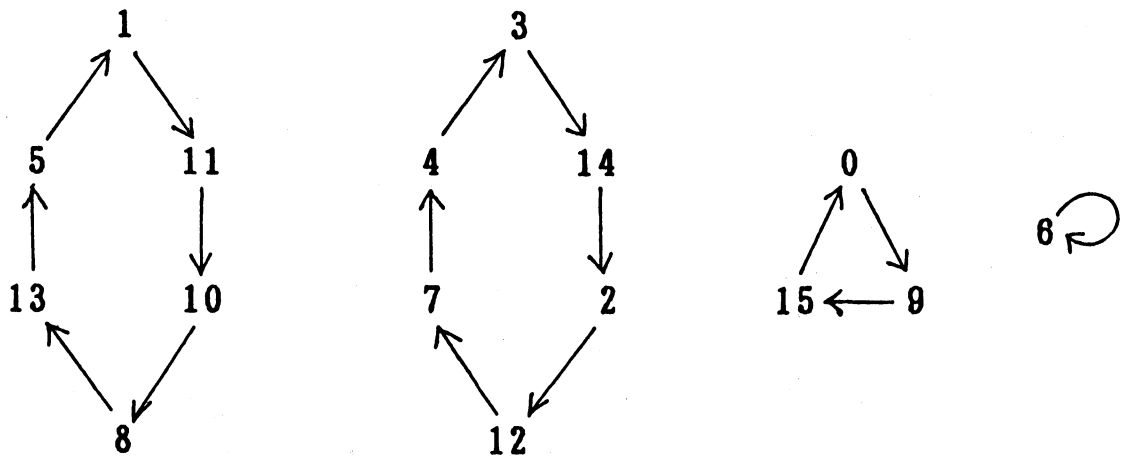
Type of scheme : $\{4_2(3), 2_2(1), 1_2(2)\}$

$K_1(3)$:



Type of scheme : $\{1_2[2[2]](1)\}$

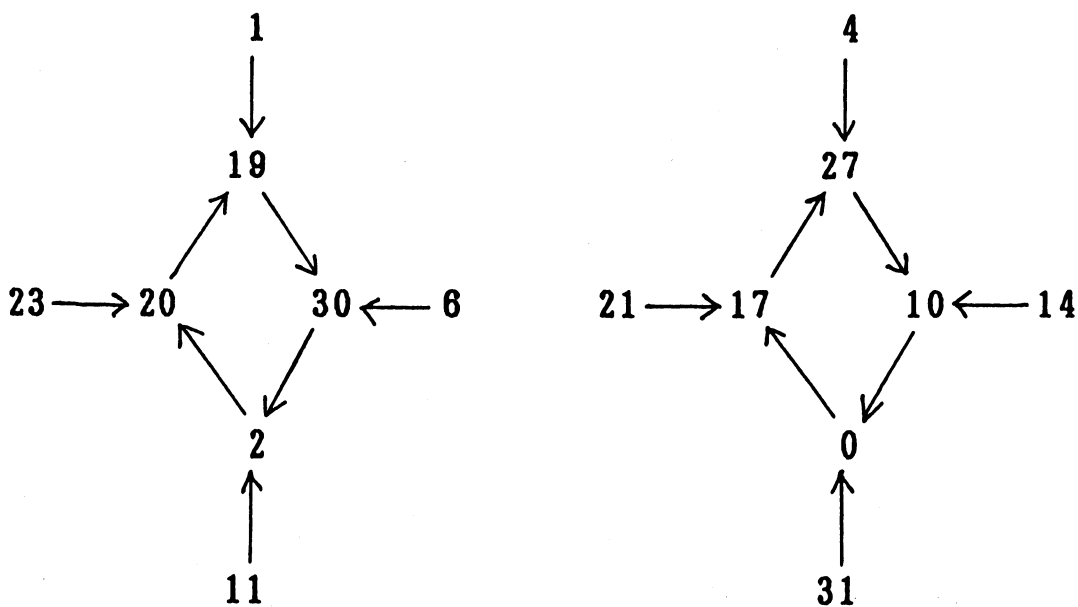
$K_1(4)$:



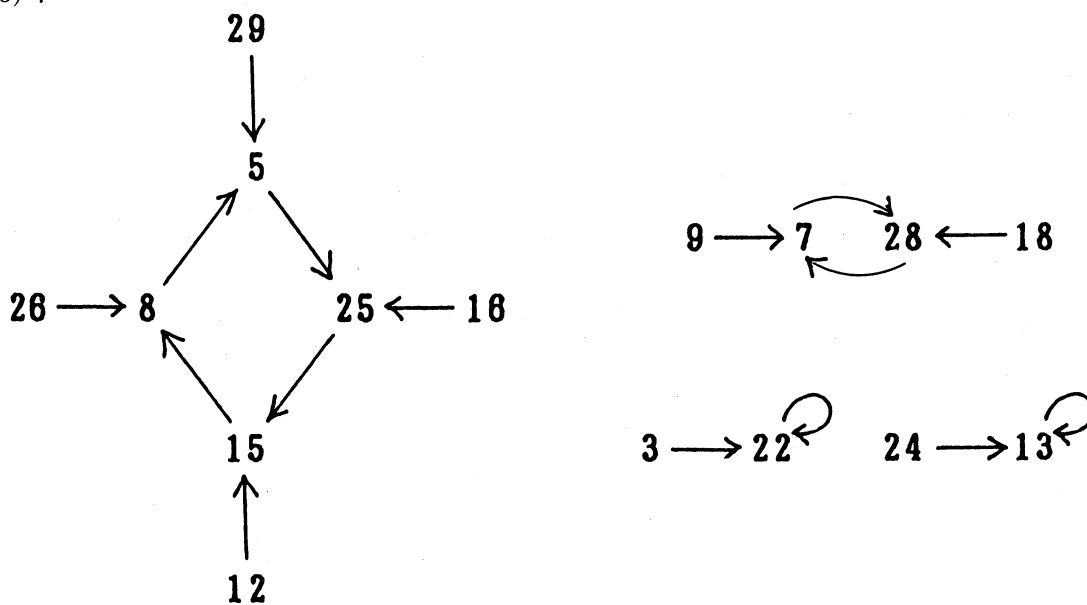
Type of scheme : $\{6(2), 3(1), 1(1)\}$

The diagram, thus constructed, is called a configuration diagram of scheme $K_0(n)$. The configuration diagram of scheme $K_1(n)$ is constructed by using the mapping g in place of the mapping f .

We shall show examples of configuration diagrams of $K_0(n)$ and $K_1(n)$ for $n=3, 4$ and 5.



$K_1(5)$:



Type of scheme : $\{4_2(3), 2_2(1), 1_2(2)\}$

In the above diagrams we show type of schemes by using notations defined in [2].

Proposition 2.3 *If n is odd, then there exists a nonzero vector y in X such that*

$$f(x+y \bmod 2) = f(x)$$

and

$$g(x+y \bmod 2) = g(x)$$

for every x in X .

Proof Let us take a vector

$$y = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Then we have $f(y) = Ay \pmod{2} = 0$. So we get, by additivity of the mapping f ,

$$\begin{aligned} f(x+y \pmod{2}) &= f(x) + f(y) \pmod{2} \\ &= f(x), \end{aligned}$$

and, by the formula (1.1)

$$\begin{aligned} g(x+y \pmod{2}) &= f(x+y \pmod{2}) + d \pmod{2}, \\ &= f(x) + f(y) + d \pmod{2}, \\ &= f(x) + d \pmod{2}, \\ &= g(x) \pmod{2}, \end{aligned}$$

The conclusion holds. □

Next we shall study fixed points of the mapping f and g .

Proposition 2.4 *Fixed points of the mapping f , that is, the vector x in X satisfying the relation $f(x) = x$, are as follows:*

- (i) 0 (if $n \equiv 0 \pmod{3}$),
- (ii) 0 (if $n \equiv 1 \pmod{3}$),
- (iii) 0 and

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad (\text{if } n \equiv 2 \pmod{3})$$

Proof The relation $f(x) = x$ is written as follows:

$$\left\{ \begin{array}{l} x_2 = x_1, \\ x_1 + x_3 \bmod 2 = x_2, \\ x_2 + x_4 \bmod 2 = x_3, \\ \vdots \\ x_{i-1} + x_{i+1} \bmod 2 = x_i, \\ \vdots \\ x_{n-2} + x_n \bmod 2 = x_{n-1}, \\ x_{n-1} = x_n \end{array} \right. \quad (2.1)$$

By the equation (2.1), the zero vector is a fixed point of the mapping f for every n , and the fixed point the first element of which is 1 is a vector in which the second element is 1, the third element is 0, and the last two elements are equal.

Such a fixed point appears only if $n \equiv 2 \pmod{3}$, and the form of the fixed point is

$$\left(\begin{array}{c} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{array} \right)$$

Proposition 2.5 *Fixed points of the mapping g are as follows:*

$$(i) \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ \vdots \\ \vdots \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{if } n \equiv 0 \pmod{3},$$

$$(ii) \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{if } n \equiv 1 \pmod{3},$$

and

$$(iii) \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ \vdots \\ \vdots \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{if } n \equiv 2 \pmod{3}$$

Proof The relation $g(x) = x$ is written as follows:

$$\left\{ \begin{array}{l} 1 + x_2 \text{ mod } 2 = x_1, \\ x_1 + x_3 \text{ mod } 2 = x_2, \\ x_2 + x_4 \text{ mod } 2 = x_3, \\ \vdots \\ \vdots \\ x_{i-1} + x_{i+1} \text{ mod } 2 = x_i, \\ \vdots \\ \vdots \\ x_{n-2} + x_n \text{ mod } 2 = x_{n-1}, \\ x_{n-1} + 1 \text{ mod } 2 = x_n \end{array} \right. \quad (2.2)$$

From the equation (2.2), we get

$$x_1 = \bar{x}_2,$$

where \bar{x}_2 is a *mod 2*-complement of x_2 , that is

$$\bar{x}_2 = \begin{cases} 1 & (\text{if } x_2 = 0), \\ 0 & (\text{if } x_2 = 1). \end{cases}$$

Finally we get

$$x_{n-1} = \bar{x}_n,$$

and

$$x_{n-2} = \begin{cases} x_n & (\text{if } x_{n-1} = 0, \text{ that is, } x_n = 1) \\ \bar{x}_n & (\text{if } x_{n-1} = 1, \text{ that is, } x_n = 0) \end{cases}$$

Combining these results, it follows that

(i) Fixed point the first and the last three elements of which are both 1, 0 and 1 appears only if $n \equiv 0 \pmod{3}$.

(ii) Fixed point the first and the last three elements of which are 0, 1 and 1, and 1, 1 and 0, respectively, appears only if $n \equiv 1 \pmod{3}$.

(iii) Fixed point the first and the last three elements of which are 1, 0 and 1, and 1, 1 and 0, or, 0, 1 and 1 and 1, 0 and 1, respectively, appears only if $n \equiv 2 \pmod{3}$.

So we get the conclusion. \square

3. Characterization number $h(n)$

The mappings f and g have the following property.

Proposition 3.1

$$g^p(x) = f^p(x) + g^p(0) \quad \text{for each } x \in X (p=1, 2, \dots)$$

Proof (by induction)

When $p=1$, $g(x) = f(x) + d \pmod{2}$ (by definition)
 $= f(x) + g(0) \pmod{2}$ ($x \in X$)

Assume that the relation holds for $p-1$, that is,

$$g^{p-1}(x) = f^{p-1}(x) + g^{p-1}(0) \quad (x \in X)$$

Then, for each x in X we have

$$\begin{aligned} g^p(x) &= g(g^{p-1}(x)) \\ &= f(g^{p-1}(x)) + d \quad \pmod{2} \\ &= f(f^{p-1}(x) + g^{p-1}(0)) + d \quad \pmod{2} \end{aligned}$$

By additivity of the mapping f ,

n	h(n)	n	h(n)
=====		=====	
1	1	28	16383
2	2	29	61
3	x	30	31
4	3	31	x
5	4	32	62
6	7	33	30
7	x	34	4095
8	14	35	x
9	6	36	87381
10	31	37	1022
11	x	38	8190
12	63	39	x
13	14	40	1023
14	30	41	252
15	x	42	127
16	15	43	x
17	28	44	8190
18	511	45	4094
19	x	46	8388607
20	126	47	x
21	62	48	2097151
22	2047	49	2046
23	x	50	510
24	1023	51	x
25	126	52	67108863
26	1022	53	2044
27	x	54	1048575
		55	x

$$\begin{aligned} &= f^p(x) + f(g^{p-1}(0)) + d && \text{mod } 2 \\ &= f^p(x) + g(g^{p-1}(0)) && \text{mod } 2 \\ &= f^p(x) + g^p(0) \end{aligned}$$

The induction holds. □

So we immediately get the following theorem.

Theorem 3.2. $f^p = g^p$ if and only if $g^p(0) = 0$.

Let us denote the least positive number p such that $g^p(0) = 0$, if exists, by $h(n)$. We call the number $h(n)$, if exists, the characterization number of schemes $K_0(n)$ and $K_1(n)$. By using computer we get the following table of $h(n)$ (see p. 122.).

The table shows the following relation

- (1) The number $h(n)$ exists if and only if $n \not\equiv 3 \pmod{4}$.
- (2) If n is even, then we have

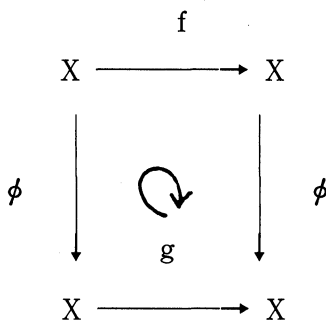
$$h(2n+1) = 2h(n)$$

the proofs of which are obtained in [3], using slightly changed form of CA-90(n).

4. Isomorphism between schemes $K_0(n)$ and $K_1(n)$.

The schemes $K_0(n)$ and $K_1(n)$ have the following relation.

Theorem 4.1 For two schemes $K_0(n) = \langle X, f \rangle$ and $K_1(n) = \langle X, g \rangle$, there exists a bijection ϕ from X onto X such that the following diagram commutes, and, if p is a fixed point of the mapping f , then $\phi(p)$ is a fixed point of the mapping g



Proof Let us take a mapping ϕ as follows:

$$\phi(x) = x + q \pmod{2} \quad (x \in X)$$

where q is a fixed point of the mapping g , which exists for each n (proposition 2.5).

So we get for each x in X

$$\begin{aligned} (\phi \cdot f)(x) &= f(x) + q && \text{mod } 2, \\ &= f(x) + g(q) && \text{mod } 2, \\ &= f(x) + f(q) + d && \text{mod } 2, \\ &= f(x + q \pmod{2}) + d && \text{mod } 2, \\ &= g(x + q \pmod{2}) && \text{mod } 2, \\ &= g(\phi(x)) \\ &= (g \cdot \phi)(x) \end{aligned}$$

And, if p is a fixed point of the mapping f , that is $f(p) = p$, then we have

$$\begin{aligned} g(\phi(p)) &= \phi(f(p)) && \text{(the commutativity),} \\ &= \phi(p) \end{aligned}$$

So the vector $\phi(p)$ is a fixed point of the mapping g . □

We can get similar results for schemes representing finite cellular automata $CA-150(n)$ with both boundary conditions of 0-type and 1-type, respectively.

References

- [1] S. FUJINO: *On the behavior of linear finite cellular automata CA-90(m)*, *RMC* **63-03J** (1988), pp. 25
- [2] T. KITAGAWA and S. FUJINO: *On modular type iteration scheme*, *RMC* **64-09J** (1989), pp. 171
- [3] Y. KAWAHARA: *On the existence of Fujino's characterization number $H(m)$ associated with finite cellular automata CA-90(m-1)*, *RMC* **63-08J** (1988), pp. 17